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Existence and properties of positive solutions for Caputo fractional difference equation and applications

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Abstract

This paper deals with a typical Caputo fractional differential equation. This equation appears in important applications such as modeling of medicine distributed throughout the body via injection and equation for general population growth. We use the fixed point theory of concave operators in specific normed spaces to find a parameter interval for which the unique positive solution exists. Some properties of positive solutions are studied and illustrative examples are given.

Keywords. Green's function, Positive solution, Fixed point theorem, Fractional difference equation.

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1. Introduction

There is no doubt that the topic of fractional calculus is one of the useful fields of applied mathematics which has applications in the areas such as engineering, economics, control theory, and other fields. In recent years the fractional differential equations have been of great interest, there is a great deal of work focused on studying the existence and uniqueness of solutions [1, 2, 7, 8].

Discrete fractional difference equations appear in applications. For example, the mathematical model of discrete fractional difference equation is given by

$$\begin{cases} \Delta_c^{\delta} y(t) = -\mu y(t+\delta-1), \\ y(0) = a_0, \end{cases}$$

where y is the concentration of the drug in the body at time t after injection, μ is the constant first-order elimination rate of the drug (the negative sign means the drug is eliminated from the body) and a_0 is the initial concentration, see [4] for more details.

This kind of application motivates us to consider the existence and uniqueness of positive solutions for more general fractional differential equations of the following form:

$$\begin{cases} -\Delta_c^{\delta} y(t) = \mu f(t+\delta-1, y(t+\delta-1)), \quad t \in \{0, 1, ..., T+1\} \\ y(\delta-n) = a_0, \\ \Delta y(\delta+T) = \Delta^j y(\delta-n) = 0, j = 2, 3, ..., n-1, \end{cases} \tag{1.1}$$

where $\delta \in (n-1, n]$ and $a_0 \geq 0, T \geq n, \mu$ are real numbers, Δ_c^{δ} is the standard Caputo difference, $f : [\delta-(n-1), \delta+T]_{\mathbb{N}_{\delta-(n-1)}} \times [0, \infty) \to [0, \infty)$ is continuous and increasing.

We arrange the paper in the following manner. Section 2 gives some preliminary results and notations. Section 3 is devoted to find an interval for parameter μ, for which the unique positive solution of (1.1) exists. In section 4, we study a standard equation for general population growth and the model with the discrete fractional calculus of medicine

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distributing throughout the body via injection. There are several recent areas of specialized research in mathematical biology: Enzyme kinetics, biological tissue analysis, cancer modeling, heart and arterial disease modeling being among the popular ones, see [6]. There are three versions of fractional calculus: continuous, discrete, and quantum. In this paper, we focus on the discrete version.

2. Preliminaries and basic notations

Definition 2.1. [10] Let $\mathbb{N}_a := \{a, a+1, ...\}$, $a \in \mathbb{R}$. The difference operator Δ acts on given function f as follows:

$$\Delta f(t) := f(t+1) - f(t), t \in \mathbb{N}_a.$$

Definition 2.2. [10] Suppose $t > 1, \delta \in \mathbb{R}$. The falling factorial power $t^{\underline{\delta}}$ is given by:

$$t^{\underline{\delta}} = \frac{\Gamma(t+1)}{\Gamma(t+1-\delta)}.$$

Theorem 2.3. *[5] According to definition of Δ and falling fractional power we have:*

$$\Delta t^{\underline{\delta}} = \delta t^{\underline{\delta-1}}.$$

Definition 2.4. [5] The fractional sum of order δ for a given function h, for $\delta > 0$, is defined by

$$\Delta^{-\delta} h(t) := \frac{1}{\Gamma(\delta)} \sum_{k=a}^{t-\delta} (t - \sigma(k))^{\underline{\delta-1}} h(k),$$

for $t \in \{\delta + a, \delta + a + 1, ...\} := \mathbb{N}_{\delta+a}$ and $\sigma(k) = k + 1$. The δth fractional difference of order δ, for $\delta > 0$, is defined by $\Delta^{\delta} h(t) = \Delta^n \Delta^{\delta-n} h(t)$, where $t \in \mathbb{N}_{a+n-\delta}$ and $\delta \in \mathbb{N}$ is such that $0 \leq n-1 < \delta \leq n$.

The Caputo fractional difference for a given function h for $\delta > 0$, is defined by

$$\Delta_c^{-\delta} h(t) := \Delta^{-(n-\delta)} \Delta^n h(t) = \frac{1}{\Gamma(n-\delta)} \sum_{k=a}^{t-(n-\delta)} (t - \sigma(k))^{\underline{n-\delta-1}} \Delta^n h(k), \tag{2.1}$$

where $0 \leq n-1 < \delta \leq n$.

Lemma 2.5. *[3] If $\delta > 0$ and h is defined on $\mathbb{N}_a$, then*

$$\Delta_{a+(n-\delta)}^{-\delta} \Delta_c^{\delta} h(t) = h(t) - \sum_{k=0}^{n-1} c_k (t-a)^{\underline{k}}, \tag{2.2}$$

where $c_k \in \mathbb{R}, k = 0, 1, 2, ..., n-1$, and $n - 1 < \delta \leq n$.

Lemma 2.6. *[9] Let $h : \mathbb{N}_{a+\delta} \times \mathbb{N}_a \to \mathbb{R}$ be given. Then*

$$\Delta\Big(\sum_{k=a}^{t-\delta} h(t,k)\Big) = \sum_{k=a}^{t-\delta} \Delta_t h(t,k) + h(t+1, t+1-\delta),$$

for $t \in \mathbb{N}_{a+\delta}$.

Let $\mathbb{E}$ be a Banach space and $\mathbb{P} \subset \mathbb{E}$. Then $\mathbb{P}$ is called a cone if it satisfies in the following conditions:
(i) if $a \in \mathbb{P}, \kappa \geq 0$ then $\kappa a \in \mathbb{P}$,
(ii) if $a \in \mathbb{P}$ and $-a \in \mathbb{P}$, then $a = 0$.
We say that $(\mathbb{E}, \|.\|)$ is partially ordered by a cone $\mathbb{P}$ and use the notation $a \preceq b$ if $b - a \in \mathbb{P}$. If $a \preceq b$ and $a \neq b$, then we use the notation $a \prec b$. Let $\mathbb{P}^0$ be the set of interior points of $\mathbb{P}$. If $\mathbb{P}^0 \neq \emptyset$, then $\mathbb{P}$ is called solid. If $b - a \in \mathbb{P}^0$ then we use the notation $a \ll b$. The cone $\mathbb{P}$ is called normal if there exists a positive number $M > 0$ such that, for all $a, b \in \mathbb{E}$, if $0 \preceq a \preceq b$ then $\|a\| \leq M\|b\|$. The set of $[a_1, a_2] = \{a \in \mathbb{E} | a_1 \leq a \leq a_2\}$ is called the interval between $a_1, a_2 \in \mathbb{E}$. If $T : \mathbb{D} \to \mathbb{E}$ satisfies in $T(\lambda u + (1-\lambda)v) \geq \lambda Tu + (1-\lambda)Tv$, for $u, v \in \mathbb{D}$, where $\mathbb{D}$ is convex and $\lambda \in [0, 1]$,

