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Approximate price of the option under discretization by applying fractional quadratic interpolation

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Abstract

The time-fractional Black-Scholes model (TFBSM) governing European options in which the temporal derivative is focused on the Caputo fractional derivative with $0 < \beta \leq 1$ is considered in this article. Approximating financial options with respect to their hereditary characteristics can be well understood and explained due to its outstanding memory effect current in fractional derivatives. Compelled by the stated cause, It is important to find reasonably accurate and successful numerical methods when approaching fractional differential equations. The simulation model given here is developed in two ways: one, the semi-discrete is produced in the time using a quadratic interpolation with the order of precision $\tau^{3-\alpha}$ in the case of a smooth solution, and subsequently, the unconditional stability and convergence order are investigated. The spatial derivative variables are simulated using the collocation approach based on a Legendre basis for the designed full-discrete scheme. Last, we employ various test problems to demonstrate the suggested design's high precision. Moreover, the obtained results are compared to those obtained using other methodologies, demonstrating that the proposed technique is highly accurate and practicable.

Keywords. Time fractional Black-Scholes model, Square interpolation, Legendre Polynomials, Collocation method.
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1. INTRODUCTION

Economic and investment market participants are always faced with risks that may expose them to losses due to market conditions, fluctuations, and uncertainty about the future market situation. To this end, efforts have always been made to adopt appropriate strategies to cover these risks, in other words, to manage the risks facing capital market participants. One of the tools created for this purpose is derivatives.

Derivatives are traded in the form of four main derivative contracts, including futures contracts, futures contracts, option contracts, and exchange contracts depending on the market situation and conditions.

Derivative contracts are a type of financial contract that derives their value from the underlying asset.

The importance of the option as one of the financial derivatives has led to the consideration of how to evaluate an option in terms of the theory and reality of an issue. Thus, when one has assumed the obligation of pricing options, it is necessary to have a good understanding of the scheme. In the early 1970s, Fisher, Mirren, and Robert took a major step in option pricing [5, 23]. The result was the Black-Scholes model, which was a well-known model. The accuracy and reliability of this model in predicting the price of options will significantly increase options trading [1, 12, 31].

Fractional Partial Differential Equations, In fact, generalized partial differential equations are of the correct order that is used to model financial problems, magnetism, fluid mechanics, and so on. Therefore, due to the large and important application of such differential equations in science and engineering, many researchers have attracted their attention. Thus, the methods of solving these equations are very important. Although many of these differential equations can be solved by analytical methods, most of them do not have analytical solution methods, or accurate

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and effective methods for solving partial differential equations of the fraction are easily obtained. they will not come. Therefore, numerical methods are used to solve them. Numerical methods for solving fractional derivatives and integrals, due to the non-locality of them are very effective. In order to solve fractional differential equations, spectral methods, meshless methods, finite difference methods, finite volume methods, and finite element methods have been presented. For references to these studies see e.g. [7, 19, 24, 25, 37, 38].

In recent years, the Black-Scholes model [11, 26] has been expanded by additional researchers since fractional integrals and derivatives are valuable resources for describing the inherited and memory characteristics of different sub-positions. The pricing of the European call option, for example, was first carried out using the fractional Black-Scholes equation [22]. Nikolai et al. in [17] using spectral techniques, gives specifically strong solutions for fractional Pearson diffusions. It also introduces stochastic solutions, using a robust time shift non-Markovian inverse. Another investigation into this model was conducted by Zhang et al. in 2016, which showed the numerical simulation of the Black-Scholes fractional model (TFBSM) on European options [14]. In addition, The analysis suggested by Song and Wang in [33] deals with the pricing problems of the put alternative based on the Black-Scholes time-fractional equation, where so-called modified Riemann-Liouville fractional derivative is the fractional derivative.

In current study, we explore the Black-Scholes model in the temporal fractional sense (BSM-TFS) with $0 < \beta \le 1$ and the initial condetion $v(x, 0) = p(x),\ 0 < x < 1$, as below

$$ {}_0\mathcal{D}_t^{\beta} v(x,t) = \mu \frac{\partial^2 v(x,t)}{\partial x^2} + \eta \frac{\partial v(x,t)}{\partial x} - \rho v(x,t) + q(x,t),\ 0 < t \le T, \tag{1.1} $$

where $q(x, t)$ is the source term and $\mu = \frac{1}{2}\sigma^2 > 0$, $\eta = \rho - \mu$ and $\rho > 0$ are the given constant. The left Caputo fractional derivative ${}_0\mathcal{D}_t^{\beta}, 0 < \beta < 1$ is defined as

$$ {}_0\mathcal{D}_t^{\beta} v(x,t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-\tau)^{-\beta} \frac{\partial v(x,\tau)}{\partial \tau} d\tau. $$

In Eq. (1.1), the boundary conditions with $0 < t \le T$ are

$$ v(0,t) = p_0(t), \quad v(1,t) = p_1(t), \quad 0 < t \le T. \tag{1.2} $$

It is very difficult to achieve an effective solution to these issues because of the memory characteristic of fractional derivatives. Therefore, The researchers looked at different ways to evaluate these issues [8, 16, 27, 40]. Many studies have been done to find a solution for the analytical form [4, 28, 29, 32]. However, due to the uncertain trend and complexity of analytical methods, the development of applied numerical methods to solve the Black-Scholes model was considered. Some of those methods are listed below. In the model presented in [3] for numerical approximation, a different explicit scheme is used and its convex composition determines its stability. Salahuddin et al. in [36], analytically as well as numerically, examined some solution methods for the Black-Scholes model with European call and put options. Using various weights for numerical proximations, they research a weighted average strategy. In 2006 Phaochoo et al. present a computational procedure for solving a fractional Black-Scholes equation based on the meshless local Petrov-Galerkin (MLPG) method [30]. The meshless local Petrov-Galerkin method is a completely meshless approach containing not only meshless interpolation for the test functions but also meshless interpolation for the test functions.

The rest of the paper is formed as below. In section 2, space and temporal discretization for the European option are evaluated in which the time variable is based on a quadratic interpolation. Description of the new strategy will explain in section 3. The convergence analysis will discuss in section 4. In the last section, we will report some numerical examples to demonstrate the efficiency of the new scheme.

2. Space and temporal discretization for European option

In this section, a implicit numerical scheme with order accuracy of $\mathcal{O}(\tau^{3-\alpha})$ in time is constructed. Afterwards, for obtaining the full discrete, the function $v(x, t)$ approximate in x-variable with shifted Legendre polynomials. Then, the unconditional stability and the convergence order of the time-discrete scheme will discuss in the next section.

2.1. **Review of time discretization.** The structure of the numerical approximate method for the Caputo fractional derivative has been performed using quadratic interpolation in many papers including [2, 9, 20] and [21]. To obtain explicit recursion formula, the structure is established by using three points that show the involved $\mathcal{O}(\tau^{3-\beta})$ with $0<\beta<1$. As an application, we use the defined explicit formula in [15] to get semi-discrete of Eq. (1.1). Suppose a uniform mesh be $t_j = j\tau,\ j=0,1,\dots,J$ with $\tau=\frac{T}{J}$ in which $J\in\mathbb{Z}^+$. Then, we can discretize the Caputo derivative for $0<\beta<1$ by the quadratic scheme that is defined in [15] as

$$ {}_0\mathcal{D}_t^{\beta} v(x,t) = \frac{\tau^{-\beta}}{\Gamma(2-\beta)} \sum_{j=0}^{J} \lambda_{J,j} u(x,t_j) + \mathcal{O}(\tau^{3-\beta}), \tag{2.1} $$

where the coefficients $\lambda_{J,j}$ can be expressed as following

$$ \lambda_{J,j} = \begin{cases} for J=1, \begin{cases} -\mathcal{A}_1, & j=0,\\ \mathcal{A}_1, & j=1, \end{cases} \\ for J=2, \begin{cases} -\mathcal{A}_2+\mathcal{B}_{2,2}, & j=0,\\ \mathcal{A}_2+\mathcal{C}_{2,2}, & j=1,\\ \mathcal{D}_{2,2}, & j=2. \end{cases} \\ for J\geq 3, \begin{cases} -\mathcal{A}_J+\mathcal{B}_{J,j+2}, & j=0,\\ \mathcal{A}_J+\mathcal{B}_{J,j+2}+\mathcal{C}_{J,j+1}, & j=1,\\ \mathcal{B}_{J,j+2}+\mathcal{C}_{J,j+1}+\mathcal{D}_{J,j}, & 2\leq j\leq J-2,\\ \mathcal{C}_{J,j+1}+\mathcal{D}_{J,j}, & j=J-1,\\ \mathcal{D}_{J,j}, & j=J, \end{cases} \end{cases} $$

in which

$$ \begin{aligned} \mathcal{A}_J &= J^{1-\beta}-(J-1)^{1-\beta},\\ \mathcal{B}_{J,j} &= \frac{1}{2-\beta}\big[(J-j+1)^{1-\beta}(J-j+\frac{\beta}{2})-(J-j)^{1-\beta}(J-j-\frac{\beta}{2}+1)\big],\\ \mathcal{C}_{J,j} &= \frac{2}{2-\beta}\big[(J-j)^{1-\beta}(J-j-\beta+2)-(J-j+1)^{2-\beta}\big],\\ \mathcal{D}_{J,j} &= \frac{1}{2-\beta}\big[(J-j+1)^{1-\beta}(J-j-\frac{\beta}{2}+2)-(J-j)^{1-\beta}(J-j-\frac{3\beta}{2}+3)\big]. \end{aligned} $$

2.2. **Review of Space discretization.** Now, Let me for remembering the polynomials of Legendre $\mathrm{L}_i(x), i=0,1,\dots$ that are represented with the corresponding recurrence equation on $L_2([-1,1])$ as

$$ \begin{cases} \mathrm{L}_{i+1}(x) = \frac{2i+1}{i+1}x\mathrm{L}_i(x) - \frac{i}{i+1}\mathrm{L}_{i-1}(x),\ i=1,2,\dots,\\ \mathrm{L}_0(x)=1,\ \mathrm{L}_1(x)=x. \end{cases} $$

The Legendre polynomial has the extension as follows, employing the Rodrigues' formula $\mathrm{L}_i(x) = \frac{(-1)^i}{2^i i!}\frac{\partial^i(1-x^2)^i}{\partial x^i}$.