then T is called a concave operator. Suppose e is a nonzero element of $\mathbb{P}$. Define $\mathbb{E}_e = \{t \in \mathbb{E} : \exists k > 0, -ke \leq t \leq ke\}$ with norm

$$\|t\|_e = \inf\{k > 0 : -ke \leq t \leq ke\}.$$

Lemma 2.7. *[11] If $\mathbb{P}$ is a normal cone, then*
(i) $\mathbb{E}_e$ is a Banach space with respect to $\|.\|_e$. Assume that there exists a positive number $M > 0$ such that $\|t\| \leq M\|t\|_e, \forall t \in \mathbb{E}_e$.
(ii) $\mathbb{P}_e = \mathbb{E}_e \cap \mathbb{P}$ is a normal solid cone in $\mathbb{E}_e$.

Lemma 2.8. *[11] Assume that $\mathbb{P}$ is a normal solid cone and $A : \mathbb{P} \to \mathbb{P}$ be concave. If $A(0) > 0$ then*
(i) there exists a positive number μ^*, such that for $0 \leq \mu < \mu^*$ there exists a solution $y(\mu)$ in $\mathbb{P}$ that is unique solution of the equation

$$y = \mu A y. \tag{2.3}$$

The equation (2.3) does not have positive solution in $\mathbb{P}$ for $\mu \geq \mu^*$,
(ii) consider the iterative sequence $y_n(\mu) = \mu A y_{n-1}(\mu), n \geq 1$ where $y_0(\mu) = y_0 \in \mathbb{P}$. Then $y_n(\mu) \to y(\mu)$ as $n \to \infty$, for $0 < \mu < \mu^*$,
(iii) the function $y(.)$ is continuous and strictly increasing on $[0, \mu^*)$,
(iv) $\|y(\mu)\| \to +\infty$, as $\mu \to \mu^* - 0$,
(v) suppose that there exist $\nu_0 \in \mathbb{P}$ and $\mu_0 > 0$ where $\mu_0 A \nu_0 \leq \nu_0$, then $\mu^* > \mu_0$.

Lemma 2.9. *[11] Assume $\mathbb{P}$ is a normal solid cone and $A : \mathbb{P}^0 \to \mathbb{P}^0$ is an increasing operator. Assume that there is positive number r such that $0 < r < 1$, and*

$$A(ty) \geq t^r Ay, \quad \forall x \in \mathbb{P}^0, 0 < t < 1. \tag{2.4}$$

Let y_μ be unique solution of the equation $Ay = \mu y$ $(\mu > 0)$ in $\mathbb{P}^0$. Then
(i) y_μ is strictly decreasing;
(ii) y_μ is continuous (i.e., $\mu \to \mu_0$ implies $\|y_\mu - y_{\mu_0}\| \to 0$);
(iii) $\lim_{\mu\to\infty} \|y_\mu\| = 0, \lim_{\mu\to 0^+} \|y_\mu\| = +\infty$.

Lemma 2.10. (*i*) *The solution of (1.1) is represented by*

$$y(t) = \mu \sum_{k=0}^{T+1} \mathcal{K}(t,k) f(k + \delta - 1, y(k + \delta - 1)) + a_0, \tag{2.5}$$

where $\mathcal{K}(t,k)$ is the Green's fuction given by

$$\mathcal{K}(t,k) = \frac{1}{\Gamma(\delta)} \begin{cases} (\delta-1)(t-\delta+n)(T+\delta-k-1)^{\underline{\delta-2}} - (t-k-1)^{\underline{\delta-1}} & ,0 \leq k < t-\delta+1, \\ \\ (\delta-1)(t-\delta+n)(T+\delta-k-1)^{\underline{\delta-2}} & ,0 \leq t-\delta+1 \leq k, \end{cases}$$

(*ii*) $\mathcal{K}(t,k) > 0, (t,k) \in [\delta-(n-1), \delta+T]_{\mathbb{N}_{\delta-(n-1)}} \times [0, T+1]_{\mathbb{N}_0}$.

Proof. Using Lemma 2.5

$$y(t) = -\frac{\mu}{\Gamma(\delta)} \sum_{k=0}^{t-\delta} (t-k-1)^{\underline{\delta-1}} f(k+\delta-1, y(k+\delta-1)) + c_0 + c_1 t + c_2 t^{\underline{2}} + \dots + c_{n-1} t^{\underline{n-1}},$$

for $c_i \in \mathbb{R}, i = 0, 1, 2, ..., n-1$.

On the other hand by applying Lemma 2.6 we have

$$\begin{aligned} \Delta y(t) &= -\frac{\mu}{\Gamma(\delta)} \sum_{k=0}^{t+1-\delta} (\delta-1)(t-k-1)^{\underline{\delta-2}} f(k+\delta-1, y(k+\delta-1)) + c_1 + 2c_2 t \\ &+ \dots + (n-1) c_{n-1} t^{\underline{n-2}}, \end{aligned}$$

$$\begin{aligned}\Delta^2 y(t) &= -\frac{\mu}{\Gamma(\delta)}\sum_{k=0}^{t+2-\delta}(\delta-1)(\delta-2)(t-k-1)^{\underline{\delta-3}}f(k+\delta-1,y(k+\delta-1))+2c_2\\ &+ ...+(n-1)(n-2)c_{n-1}t^{\underline{n-3}},\end{aligned}$$

$\vdots$

$$\begin{aligned}\Delta^{n-1} y(t) &= -\frac{\mu}{\Gamma(\delta)}\sum_{k=0}^{t+n-1-\delta}(\delta-1)(\delta-2)...(\delta-n+1)(t-k-1)^{\underline{\delta-n}}f(k+\delta-1,y(k+\delta-1))\\ &+ (n-1)(n-2)...c_{n-1}.\end{aligned}$$