$$ \mathrm{L}_i(x) = \frac{1}{2^i}\sum_{r=0}^{\lfloor \frac{i}{2}\rfloor} \frac{(-1)^r(2i-2r)!}{r!(i-r)!(i-2r)!}x^{i-2r}, i=0,1,\dots. \tag{2.2} $$

The polynomials $\mathrm{L}_i(x),\ i=0,1,\dots$ are orthogonal functions with respect to the weight $\frac{2}{2i+1}$ i.e.

$$ \langle \mathrm{L}_i(x), \mathrm{L}_j(x)\rangle = \frac{2i+1}{2}\int_{-1}^{1}\mathrm{L}_i(x)\mathrm{L}_j(x)dx = \begin{cases} 1, & i=j,\\ 0, & i\neq j. \end{cases} \tag{2.3} $$

To gain of this basis in interval $[0,1]$, we obtain the shifted Legendre polynomials with the variable change $x \to 2x-1$ as

$$\mathrm{L}_i^*(x) = \sum_{r=0}^{\lfloor \frac{i}{2} \rfloor} \sum_{\iota=0}^{i-2r} N_{i,r,\iota} x^{\iota}, i = 0, 1, \dots, \tag{2.4}$$

where

$$N_{i,r,\iota} = \binom{i-2r}{\iota} \frac{(-1)^{i-r-\iota} 2^{\iota-i} (2i-2r)!}{r!(i-r)!(i-2r)!}.$$

The set of $\{\mathrm{L}_i^*(x)\}_{i=0}^{\infty}$ is orthogonal function in the interval $[0,1]$. Thus we can approximate $v(x,t) \in L_2([0,1])$ by using $N+1$-term of this set as

$$v(x,t) = \sum_{i=0}^{N} a_i \mathrm{L}_i^*(x) = \Lambda \mathrm{L}^T, \tag{2.5}$$

in which the coefficients $a_i, i = 0, 1, \dots, N$ are defined via

$$a_i = (2i+1) \int_0^1 \mathrm{L}_i^*(x) v(x,t) dx. \tag{2.6}$$

The vector $\Lambda_{1\times(N+1)} = [a_0, a_1, \dots, a_N]$ and $\mathrm{L}_{1\times(N+1)} = [\mathrm{L}_0^*(x), \mathrm{L}_1^*(x), \dots, \mathrm{L}_N^*(x)]$ is unknown coefficients and basis functions, respectively. Indeed, we can obtain the closed form of the derivative of $\mathrm{L}_i^*(x)$ in Eq. (2.4) via the linearity of the derivative as bellow

$$\frac{\partial^n \mathrm{L}_i^*(x)}{\partial x^n} = \begin{cases} 0 & i = 0, 1, \dots n-1, \\ \sum_{r=0}^{\lfloor \frac{i}{2} \rfloor} \sum_{\iota=n}^{i-2r} N_{i,r,\iota}^n x^{\iota-n}, & o.w, \end{cases} \tag{2.7}$$

where

$$N_{i,r,\iota}^n = \frac{(-1)^{i-r-\iota} 2^{\iota-i} (2i-2r)!}{r!(i-r)!(i-2r-\iota)!(\iota-n)!}.$$

As a result, the derivative of function $v(x,t)$ can be written as the following operation

$$\frac{\partial^n v(x,t)}{\partial x^n} = \sum_{i=0}^{N} a_i \frac{\partial^n \mathrm{L}_i^*(x)}{\partial x^n} = \Lambda \mathfrak{L}_n^T, \tag{2.8}$$

where the entries of the vector $\mathfrak{L}_{\mathfrak{n}1\times(N+1)} = [\frac{\partial^n \mathrm{L}_0^*(x)}{\partial x^n}, \frac{\partial^n \mathrm{L}_1^*(x)}{\partial x^n}, \dots, \frac{\partial^n \mathrm{L}_N^*(x)}{\partial x^n}]$ are obtained from the relation (2.7).

3. Description of the new strategy

We develop a general numerical structure to solve (1.1) in this segment. The framework is developed for the all options although this method is so simplify to gain the price of option based on European option. According to the above section, we substitute (2.1) and (2.7) in (1.1) at point (x, t_j) as below

$$(\lambda_{j,j} + \overline{\rho}\tau^{\beta}) v^j - \overline{\mu}\tau^{\beta} \frac{\partial^2 v^j}{\partial x^2} - \overline{\eta}\tau^{\beta} \frac{\partial v^j}{\partial x} = \sum_{k=0}^{j-1} \overline{\lambda}_{j,k} v^k + \tau^{\beta} \overline{q}^j + \tau^{\beta} \mathcal{R}^j, \tag{3.1}$$

where $v^j = v(x,t_j), \overline{\mu} = \mu\Gamma(2-\beta), \overline{\eta} = \eta\Gamma(2-\beta), \overline{\rho} = \rho\Gamma(2-\beta), \overline{q}^j = \Gamma(2-\beta) q(x,t_j)$ and $\overline{\lambda}_{j,k} = -\lambda_{j,k}$. Also, the positive constant C exists such that $\mathcal{R}^j \leq C\mathcal{O}(\tau^{3-\beta})$. With describing V^j as the approximation of v^j and removing $\mathcal{R}^j$, one can arrive at the following semi-discrete scheme as

$$(\lambda_{j,j} + \overline{\rho}\tau^{\beta}) V^j - \overline{\mu}\tau^{\beta} \frac{\partial^2 V^j}{\partial x^2} - \overline{\eta}\tau^{\beta} \frac{\partial V^j}{\partial x} = \sum_{k=0}^{j-1} \overline{\lambda}_{j,k} V^k + \tau^{\beta} \overline{q}^j. \tag{3.2}$$

To achieve the full-discrete scheme, we apply the approximating operation (2.8) based on the Legendre polynomials. As a result we have

$$(\lambda_{j,j}+\overline{\rho}\tau^{\beta})\Lambda^{j}\mathrm{L}^{T}-\overline{\mu}\tau^{\beta}\Lambda^{j}\mathfrak{L}_{2}^{T}-\overline{\eta}\tau^{\beta}\Lambda^{j}\mathfrak{L}_{1}^{T}=\sum_{k=0}^{j-1}\overline{\lambda}_{j,k}\Lambda^{k}\mathrm{L}^{T}+Q^{j}, \tag{3.3}$$

By using the collocation points $0=x_0<x_1<\ldots<x_{N-1}<x_N=1$ that are the roots of the Legendre polynomial, the above equation can be converted into $N-1$ equations and $N+1$ unknowns. To obtain a linear system, we must apply the boundary conditions that can be determined by substituting (1.2) in (2.5) as below

$$\sum_{i=0}^{N}a_i\mathrm{L}_i^{*}(0)=\sum_{i=0}^{N}(-1)^i a_i=p_0(t),\ \sum_{i=0}^{N}a_i\mathrm{L}_i^{*}(1)=\sum_{i=0}^{N}a_i=p_1(t). \tag{3.4}$$

Then the linear system of $N+1$ equations in $N+1$ variables is obtained as

$$A\Lambda^{j}=B\Lambda^{j-1}+Q^{j} \tag{3.5}$$

where $Q^j=\tau^{\beta}[p_0(t_j),\overline{q}(x_1,t_j),\overline{q}(x_2,t_j),\ldots,\overline{q}(x_{N-1},t_j),p_1(t_j)]^T$, the matrix entires of $A_{(N+1)\times(N+1)}$ and $B_{(N+1)\times(N+1)}$ can be exhibited as below

$$A=\begin{bmatrix} I^{0}_{1\times(N+1)}\\ \mathbf{A}_{(N-1)\times(N+1)}\\ I^{1}_{1\times(N+1)}\end{bmatrix},\ B=\begin{bmatrix}0\\ \mathbf{B}_{(N-1)\times(N+1)}\\ 0\end{bmatrix}, \tag{3.6}$$

where $I^{0}_{1\times(N+1)}=[1,1,\ldots,1]$ and $I^{1}_{1\times(N+1)}=[1,-1,\ldots,(-1)^{N-1},(-1)^N]$. The matrix entires of $\mathbf{A}$ and $\mathbf{B}$ can be dedused as below

$$\begin{cases} a_{m,n}=(\lambda_{j,j}+\overline{\rho}\tau^{\beta})\mathrm{L}_n^{*}(x_m)-\overline{\mu}\tau^{\beta}\sum_{r=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{\iota=2}^{n-2r}N^{2}_{n,r,\iota}x_m^{\iota-2}\\ \qquad -\overline{\eta}\tau^{\beta}\sum_{r=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{\iota=1}^{n-2r}N^{1}_{n,r,\iota}x_m^{\iota-1},\\ b_{m,n}=\sum_{k=0}^{j-1}\overline{\lambda}_{j,k}\mathrm{L}_n^{*}(x_m),\end{cases} \tag{3.7}$$

where $m\in\{1,2,\ldots,N-1\}$ and $n\in\{0,1,\ldots,N\}$. By solving the system (3.5), one can determine the unknown coefficients $\Lambda^j, j=1,2,\ldots,J$ in each step of time. Notice that the initial solution Λ^0 is obtained with combining relation $v(x,0)=p(x)$ with Eq. (2.5) as

$$\Lambda^0=[a_0,a_1,\ldots,a_N]^T,\ a_i=(2i+1)\int_0^1\mathrm{L}_i^{*}(x)p(x)dx,\ i=0,1,\ldots,N. \tag{3.8}$$

4. Discussion of the convergence analysis

To prove the unconditional stability and obtain the convergence order of the above scheme, we define the following functional space with the standard norms $\|f(t)\|_2=\langle f(t),f(t)\rangle^{\frac{1}{2}}$.

$$\mathcal{H}^{n}_{\Omega}(f)=\{f\in L_2(\Omega),\ \frac{\partial^n f}{\partial x^n}\in L_2(\Omega),\ \forall\ n\in\mathbb{N}\},$$

where $L^2(\Omega)$ is the space of square-integrable functions with respect to Lebesgue measure in Ω as follows

$$L_2(\Omega)=\Big\{f:\Big(\int_{\Omega}|f(t)|^2\Big)^{\frac{1}{2}}<\infty\Big\}.$$