From $\Delta^j y(\delta-n)=0, j=2,3,...,n-1$, we get $c_2=c_3=...=c_{n-1}=0$, and by $\Delta y(\delta+T)=0, y(\delta-n)=a_0$, we have

$$c_1=\frac{\mu}{\Gamma(\delta)}\sum_{k=0}^{T+1}(\delta-1)(T+\delta-k-1)^{\underline{\delta-2}}f(k+\delta-1,y(k+\delta-1)),$$

and $c_0=-(\delta-n)c_1+a_0$, then we have

$$c_0=\frac{-(\delta-n)\mu}{\Gamma(\delta)}\sum_{k=0}^{T+1}(\delta-1)(T+\delta-k-1)^{\underline{\delta-2}}f(k+\delta-1,y(k+\delta-1))+a_0.$$

Therefore, the solution of (1.1) is

$$\begin{aligned}y(t) &= -\frac{\mu}{\Gamma(\delta)}\sum_{k=0}^{t-\delta}(t-k-1)^{\underline{\delta-1}}f(k+\delta-1,y(k+\delta-1))\\ &+ \frac{-(\delta-n)\mu}{\Gamma(\delta)}\sum_{k=0}^{T+1}(\delta-1)(T+\delta-k-1)^{\underline{\delta-2}}f(k+\delta-1,y(k+\delta-1))\\ &+ \frac{t\mu}{\Gamma(\delta)}\sum_{k=0}^{T+1}(\delta-1)(T+\delta-k-1)^{\underline{\delta-2}}f(k+\delta-1,y(k+\delta-1))+a_0\\ &= \mu\sum_{k=0}^{T+1}\mathcal{K}(t,k)f(k+\delta-1,y(k+\delta-1))+a_0.\end{aligned}$$

Thus part (i) is proved. For (ii), if $0\le t-\delta+1\le k<T+1$ then $\mathcal{K}(t,k)>0$, it is immediate from the representation of $\mathcal{K}(t,k)$. Thus, it is sufficient to show that $\mathcal{K}(t,k)>0$, for $0\le k<t-\delta+1<T+1$. One can see that $\Delta_t\mathcal{K}(t,k)\ge 0$, for $0\le k<t-\delta+1<T+1$, we have

$$\begin{aligned}\Delta_t\mathcal{K}(t,k) &= (\delta-1)(T+\delta-k-1)^{\underline{\delta-2}}-(\delta-1)(t-k-1)^{\underline{\delta-2}}\\ &\ge (\delta-1)((T+\delta-k-1)^{\underline{\delta-2}}-(\delta-1)(T+\delta-k-1)^{\underline{\delta-2}}=0.\end{aligned}$$

Thus

$$\begin{aligned}\min_{k+\delta-1\le t\le T+\delta}\mathcal{K}(t,k) &= \mathfrak{K}(k+\delta-1,k)\\ &= (\delta-1)(k+\delta-1)(T+\delta-k-1)^{\underline{\delta-2}}-(\delta-2)^{\underline{\delta-1}}\\ &= (\delta-1)(k+\delta-1)(T+\delta-k-1)^{\underline{\delta-2}}>0.\end{aligned}$$

Thus part (ii) is proved. $\square$

3. Existence results

We define the Banach space

$$\mathbb{E} = \Big\{y : [\delta - n, \delta + T]_{\mathbb{N}_{\delta-n}} \to \mathbb{R}\Big\},$$

with norm

$$\|y\|_{\mathbb{E}} = \max |y(t)|, \qquad t \in [\delta - n, \delta + T]_{\mathbb{N}_{\delta-n}}.$$

Consider the set

$$\mathbb{P} = \Big\{y \in \mathbb{E} : y(t) \geq 0, t \in [\delta - n, \delta + T]_{\mathbb{N}_{\delta-n}}\Big\},$$

clearly $\mathbb{P}$ is a normal solid cone in $\mathbb{E}$. Let

$$e(t) = \sum_{k=0}^{T+1} \mathcal{K}(t, k).$$

Using Lemma 2.10 we have $\mathcal{K}(t, k) > 0$. Therefore, $e(t) > 0$, thus $e \in \mathbb{P} \setminus \{0\}$. Suppose

$$\mathbb{E}_e = \big\{y \in \mathbb{E} : \exists\tau > 0, -\tau e(t) \leq y(t) \leq \tau e(t), \forall t \in [\delta - n, \delta + T]_{\mathbb{N}_{\delta-n}}\big\},$$

with norm

$$\|y\|_{\mathbb{E}_e} = \inf \big\{\tau > 0 : -\tau e(t) \leq y(t) \leq \tau e(t), \forall t \in [\delta - n, \delta + T]_{\mathbb{N}_{\delta-n}}\big\}.$$

Let $\tilde{\mathbb{P}} = \mathbb{E}_e \cap \mathbb{P}$. Using Lemma 2.7, implies that $\mathbb{E}_e$ is a Banach space, $\tilde{\mathbb{P}}$ is a normal solid cone in $\mathbb{E}_e$ and

$$\tilde{\mathbb{P}}^0 = \big\{y \in \mathbb{E}_e : \exists\theta > 0 \text{ such that } y(t) \geq \theta e(t), \forall t \in [\delta - n, \delta + T]_{\mathbb{N}_{\delta-n}}\big\}.$$

Besides, there exists a positive number M such that

$$\|y\|_{\mathbb{E}} \leq M\|y\|_{\mathbb{E}_e}, \forall y \in \mathbb{E}_e.$$

Theorem 3.1. *Suppose that f is concave and there are positive numbers $\sigma > 0, \alpha > 0$ such that $f(t, 0)$ is lower bounded by σ and $f(t, 1)$ is upper bounded by α, repectively, for $t \in [\delta - (n-1), \delta + T]_{\mathbb{N}_{\delta-(n-1)}}$. Then*
(i) there exists a positive number μ^* that (1.1) has exactly one solution $y_\mu > 0$ in $\tilde{\mathbb{P}}$, for $\mu \in (0, \mu^*)$, If $\mu \geq \mu^*$, then (1.1) does not have positive solution in $\tilde{\mathbb{P}}$;
(ii) define

$$y_m(t) = \mu \sum_{k=0}^{T+1} \mathcal{K}(t, k) f(k + \delta - 1, y_{m-1}(k + \delta - 1)), \quad m \geq 1,$$

where $y_0 \in \tilde{\mathbb{P}}$ and $\mu \in (0, \mu^*)$, then

$$\max_{t\in[\delta-n,\delta+T]_{\mathbb{N}_{\delta-n}}} |y_m(t) - y_\mu(t)| \to 0, \quad as \quad m \to \infty;$$

(iii) $\max_{t\in[\delta-n,\delta+T]_{\mathbb{N}_{\delta-n}}} |y_\mu(t) - y_{\mu_0}(t)| \to 0$, as $\mu \to \mu_0$, for $\mu \in (0, \mu^*)$. If $0 < \mu_1 < \mu_2 < \mu^*$, then $y_{\mu_1}(t) \leq y_{\mu_2}(t)$ and $y_{\mu_1}(t) \neq y_{\mu_2}(t)$, $\forall t \in [\delta - n, \delta + T]_{\mathbb{N}_{\delta-n}}$;

(iv) $y_{l\mu}(t) \leq l y_\mu(t), l \in [0, 1], t \in [\delta - n, \delta + T]_{\mathbb{N}_{\delta-n}}$, where $\mu \in (0, \mu^*)$;

(v) if $\lim_{y\to\infty} \dfrac{f(t, y)}{y} = 0$ uniformly on $[\delta - (n-1), \delta + T]_{\mathbb{N}_{\delta-(n-1)}}$ then $\mu^* = \infty$.

Proof. Using Lemma 2.10 we conclude that $y(t)$ is the solution of (1.1) if and only if

$$y(t) = \mu \sum_{k=0}^{T+1} \mathcal{K}(t,k) f(k+\delta-1, y(k+\delta-1)) + a_0,$$

where $\mathcal{K}(t,k)$ is defined by Lemma 2.10. Without loss of generality, we define the operator A, without a_0 by

$$(Ay)(t) = \sum_{k=0}^{T+1} \mathcal{K}(t,k) f(k+\delta-1, y(k+\delta-1)).$$

Clearly y is the solution of (1.1) if and only if $y = \mu Ay$. For $y \in \mathbb{P}$, since $\mathcal{K}(t,k)$ and $f(t,x)$ are continuous, then Ay is continuous. Positivity of $\mathcal{K}(t,k)$ and $f(t,x)$ implies that $(Ay)(t) \geq 0, t \in [\delta-n, \delta+T]_{\mathbb{N}_{\delta-n}}$. So $Ay \in \mathbb{P}$. Concavity of $f(t,\cdot)$ for $y > 1$ implies

$$f(t,1) = f(t, \frac{1}{y}.y + (1-\frac{1}{y}).0) \geq \frac{1}{y} f(t,y) + (1-\frac{1}{y}) f(t,0),$$

and thus using the assumption of the theorem, we get

$$f(t,y) \leq y f(t,1) - (y-1) f(t,0) \leq y f(t,1) \leq \alpha y.$$

Therefore, for $y \in \mathbb{P}$, we have

$$0 \leq \sum_{k=0}^{T+1} \mathcal{K}(t,k) f(k+\delta-1, y(k+\delta-1)) \leq \sum_{k=0}^{T+1} \mathcal{K}(t,k) f(k+\delta-1, \|y\|_E) \leq M_1 e(t),$$

where

$$M_1 = \max_{t \in [\delta-(n-1), \delta+T]_{\mathbb{N}_{\delta-(n-1)}}} f(t, \|y\|_{\mathbb{E}}) \leq \max_{[\delta-(n-1), \delta+T]_{\mathbb{N}_{\delta-(n-1)}}} f(t, \|y\|_{\mathbb{E}}+1) \leq \alpha(\|y\|_{\mathbb{E}}+1).$$

Thus $0 \leq (Ay)(t) \leq M_1 e(t)$ so $Ay \in \mathbb{E}_e$, then $Ay \in \tilde{\mathbb{P}}$. Hence A is defined from $\tilde{\mathbb{P}}$ to $\tilde{\mathbb{P}}$. Concavity of $f(t,\cdot)$ implies A is concave.

By assumption of the theorem $A(0) \geq \sigma e(t) > 0$, so $A(0) \in \tilde{\mathbb{P}}^0$. Using Lemma 2.8 there is a positive number μ^* such that $y = \mu Ay$ has a unique solution y_μ for $\mu \in (0,\mu^*)$. Thus if $\mu \geq \mu^*, y = \mu Ay$ does not have solution in $\tilde{\mathbb{P}}$.

Let $y_0 \in \tilde{\mathbb{P}}$, and define $y_m = \mu A y_{m-1}, m = 1,2,3,...$, then, $y_m \to y_\mu$, as $m \to \infty$ for $\mu \in (0,\mu^*)$ y_μ is continuous, and strictly increasing. Moreover $y_{(l\mu)} \leq l y_\mu$ for $0 < \mu < \mu^*, 0 \leq l \leq 1$; $\|y_\mu\|_{\mathbb{E}_e} \to \infty$, as $\mu \to \mu^* - 0$, if there exist $\nu_0 \in \tilde{\mathbb{P}}$ and $\mu_0 > 0$ such that $\mu_0 A\nu_0 \leq \nu_0$ then $\mu^* > \mu_0$. This means,
(i) for $\mu \in (0,\mu^*)$, the problem (1.1) has exactly one positive solution y_μ in $\tilde{\mathbb{P}}$, and for $\mu \geq \mu^*$, the problem (1.1) does not have positive solution in $\tilde{\mathbb{P}}$;
(ii) let $y_0 \in \tilde{\mathbb{P}}$ and for $\mu \in (0,\mu^*)$, and define

$$y_m = \mu \sum_{k=0}^{T+1} \mathcal{K}(t,k) f(k+\delta-1, y_{m-1}(k+\delta-1)), m \geq 1,$$

then

$$\max_{t \in [\delta-n, \delta+T]_{\mathbb{N}_{\delta-n}}} |y_m(t) - y_\mu(t)| \to 0, \quad as \quad m \to \infty,$$

(iii) $\max_{t \in [\delta-n,\delta+T]_{\mathbb{N}_{\delta-n}}} |y_\mu(t) - y_{\mu_0}(t)| \to 0, \quad as \quad \mu \to \mu_0$, where $0 < \mu_0 < \mu^*$, and if $0 < \mu_1 < \mu_2 < \mu^*$, then $\exists \theta > 0$, $y_{\mu_2}(t) - y_{\mu_1}(t) \geq \theta e(t), \forall t \in [\delta-n, \delta+T]_{\mathbb{N}_{\delta-n}}$, thus $y_{\mu_1}(t) \leq y_{\mu_2}(t), \forall t \in [\delta-n, \delta+T]_{\mathbb{N}_{\delta-n}}$ and $y_{\mu_1}(t) \neq y_{\mu_2}(t)$,
(iv) $y_{(l\mu)}(t) \leq l y_\mu(t), l \in [0,1], t \in [\delta-n, \delta+T]_{\mathbb{N}_{\delta-n}}$, where $\mu \in (0,\mu^*)$. Thus the statements $(i),(ii),(iii),(iv)$ hold.