Before starting the relevant theorems, we write Eq. (3.3) with multiplying $\psi\in\mathcal{H}^2_{\Omega}$ as follows

$$(\lambda_{j,j}+\overline{\rho}\tau^{\beta})\langle v^j,\psi\rangle-\overline{\mu}\tau^{\beta}\langle\frac{\partial^2 v^j}{\partial x^2},\psi\rangle-\overline{\eta}\tau^{\beta}\langle\frac{\partial v^j}{\partial x},\psi\rangle=\sum_{k=0}^{j-1}\overline{\lambda}_{j,k}\langle v^k,\psi\rangle+\tau^{\beta}\langle\overline{q}^j,\psi\rangle+\tau^{\beta}\langle\mathcal{R}^j,\psi\rangle.$$

Considering $\langle \frac{\partial^2 U^1}{\partial x^2}, U^1\rangle = -\langle \frac{\partial U^1}{\partial x}, \frac{\partial U^1}{\partial x}\rangle$, $\langle \frac{\partial U^1}{\partial x}, U^1\rangle = 0$ and $\lambda_{J,J} = 1$, one can obtain

$$(\lambda_{j,j} + \overline{\rho}\tau^{\beta})\langle v^j, \psi\rangle \leq \sum_{k=0}^{j-1} \overline{\lambda}_{j,k}\langle v^k, \psi\rangle + \tau^{\beta}\langle \overline{q}^j, \psi\rangle + \tau^{\beta}\langle \mathcal{R}^j, \psi\rangle. \tag{4.1}$$

Now we are ready to prove the following theorems. In all stages of proving the following theorems, assume that $v(x,t) \in C^{4,3}([0,1][0,T])$.

Theorem 4.1. *For the semi-discrete design (3.3), stability is unconditionally achieved.*

Proof. Suppose for $j = 0, 1, \dots, J$, $V^j \in \mathcal{H}^2_{\Omega}$ be the approximation of the time-discrete scheme (3.3). Then with help of the relation (4.1) and substituting $\psi = V^j$, we gain

$$(\lambda_{j,j} + \overline{\rho}\tau^{\beta})\langle V^j, V^j\rangle \leq \sum_{k=0}^{j-1} \overline{\lambda}_{j,k}\langle V^k, V^j\rangle + \tau^{\beta}\langle \overline{q}^j, V^j\rangle. \tag{4.2}$$

To continue the proof, the mathematical induction on j is applied. When $j = 1$, we yield

$$(\lambda_{1,1} + \overline{\rho}\tau^{\beta})\langle V^1, V^1\rangle \leq \overline{\lambda}_{1,0}\langle V^0, V^1\rangle + \tau^{\beta}\langle \overline{q}^1, V^1\rangle. \tag{4.3}$$

By employing the Cauchy-Schwarz inequality principle, $\lambda_{1,1} = 1$, $\overline{\lambda}_{1,0} = 1$ and $1 < 1 + \overline{\rho}\tau^{\beta}$, one can be rewritten as

$$\|V^1\|_2 \leq \|V^0\|_2 + \tau^{\beta}\|\overline{q}^1\|_2 \leq \|V^0\|_2 + \max_{\forall x \in (0,1)} \|\overline{q}^1\|_2,$$

Now let for $j = 1, 2, \dots, J-1$, we have

$$\|V^j\|_2 \leq \|V^0\|_2 + \max_{\forall x \in (0,1)} \|\overline{q}^j\|_2. \tag{4.4}$$

The apply of the Cauchy-Schwarz inequality in Eq. (4.2) for $j = J$, concludes that

$$\|V^J\|_2 \leq \sum_{k=0}^{J-1} \overline{\lambda}_{J,k}\|V^k\|_2 + \tau^{\beta}\|\overline{q}^J\|_2.$$

The above inequality can be asserted as following by using relation (4.4).

$$\|V^J\|_2 \leq \sum_{k=0}^{J-1} \overline{\lambda}_{J,k}\big(\|V^0\|_2 + \max_{\forall x \in (0,1)} \|\overline{q}^1\|_2\big) + \tau^{\beta}\|\overline{q}^J\|_2. \tag{4.5}$$

From the coefficients of Eq. (2.1), we can easily compute $\mathcal{B}_{J,j} + \mathcal{C}_{J,j} = -\mathcal{D}_{J,j}$, $\forall J \geq 2, J \geq j$. Furthermore, the following relation is established

$$\sum_{j=0}^{J-1} \lambda_{J,j} = -\mathcal{D}_{J,J}.$$

In other hand, we have $1 < \mathcal{D}_{J,J} = \frac{4-\beta}{4-2\beta} \leq \frac{3}{2}$. Based on these relation, Eq. (4.5) can be obtained as

$$\|V^J\|_2 \leq C\big(\|V^0\|_2 + \max_{\forall x \in (0,1)} \|\overline{q}^j\|_2\big),$$

in which the value of C is a positive constant. Hence the proof is completed. □

Theorem 4.2. *Let $\varepsilon^j = |v^j - V^j|, j = 1, 2, \dots, J$ where V^j and v^j be the approximation and exact solution of Eq. (1.1), respectively. Then the convergence order of the time-discrete scheme (3.2) is $\mathcal{O}(\tau^{3-\beta})$.*

Proof. The approximation of Eq. (4.1) is as below

$$(\lambda_{j,j} + \overline{\rho}\tau^{\beta})\langle V^j, \psi\rangle \leq \sum_{k=0}^{j-1} \overline{\lambda}_{j,k}\langle V^k, \psi\rangle + \tau^{\beta}\langle \overline{q}^j, \psi\rangle$$

TABLE 1. Comparison of the error and convergence rate at $T = 1$ for Example 5.1.

	Method of [39] for $N = 100$ and $\beta = 0.7$		Method of [6] for $N = 100$ and $\beta = 0.7$		Method of [10] for $N = 100$ and $\beta = 0.7$		Calculated results for $N = 7$ and $\beta = 0.7$		
J	L_∞	$\mathcal{CR}$	L_∞	$\mathcal{CR}$	L_∞	$\mathcal{CR}$	L_∞	$\mathcal{CR}$	$\mathcal{CO}$
10	$3.7000E{-3}$		$5.200E{-3}$		$4.4520E{-4}$		$2.73216E{-4}$		
20	$1.5000E{-3}$	2.4116	$2.0700E{-3}$	2.4284	$1.8340E{-4}$	2.4275	$5.38425E{-5}$	5.07437	2.34323
40	$6.2714E{-4}$	2.4453	$8.3000E{-4}$	2.4453	$7.5160E{-4}$	2.4401	$1.07775E{-5}$	4.99581	2.32072
80	$2.5377E{-4}$	2.4794	$3.3000E{-4}$	2.4453	$3.0280E{-5}$	2.4840	$2.17291E{-6}$	4.95995	2.31033
160	$1.0063E{-4}$	2.5140	$1.3000E{-4}$	2.4794	$1.2130E{-6}$	2.4963	$4.39651E{-7}$	4.94235	2.30520
320	–	–	$5.0000E{-5}$	2.4667	$4.8510E{-6}$	2.5005	$8.88720E{-8}$	4.94702	2.30656
$\mathcal{CR}$		2.4623		2.4623		2.4623		4.92460	2.3

Subtracting the above relation from Eq. (4.1) and subtracting $\psi = \varepsilon^j$, the following form concludes.

$$(\lambda_{j,j} + \overline{\rho}\tau^\beta)\langle\varepsilon^j, \varepsilon^j\rangle \le \sum_{k=0}^{j-1} \overline{\lambda}_{j,k}\langle\varepsilon^k, \varepsilon^j\rangle + \tau^\beta\langle\mathcal{R}^j, \varepsilon^j\rangle.$$

Similar to the ideas of the proof for the first theorem, $\overline{\rho}\tau^\beta \le \lambda_{J,J} + \overline{\rho}\tau^\beta$ and $\|\varepsilon^0\|_2 = 0$, we have

$$\overline{\rho}\tau^\beta\|\varepsilon^k\|_2 \le C\tau^\beta\|\mathcal{R}^k\|_2 \Longrightarrow \|\varepsilon^k\|_2 \le C^{'}\|\mathcal{R}^k\|_2,$$

where $C^{'} = \frac{C}{\overline{\rho}}$. Hence completes the proof. □

5. Discussion of numerical results

This segment discusses, with using instances, the accuracy and benefit of the established strategy for the specified form of price barrier option by TFBSM, which is one of the most common financial market templates. The order (indicated via $\mathcal{C}_\mathcal{O}$) and the rate (indicated via $\mathcal{CR}$) of convergence is calculated by the following relationship, respectively,

$$\mathcal{C}_\mathcal{O} = \log_2(\frac{e_{i+1}}{e_i}),\ \mathcal{CR} = 2^{\mathcal{CO}}$$

where e_{i+1} and e_i are errors correspond with mesh size $2J$ and J, respectively. The numerical findings provide support for theoretical analysis . In this article, Wolfram software is used for computation.

Example 5.1.

$$\begin{cases} {}_0\mathcal{D}_t^\beta v(x,t) = \mu\frac{\partial^2 v(x,t)}{\partial x^2} + \eta\frac{\partial v(x,t)}{\partial x} - \rho v(x,t) + q(x,t),\ 0 < t \le T, \\ v(x,0) = x^2(1-x), \\ v(0,t) = 0, v(1,t) = 0, \end{cases} \tag{5.1}$$

where $\sigma = 0.25, \mu = \frac{1}{2}\sigma^2, \eta = \rho - \mu, \rho = 0.05$ *and the source term* $q(x,t) = (\frac{2t^{2-\beta}}{\Gamma(3-\beta)} + \frac{2t^{1-\beta}}{\Gamma(2-\beta)})x^2(1-x) - (t+1)^2[\mu(2-6x) + \eta(2x-3x^2) - \rho x^2(1-x)]$. *The exact solution is* $v(x,t) = (t+1)^2x^2(1-x)$.