To prove (v), let $\eta = \|e\|_{\mathbb{E}}$. Let $\mu > 0$ be an arbitrary positive real number. From $\lim_{y\to\infty} \dfrac{f(t,y)}{y} = 0$ we may choose a $B > 0$ large so that $f(t, B) \leq (\eta\mu)^{-1}B, \forall t \in [\delta-(n-1), \delta+T]_{\mathbb{N}_{\delta-(n-1)}}$. Set $\nu_0(t) = \eta^{-1}Be(t)$, thus

$$\begin{aligned}\mu(A\nu_0)(t) &= \mu\sum_{k=0}^{T+1}\mathcal{K}(t,k)f(k+\delta-1, \eta^{-1}Be(k+\delta-1))\\ &\leq \mu\sum_{k=0}^{T+1}\mathcal{K}(t,k)f(k+\delta-1, B) \leq \eta^{-1}Be(t) = \nu_0(t),\end{aligned}$$

$$\nu_0(t) - \mu(A\nu_0)(t) = \sum_{k=0}^{T+1}\mathcal{K}(t,k)[\eta^{-1}B - \mu f(k+\delta-1, \eta^{-1}Be(k+\delta-1))] \leq M_2 e(t),$$

where

$$M_2 = \sup_{t\in[\delta-(n-1),\delta+T]_{\mathbb{N}_{\delta-(n-1)}}}[\eta^{-1}B + \mu f(t, B)] \leq \eta^{-1}B + \mu\alpha(B+1).$$

Therefore, $\nu_0 - \mu(A\nu_0) \in \tilde{\mathbb{P}}$.

That is, $\mu A\nu_0(t) \leq \nu_0$. Using Lemma 2.8 (v), implies that $\mu^* > \mu$. Since μ is any positive real number, we conclude $\mu^* = \infty$. □

Theorem 3.2. *Suppose that following conditions hold*
(i) there is a positive number r such that $0 < r < 1$, and

$$f(t, \tau x) \geq \tau^r f(t, x), \quad \forall t \in [\delta-(n-1), \delta+T]_{\mathbb{N}_{\delta-(n-1)}}, \quad x \geq 0, \quad \tau \in (0,1),$$

(ii) there exists a positive number γ such that $f(t, 1) \leq \gamma$,
(iii) $\min_{t\in[\delta-(n-1),\delta+T]_{\mathbb{N}_{\delta-(n-1)}}} f(t, e(t)) > 0$, where $e(t)$ is given in Theorem 3.1. Then we have:
(a) the problem (1.1) has exactly one positive solution y_μ in $\tilde{\mathbb{P}}^0$, for $\mu > 0$,
(b) If $0 < \mu_1 < \mu_2$, then $y_{\mu_1}(t) \leq y_{\mu_2}(t), t \in [\delta-n, \delta+T]_{\mathbb{N}_{\delta-n}}$ and $y_{\mu_1}(t) \neq y_{\mu_2}(t)$,
(c) if $\mu \to \mu_0 (\mu_0 > 0)$ then $\max_{t\in[\delta-n,\delta+T]_{\mathbb{N}_{\delta-n}}} |y_\mu(t) - y_{\mu_0}(t)| \to 0$,
(d) if $\mu \to +\infty$ then $\max_{t\in[\delta-n,\delta+T]_{\mathbb{N}_{\delta-n}}} |y_\mu(t)| \to +\infty$, if $\mu \to 0^+$ then $\max_{t\in[\delta-n,\delta+T]_{\mathbb{N}_{\delta-n}}} |y_\mu(t)| \to a_0$.

Proof. Let A be defined as in Theorem 3.1. If $y > 1$, then

$$f(t,1) = f(t, \frac{1}{y}.y) \geq (\frac{1}{y})^r f(t,y),$$

thus

$$f(t,y) \leq y^r f(t,1) \leq \gamma y^r.$$

Therefore, for $y \in P$, we have

$$\begin{aligned}0 &\leq \sum_{k=0}^{T+1}\mathcal{K}(t,k)f(k+\delta-1, y(k+\delta-1))\\ &\leq \sum_{k=0}^{T+1}\mathcal{K}(t,k)f(k+\delta-1, \|y\|_{\mathbb{E}}) \leq M_3 e(t),\end{aligned}$$

where

$$\begin{aligned} M_3 &= \max_{t\in[\delta-(n-1),\delta+T]_{\mathbb{N}_{\delta-(n-1)}}} f(t,\|y\|_{\mathbb{E}}) \\ &\leq \max_{t\in[\delta-(n-1),\delta+T]_{\mathbb{N}_{\delta-(n-1)}}} f(t,\|y\|_{\mathbb{E}}+1) \leq \gamma(\|y\|_{\mathbb{E}}+1)^r. \end{aligned}$$

Thus, $0 \leq Ay(t) \leq M_3 e(t), t \in [\delta-n, \delta+T]_{\mathbb{N}_{\delta-n}}$, which implies $Ay \in \mathbb{E}_e$. Therefore $Ay \in \mathbb{E}_e \cap \mathbb{P} = \tilde{\mathbb{P}}$. Clearly $A : \tilde{\mathbb{P}} \to \tilde{\mathbb{P}}$.

If $y \in \tilde{\mathbb{P}}^0$, $\exists \delta > 0$ such that $y(t) \geq \delta e(t) \geq 0, t \in [\delta-n, \delta+T]_{\mathbb{N}_{\delta-n}}$. So if we take $\tau \in (0,1)$ such that $\tau < \delta$, then we have

$$\begin{aligned} (Ay)(t) &= \sum_{k=0}^{T+1} \mathcal{K}(t,k) f(k+\delta-1, y(k+\delta-1)) \geq \sum_{k=0}^{T+1} \mathcal{K}(t,k) f(k+\delta-1, \theta e(k+\delta-1)) \\ &\geq \tau^r \sum_{k=0}^{T+1} \mathcal{K}(t,k) f(k+\delta-1, e(k+\delta-1)). \end{aligned}$$

Let $a = \min_{t\in[\delta-(n-1),\delta+T]_{\mathbb{N}_{\delta-(n-1)}}} f(t, e(t))$. Thus $a > 0$ and $Ay(t) \geq a\tau^r e(t),\ t \in [\delta-n, \delta+T]_{\mathbb{N}_{\delta-n}}$. Therefore, $A : \tilde{\mathbb{P}}^0 \to \tilde{\mathbb{P}}^0$. The increasing property of $f(t,\cdot)$ implies that the operator A is increasing. If $y \in \tilde{\mathbb{P}}^0$ and $\tau \in (0,1)$, then

$$\begin{aligned} A(\tau y)(t) &= \sum_{k=0}^{T+1} \mathcal{K}(t,k) f(k+\delta-1, \tau y(k+\delta-1)) \geq \sum_{k=0}^{T+1} \mathcal{K}(t,k) \tau^r f(k+\delta-1, y(k+\delta-1)) \\ &\geq \tau^r \sum_{k=0}^{T+1} \mathcal{K}(t,k) f(k+\delta-1, y(k+\delta-1)) = \tau^r (Ay)(t). \end{aligned}$$

Thus, A satisfies in (2.4). Consider the following equation

$$F(y)(t) = \lambda y(t). \tag{3.1}$$

Using Lemma 2.9, for any $\lambda > 0$, (3.1) has exactly one solution y_λ in $\tilde{\mathbb{P}}^0$, y_λ is strictly decreasing, i.e., $0 < \lambda_1 < \lambda_2$ implies $y_{\lambda_1} \gg y_{\lambda_2}$, y_λ is continuous, i.e. $\lambda \to \lambda_0 (\lambda_0 > 0)$ implies $\|y_\lambda - y_{\lambda_0}\| \to 0, \lim_{\lambda\to\infty} \|y_\lambda\| = 0, \lim_{\lambda\to 0^+} \|y_\lambda\| = +\infty$.

Let $\mu = \frac{1}{\lambda}$, $\mu_0 = \frac{1}{\lambda_0}$, $\mu_1 = \frac{1}{\lambda_1}$, $\mu_2 = \frac{1}{\lambda_2}$. Then (3.1) is changed to
$y(t) = \mu(Ay)(t)$. Using Lemma 2.10, y is the solution of the problem (1.1) if and only if $y = \mu Ay$. Therefore,
(a) the problem (1.1) has exactly one positive solution y_μ in $\tilde{\mathbb{P}}^0$, for $\mu > 0$,
(b) if $0 < \mu_1 < \mu_2$, then $\exists \theta > 0$ such that $y_{\mu_1} - y_{\mu_2} \geq \theta e(t), t \in [\delta-n, \delta+T]_{\mathbb{N}_{\delta-n}}$ and thus $y_{\mu_1} \leq y_{\mu_2}, \forall t \in [\delta-n, \delta+T]_{\mathbb{N}_{\delta-n}}$ and $y_{\mu_1} \neq y_{\mu_2}$,
(c) if $\mu \to \mu_0 (\mu_0 > 0)$ then $\max_{t\in[\delta-n,\delta+T]_{\mathbb{N}_{\delta-n}}} |y_\mu(t) - y_{\mu_0}(t)| \to 0$,
(d) if $\mu \to +\infty$ then $\max_{t\in[\delta-n,\delta+T]_{\mathbb{N}_{\delta-n}}} |y_\mu(t)| \to +\infty$, if $\mu \to 0^+$ then
$\max_{t\in[\delta-n,\delta+T]_{\mathbb{N}_{\delta-n}}} |y_\mu(t)| \to a_0$. □

4. Examples

Example 4.1. Consider the problem

$$\begin{cases} \Delta_c^{1.8} y(t) + \mu((y(t+\delta-1))^{\frac{1}{3}} + e^t) = 0 \\ y(\delta-2) = a_0, \\ \Delta y(10.8) = 0. \end{cases} \tag{4.1}$$

where $\delta = 1.8, T = 9, \mu > 0$, is an integer and

$$f(t,y) = ((y(t))^{\frac{1}{3}} + e^t), \quad \forall t \in [0.8, 10.8]_{\mathbb{N}_{0.8}}$$

Clearly $f(t,y) : [0.8, 10.8]_{\mathbb{N}_{0.8}} \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, increasing and concave. Take $\sigma \in (0,1), \alpha \geq 1 + e^{10}$. Then

$$f(t,0) = e^t \geq 1 \geq \sigma, \quad f(t,1) = 1 + e^t \leq 1 + e^{10} \leq \alpha.$$

It can be seen that $\lim_{y\to\infty} \dfrac{f(t,y)}{y} = 0$ uniformly on $[0.8, 10.8]_{\mathbb{N}_{0.8}}$. Using Theorem 3.1, $\mu^* = \infty$. Therefore
(i) the problem(4.1) has a unique positive solution y_μ in $\tilde{\mathbb{P}}$, for $\mu \in (0, +\infty)$,
(ii) If $y_0 \in \tilde{\mathbb{P}}$ and for $\mu \in (0, +\infty)$, define

$$y_m(t) = a_0 + \mu \sum_{k=0}^{T+1} \mathcal{K}(t,k) f(k + \delta - 1, y_{m-1}(k + \delta - 1)), m = 1, 2, 3, ...,$$

therefore

$$\max_{t\in[-0.2,10.8]_{\mathbb{Z}_{-0.2}}} |y_\mu(t) - y_{\mu_0}(t)| \to 0, \ as \ \mu \to \mu_0.$$

(iii) $\max_{t\in[-0.2,10.8]_{\mathbb{Z}_{-0.2}}} |y_\mu(t) - y_{\mu_0}(t)| \to 0, \ as \ \mu \to \mu_0$, where $\mu \in (0, +\infty)$. If $0 < \mu_1 < \mu_2 < +\infty$, then $y_{\mu_1}(t) \leq y_{\mu_2}(t)$, for all $t \in [-0.2, 10.8]_{\mathbb{Z}_{-0.2}}$ and $y_{\mu_1}(t) \neq y_{\mu_2}(t)$,
(iv) $y_{l\mu}(t) \leq l y_\mu(t), l \in [0,1], t \in [-0.2, 10.8]_{\mathbb{Z}_{-0.2}}$, where $\mu \in (0, +\infty)$.

Example 4.2. In this example we consider the problem

$$\begin{cases} \Delta_c^{\frac{15}{7}} y(t) + \mu((y(t+\delta-1))^{\frac{1}{3}} + e^t) = 0 \\ y(\frac{-6}{7}) = a_0, \\ \Delta y(5) = \Delta^2 y(\frac{-6}{7}) = 0, \end{cases} \tag{4.2}$$

where $\delta = \frac{15}{7}, T = \frac{20}{7}, \mu > 0$, is an integer and

$$f(t,y) = \frac{y}{2 + \sin t} + e^t, \quad \forall t \in [\frac{1}{7}, 5]_{\mathbb{N}_{\frac{1}{7}}}$$

Clearly $f(t,y) : [\frac{1}{7}, 5]_{\mathbb{N}_{\frac{1}{7}}} \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, increasing and concave and

$$f(t,0) = e^t \geq e^{\frac{1}{7}} \geq \sigma, \quad f(t,1) = \frac{1}{2 + \sin t} + e^t \leq e^5 + 1 \leq \alpha.$$