Tables 1 and 2 list the results of numerical scheme for $N = 7$, $\beta = 0.2, \beta = 0.7$ and $\beta = 0.8$ at final time $T = 1$ where results confirm the convergence order, $\mathcal{O}(\tau^{3-\beta})$, of our numerical scheme and it is close to the time order of convergence (TOC). In Table 1, calculated errors and convergence rates of the presented method are compared with the implicit finite difference scheme [39], compact finite deference approach [6] and MLS approach in the temporal and space variable [10]. In Table 2, the results calculated with $\beta = 0.2$ and $\beta = 0.8$ at $T = 1$. From these tables, one can be derived that our results are better than the mentioned methods. In addition, more highly accurate results are given with very low space size.

Example 5.2. *In second example, the constant values of Eq.* (1.1) *are considered as* $\mu = 1, \eta = \rho - \mu$ *and* $\rho = 0.5$. *The initial and nonhomogeneous boundary conditions are achieved from the exact solution* $u(x,t) = (t+1)^2(x^3+x^2+1)$, *in addition, the source term is* $q(x,t) = (\frac{2t^{2-\beta}}{\Gamma(3-\beta)} + \frac{2t^{1-\beta}}{\Gamma(2-\beta)})(x^3+x^2+1) - (t+1)^2[\mu(2+6x) + \eta(2x+3x^2) - \rho(x^3+x^2+1)]$.

TABLE 2. Computational results with $N = 7$ for Example 5.1 at $T = 1$.

	$\beta = 0.2$				$\beta = 0.8$			
J	L_∞	$\mathcal{C_O}$	L_2	$\mathcal{C_O}$	L_∞	$\mathcal{C_O}$	L_2	$\mathcal{C_O}$
15	$2.51079E{-6}$		$7.81965E{-6}$		$1.96399E{-4}$		$6.11465E{-4}$	
30	$3.48600E{-7}$	2.84850	$1.08615E{-6}$	2.84788	$4.20490E{-5}$	2.22365	$1.30993E{-4}$	2.22278
60	$4.89437E{-8}$	2.83238	$1.52555E{-7}$	2.83182	$9.08013E{-6}$	2.21129	$2.82950E{-5}$	2.21088
120	$6.92266E{-9}$	2.82173	$2.15849E{-8}$	2.82123	$1.96861E{-6}$	2.20553	$6.13532E{-6}$	2.20533
240	$9.15870E{-10}$	2.91811	$2.85872E{-9}$	2.91658	$4.27694E{-7}$	2.20253	$1.33303E{-6}$	2.20243
TOC		2.8		2.8		2.2		2.2

TABLE 3. Comparison of the error and convergence rate at $T = 1$ for Example 5.1 with the presented methods in [39], [6] and [10].

	Method of [39] for $N = 100$ and $\beta = 0.7$		Method of [6] for $N = 100$ and $\beta = 0.7$		Method of [10] for $N = 100$ and $\beta = 0.7$		Calculated result for $N = 7$ and $\beta = 0.7$	
J	L_∞	$\mathcal{CR}$	L_∞	$\mathcal{CR}$	L_∞	$\mathcal{CR}$	L_∞	$\mathcal{CR}$
10	$5.5000E{-3}$		$3.500E{-3}$		$3.7860E{-4}$		$2.88154E{-5}$	
20	$2.2000E{-3}$	2.4967	$1.4400E{-3}$	2.5140	$1.5720E{-4}$	2.4084	$4.93088E{-6}$	5.84386
40	$8.9427E{-4}$	2.4623	$5.9000E{-4}$	2.4794	$6.4420E{-5}$	2.4402	$9.05585E{-7}$	5.44497
80	$3.5176E{-4}$	2.5491	$2.4000E{-4}$	2.5315	$2.6350E{-5}$	2.4448	$1.73895E{-7}$	5.20767
160	$1.3065E{-4}$	2.6945	$9.5000E{-5}$	2.5669	$1.0590E{-5}$	2.4882	$3.42668E{-8}$	5.07473
$\mathcal{CR}$		2.4623		2.4623		2.4623		4.9246

TABLE 4. Computational results with $N = 7$ for Example 5.1 at $T = 1$.

	$\beta = 0.5$				$\beta = 0.8$			
J	L_∞	$\mathcal{C_O}$	L_2	$\mathcal{C_O}$	L_∞	$\mathcal{C_O}$	L_2	$\mathcal{C_O}$
8	$3.30838E{-5}$		$1.06586E{-4}$		$5.64401E{-8}$		$1.79963E{-4}$	
16	$4.67087E{-6}$	2.82436	$1.50084E{-5}$	2.82818	$1.05280E{-5}$	2.42250	$3.34975E{-5}$	2.42557
32	$6.99419E{-7}$	2.73946	$2.24157E{-6}$	2.74319	$2.08836E{-6}$	2.33378	$6.63516E{-6}$	2.33585
64	$1.09366E{-7}$	2.67699	$3.49681E{-7}$	2.68040	$4.34536E{-7}$	2.26482	$1.37932E{-6}$	2.26617
128	$1.76712E{-8}$	2.62969	$5.63861E{-8}$	2.63263	$9.26208E{-8}$	2.23007	$2.93825E{-7}$	2.23093
256	$2.94851E{-9}$	2.58334	$9.39453E{-9}$	2.58545	$2.03885E{-8}$	2.18358	$6.47058E{-8}$	2.18299
TOC		2.5		2.5		2.2		2.2

Tables 3 and 4 demonstrat results of the numerical scheme for $N = 7$, $\beta = 0.5, \beta = 0.7$ and $\beta = 0.8$ at final time $T = 1$. In Table 3, the comparison of the error and convergence rate of the current method has been presented with the implicit finite difference scheme [39], compact finite deference approach [6] and MLS approach [10]. In Table 4, the results calculated with $\beta = 0.2$ and $\beta = 0.8$ at $T = 1$. One may infer from all these tables that our findings are better than the described approaches. Furthermore, with quite low space size, more reasonably precise results are provided.

Example 5.3. *In third example with homogeneous boundary conditions, we consider the constant values of Eq.* (1.1) *as* $\rho = 0.02, \sigma = 0.8, \mu = \frac{\sigma^2}{2}$ *and* $\eta = \rho - \mu$. *The initial condition is achieved from the exact solution* $u(x,t) = (t^3 + 1)x^4(x-1)$, *in addition, the source term is* $q(x,t) = \frac{6t^{3-\beta}}{\Gamma(4-\beta)}(x^5-x^4)-(t^3+1)\Big(\mu(20x^3-12x^2)+\eta(5x^4-4x^3)-\rho(x^5-x^4)\Big)$.

The results of this example have been showed in Table 5 and 6 with $N = 5$ at $T = 1$. Table 5 shows the comparison of the error and convergence order the our results than a compact quadratic spline collocation method [35]. According to this table, our findings are better than the described method. Moreover, the evaluated temporal order and rate with $\beta = 0.3$ and $\beta = 0.9$ are computed in Table 6 that supports the theoretical analysis.

6. CONCLUSION

The time-fractional Black-Scholes model includes considerations of its classic. The existence of the fractional-order derivative of this model instead of the integer-order instance leads to complex specific and numerical solutions. Therefore, a simulation model for solving TFBSM is provided in the proposed investigation. First of all, the discretization procedure in a temporal direction by the quadratic interpolation (accuracy order of $\mathcal{O}(\tau^{3-\beta})$) is explained. Then we

TABLE 5. Comparison of the error and convergence rate at $T = 1$ for Example 5.3 with the presented methods in [35].

	Method of [35] for $N = 200$ and $\beta = 0.2$		Current method for $N = 7$ and $\beta = 0.2$		Method of [35] for $N = 200$ and $\beta = 0.8$		Current method for $N = 7$ and $\beta = 0.8$	
J	$Max - error$	$\mathcal{C}_{\mathcal{O}}$	$Max - error$	$\mathcal{C}_{\mathcal{O}}$	$Max - error$	$\mathcal{C}_{\mathcal{O}}$	$Max - error$	$\mathcal{C}_{\mathcal{O}}$
10	$5.3144E{-}6$		$5.29194E{-}6$		$1.2014E{-}4$		$1.19851E{-}4$	
20	$8.1141E{-}7$	2.7114	$8.08092E{-}7$	2.71120	$2.6248E{-}5$	2.1944	$2.61872E{-}5$	2.19431
40	$1.2242E{-}7$	2.7286	$1.22046E{-}7$	2.72709	$5.7259E{-}6$	2.1966	$5.71298E{-}6$	2.19655
80	$1.8200E{-}8$	2.7498	$1.81751E{-}8$	2.74739	$1.2477E{-}6$	2.1982	$1.24507E{-}6$	2.19802
160	$2.5779E{-}9$	2.8197	$2.56779E{-}9$	2.82336	$2.7161E{-}7$	2.1997	$2.71185E{-}7$	2.19888

TABLE 6. Computational results with $N = 7$ for Example 5.3 at $T = 1$.

	$\beta = 0.3$			$\beta = 0.9$		
J	$Max - error$	$\mathcal{C}_{\mathcal{O}}$	$\mathcal{CR}$	$Max - error$	$\mathcal{C}_{\mathcal{O}}$	$\mathcal{CR}$
8	$1.85562E{-}5$			$2.95336E{-}4$		
16	$2.99293E{-}6$	2.63227	6.20002	$6.89501E{-}5$	2.09873	4.28333
32	$4.77678E{-}7$	2.64745	6.26558	$1.60990E{-}5$	2.09858	4.28288
64	$7.56406E{-}8$	2.65881	6.31510	$3.75748E{-}6$	2.09913	4.28452
128	$1.19054E{-}8$	2.66755	6.35347	$8.76766E{-}7$	2.09950	4.28561
256	$1.87345E{-}9$	2.66784	6.35478	$2.04644E{-}7$	2.09907	4.28434
TOC		2.7	6.49802		2.1	4.28709

have gone on how to reach the approximated solution by applying the collocation method based on a Legendre basis. Moreover, the unconditional stability of the time-discrete scheme by using the energy method and also we proved the convergence order of the time-discrete in the case of smooth solutions that is $\mathcal{O}(\tau^{3-\beta})$. Some numerical results with the analytical solution are intended to show the accuracy and convergence order of the numerical procedure, that the numerical outcomes have illustrated the accuracy of the current scheme. As mentioned in the paper, we have examined the order of convergence for the case of the smooth solution, but we know that the Caputo derivative has a singularity at point t_0 [13, 18, 34], so the study of this singularity can be examined in the next work.

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