Therefore
(i) for $\mu \in (0, +\infty)$, the y_μ in $\tilde{\mathbb{P}}$, is a unique positive solution for the problem(4.2),
(ii) if $y_0 \in \tilde{\mathbb{P}}$ and for $\mu \in (0, +\infty)$, define

$$y_m(t) = a_0 + \mu \sum_{k=0}^{T+1} \mathcal{K}(t,k) f(k + \delta - 1, y_{m-1}(k + \delta - 1)), m = 1, 2, 3, ...,$$

therefore

$$\max_{t\in[\frac{-6}{7},5]_{\mathbb{Z}_{\frac{1}{7}}}} |y_\mu(t) - y_{\mu_0}(t)| \to 0, \ as \ \mu \to \mu_0.$$

(iii) $\max_{t\in[\frac{-6}{7},5]_{\mathbb{Z}_{\frac{1}{7}}}} |y_\mu(t) - y_{\mu_0}(t)| \to 0, \ as \ \mu \to \mu_0$, where $\mu \in (0, +\infty)$. If $0 < \mu_1 < \mu_2 < +\infty$, then $y_{\mu_1}(t) \leq y_{\mu_2}(t), \forall t \in [\frac{-6}{7}, 5]_{\mathbb{Z}_{\frac{1}{7}}}$ and $y_{\mu_1}(t) \neq y_{\mu_2}(t)$,
(iv) $y_{l\mu}(t) \leq l y_\mu(t), l \in [0,1], t \in [\frac{-6}{7}, 5]_{\mathbb{Z}_{\frac{1}{7}}}$, where $\mu \in (0, +\infty)$.

Moreover $r = \frac{1}{2}$. Take $\gamma \geq 1$, then, $f(t,1) = \frac{1}{2+\sin t} + e^t \leq e^5 + 1 \leq \gamma$, $f(t, e(t)) = e(t) > 0$ and thus $\min_{t\in[\delta,\delta+T]_{\mathbb{N}_\delta}} f(t, e(t)) > 0$. Thus all requirements of Theorem 3.2 hold. Therefore:

(i) for $\mu > 0$, this problem has a unique positive solution y_μ in $\widetilde{\mathbb{P}}^0$,
(ii) if $0 < \mu_1 < \mu_2$, then $y_{\mu_1}(t) \le y_{\mu_2}(t), t \in [\frac{-6}{7}, 5]_{\mathbb{Z}_{\frac{1}{7}}}$ and $y_{\mu_1}(t) \ne y_{\mu_2}(t)$,
(iii) if $\mu \to \mu_0(\mu_0 > 0)$ then $\max_{t\in[\frac{-6}{7},5]_{\mathbb{Z}_{\frac{1}{7}}}} |y_\mu(t) - y_{\mu_0}(t)| \to 0$,
(iv) if $\mu \to +\infty$ then $\max_{t\in[\frac{-6}{7},5]_{\mathbb{Z}_{\frac{1}{7}}}} |y_\mu(t)| \to +\infty$, if $\mu \to 0^+$ then $\max_{t\in[\frac{-6}{7},5]_{\mathbb{Z}_{\frac{1}{7}}}} |y_\mu(t)| \to a_0$.

Example 4.3. We consider the fractional difference equation

$$\begin{cases} \Delta_c^\delta y(t) = \mu f(t, y(t+\delta-1)), \\ y(\delta-1) = a_0, \end{cases} \tag{4.3}$$

where $0 < \delta \le 1$, $f : [\delta, \delta+T]_{\mathbb{N}_\delta} \times [0,\infty) \to [0,\infty)$ is continuous and increasing. The sulotion $y(t)$ of problem (4.3), is given by

$$y(t) = a_0 + \frac{\mu}{\Gamma(\delta)} \sum_{k=a}^{t-\delta} (t-k-1)^{\underline{\delta-1}} f(t, y(k+\delta-1)), \tag{4.4}$$

where $\mathcal{K}(t,k) = (t-k-1)^{\underline{\delta-1}}$, which is positive, thus Theorems 3.1 and 3.2 hold.

In the special case if $f(t,y) = y(t)$ the problem (4.3) is the standard discrete equation for general population growth, that means

$$\begin{cases} \Delta_c^\delta y(t) = \mu y(t+\delta-1), \\ y(\delta-1) = a_0. \end{cases} \tag{4.5}$$

In this simplest model, μ tells us how fast the population is changing at any given population level and a_0 represents the initial population size.

To obtain an explicit formula for this solution, we use the method of successive approximation. Let $y_0(t) = a_0$ and

$$y_m(t) = a_0 + \mu \Delta_c^{-\delta} y_{m-1}(t+\delta-1).$$

Using power formula for $m = 1$ implies

$$\begin{aligned} y_1(t) &= a_0 + \mu \Delta_c^{-\delta} y_0(t+\delta-1) = a_0 + \mu \Delta_c^{-\delta} \frac{(t+\delta-1)^{\underline{\delta-1}}}{\Gamma(\delta)} a_0 \\ &= a_0 + \frac{\mu}{\Gamma(2\delta)} (t+\delta-1)^{\underline{2\delta-1}} a_0 = a_0 \big[1 + \frac{\mu}{\Gamma(2\delta)} (t+\delta-1)^{\underline{2\delta-1}}\big]. \end{aligned}$$

Using the same manner for $m = 2$ implies

$$\begin{aligned} y_2(t) &= y_1(t) + \mu \Delta_c^{-\delta} y_1(t+\delta-1) \\ &= a_0 + \frac{\mu}{\Gamma(2\delta)} (t+\delta-1)^{\underline{2\delta-1}} a_0 + \frac{\mu^2}{\Gamma(3\delta)} (t+2\delta-1)^{\underline{3\delta-1}} a_0 \\ &= a_0 \big[1 + \frac{\mu}{\Gamma(2\delta)} (t+\delta-1)^{\underline{2\delta-1}} + \frac{\mu^2}{\Gamma(3\delta)} (t+2\delta-1)^{\underline{3\delta-1}}\big]. \end{aligned}$$

Now using induction on m and letting $m \to \infty$ we obtain the solution

$$y_\mu := y(t) = a_0 + a_0 \sum_{k=1}^{\infty} \frac{\mu^k}{\Gamma((k+1)\delta)} (t + (\delta-1)k)^{\underline{k\delta+\delta-1}}. \tag{4.6}$$

For $\delta = 1$ we have $y(t) = a_0 \sum_{k=0}^{\infty} \frac{\mu^k}{k!} t^k = a_0 e^{\mu t}$ on the time scale $\mathbb{Z}$.

For this problem obviously , $f(t,y)$ is continuous, increasing and concave function with respect to t. For $0 < \tau < 1$,

$$f(t, \tau y) = \tau y(t) > \tau^{\frac{1}{2}} y(t) = \tau^r f(t,y), \quad \textit{for all } t \in [\delta, \delta+T]_{\mathbb{N}_\delta}$$

where $r = \frac{1}{2}$. Take $\gamma \geq 1$, then, $f(t,1) = 1 \leq \gamma$. Moreover, $f(t, e(t)) = e(t) > 0$ and thus $\min_{t\in[\delta,\delta+T]_{\mathbb{N}_\delta}} f(t, e(t)) > 0$. Thus all requirements of Theorem 3.2 hold. Therefore:
(i) for $\mu > 0$, this problem has exactly one positive solution y_μ in $\tilde{\mathbb{P}}^0$,
(ii) if $0 < \mu_1 < \mu_2$, then $y_{\mu_1}(t) \leq y_{\mu_2}(t), t \in [\delta-1, \delta+T]_{\mathbb{N}_{\delta-1}}$ and $y_{\mu_1}(t) \neq y_{\mu_2}(t)$,
(iii) if $\mu \to \mu_0(\mu_0 > 0)$ then $\max_{t\in[\delta-n,\delta+T]_{\mathbb{N}_{\delta-1}}} |y_\mu(t) - y_{\mu_0}(t)| \to 0$,
(iv) if $\mu \to +\infty$ then $\max_{t\in[\delta-1,\delta+T]_{\mathbb{N}_{\delta-1}}} |y_\mu(t)| \to +\infty$, if $\mu \to 0^+$
then $\max_{t\in[\delta-1,\delta+T]_{\mathbb{N}_{\delta-1}}} |y_\mu(t)| \to a_0$.

Example 4.4. This example is a sample of a mathematical model of discrete fractional differential equation given by

$$\begin{cases} \Delta_c^\delta y(t) = -\mu y(t+\delta-1), \\ y(0) = a_0, \end{cases} \tag{4.7}$$

where y is the concentration of the drug in the body at time t after injection, μ is constant first-order elimination rate of the drug (the negative sign means drug is eliminated from the body) and a_0 is the initial concentration. Therefore

(i) for $\mu > 0$, this problem has exactly one positive solution y_μ in $\tilde{\mathbb{P}}^0$,
(ii) if $0 < \mu_1 < \mu_2$, then $y_{\mu_2}(t) \leq y_{\mu_1}(t)$, and $y_{\mu_1}(t) \neq y_{\mu_2}(t)$,
(iii) if $\mu \to \mu_0(\mu_0 > 0)$ then $\max_{t\in[\delta-1,\delta+T]_{\mathbb{N}_{\delta-1}}} |y_\mu(t) - y_{\mu_0}(t)| \to 0$,
(iv) if $\mu \to +\infty$ then $\max_{t\in[\delta-1,\delta+T]_{\mathbb{N}_{\delta-1}}} |y_\mu(t)| \to +\infty$, if $\mu \to 0^+$, then $\max_{t\in[\delta-1,\delta+T]_{\mathbb{N}_{\delta-1}}} |y_\mu(t)| \to a_0$.

The solution of the problem (4.7), is given by

$$y(t) = a_0 + \frac{-\mu}{\Gamma(\delta)} \sum_{k=a}^{t-\delta} (t-k-1)^{\underline{\delta-1}} y(k+\delta-1). \tag{4.8}$$

Thus exact solution is given by

$$y_\mu := y(t) \quad = \quad a_0 + a_0 \sum_{k=1}^{\infty} \frac{(-\mu)^s}{\Gamma((k+1)\delta)} (t + (\delta-1)k)^{\underline{k\delta+\delta-1}}. \tag{4.9}$$

For $\delta = 1$ we have $y(t) = a_0 \sum_{k=0}^{\infty} \dfrac{(-\mu)^k}{k!} t^k = a_0 e^{-\mu t}$ on the time scale $\mathbb{Z}$.

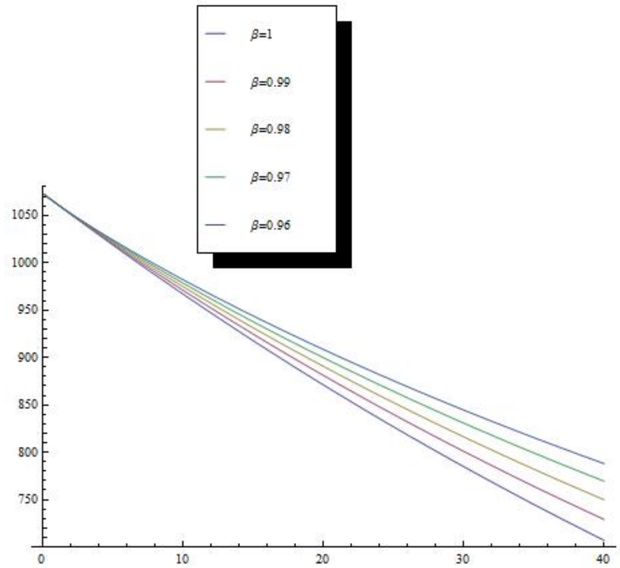

FIGURE 1. Solution of (4.7) for different values of δ, and $a_0 = 1073, \mu = 0.010434$

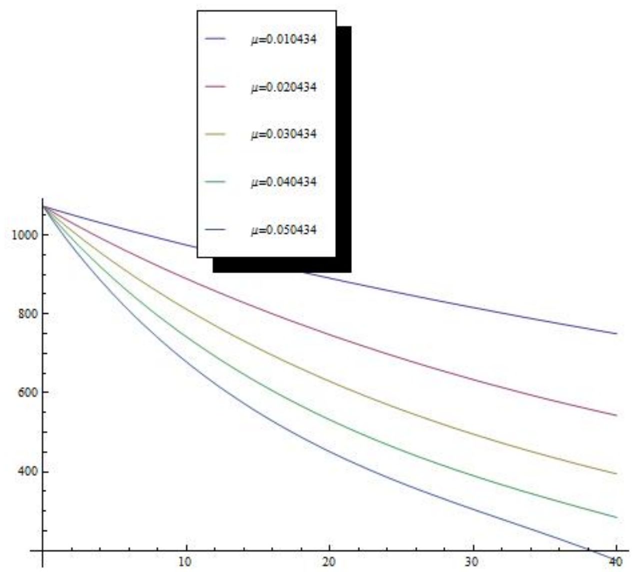

FIGURE 2. Solution of (4.7) for different values of μ, and $a_0 = 1073, \delta = 0.98$

5. Conclusion

Here we study a typical fractional-order differential equation of the form (1.1). Using the fixed point theory of concave operators, we show that this problem has exactly one positive solution dependent on the parameter μ. We compute the solution in some cases that appear in applications.

It would be interesting to consider this problem with other definitions of continuous and discrete fractional derivatives. The system of discrete fractional differential equations may be considered for further research.

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