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The generalized conformable derivative for 4α-order Sturm-Liouville problems

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Abstract

In this paper, we discuss the new generalized fractional operator. This operator similarly conformable derivative satisfies properties such as the sum, product/quotient, and chain rule. Laplace transform is defined in this case, and some of its properties are stated. In the following, the Sturm-Liouville problems are investigated, and also eigenvalues and eigenfunctions are obtained.

Keywords. Fractional Sturm-Liouville, Conformable derivative, Mittag-Leffler functions, Eigenvalues.

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1. Introduction

The creation of concept of fractional calculus was formed by L'Hopital in a letter to Leibniz in 1695. The interest in fractional differential equations grew rapidly, and various types of definitions were introduced. One thing that all these have in common, is that they consist of integral with different singular kernels. The most popular of them, we can mention to Grünwald-Letnikov, Riemann-Liouville, Caputo and Riesz, etc. Although fractional operators are linear, Unfortunately, this class of fractional derivatives is unsatisfying in some properties, such as the product rule, quotient rule, chain rule, and compositions rule. In 2014, Kalil et al.[14] and Also Katugampola, [13] by modifying the limit definition on the classic derivative, introduced a simple and local type of fractional derivative called conformable fractional derivative [11, 15, 29]. These new definitions comply with the computational relationship of the first derivative. Recently Mingarelli et al. [22] introduced the so-called "*Generalized Conformable Fractional Derivative*"(GCFD) as a unifying framework for conformable fractional methods.

In recent decades, much effort has focused on the class of well-known fractional Sturm-Liouville problem. These types of FSLPs arise in various areas of science and in many fields of engineering, for example, mechanics, electricity, chemistry, biology, economics and control theory [2, 4, 21, 24, 25, 28].
Our purpose is to investigate and discuss on eigenvalues of 4α-order of Sturm-Liouville problems with new definition (GCFD).

2. Preliminaries

In this section, we recall some definitions, notations and properties of fractional calculus theory used in this work.

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Definition 2.1. *(Ref [14]) The conformable fractional derivative of a function* $f : [0, \infty] \to \mathbb{R}$ *is defined as:*

$$D^{\alpha} f(t) = \lim_{h \to 0} \frac{f(t + ht^{1-\alpha}) - f(t)}{h}, \qquad t > 0, \tag{2.1}$$

where $0 < \alpha \leq 1$. Also Katugampola present the following definition [13].

Definition 2.2. *Let* $f : [0, \infty] \to \mathbb{R}$*. Then the fractional derivative of* f *of order* $0 < \alpha \leq 1$ *is defined by,*

$$D^{\alpha} f(t) = \lim_{h \to 0} \frac{f(te^{ht^{-\alpha}}) - f(t)}{h}, \qquad t > 0. \tag{2.2}$$

In both definitions, if $(D^{\alpha} f)(t)$ exists on $(0, \infty)$, then $D^{\alpha} f(0) = \lim_{t \to 0^+} D^{\alpha} f(t)$.
Mingarelli [22], proposed a generalized fractional derivative as follows.

Definition 2.3. *For a given function* $f : I \subseteq R \to R$*, defined on the range of an appropriate real valued function* $p : U_{\delta} \to R$ *where* $U_{\delta} = \{t, h) : t \in I = (a, b), |h| < \delta\}$*, by means of the limit*

$$D_p^{\alpha} f(t) = \lim_{h \to 0} \frac{f(p(t, h, \alpha)) - f(t)}{h}, \tag{2.3}$$

whenever the limit exists and is finite, be called the $(\alpha - p)$***-derivative*** *of* f *at* t *or* f *is* $(\alpha - p)$***-differentiable*** *at* t.

The function p must be satisfies in the following conditions.
$H1^+$ for $t \in I$ and for all sufficiently small $\varepsilon > 0$, the equation $p(t, h) = t + \varepsilon$ has a solution $h = h(t, \varepsilon)$. In addition, $h \to 0$ as $\varepsilon \to 0$.
$H1^-$ for $t \in I$ and for all sufficiently small $\varepsilon > 0$, the equation $p(t, h) = t - \varepsilon$ has a solution $h = h(t, \varepsilon)$. In addition, $h \to 0$ as $\varepsilon \to 0$.
$H_2 \frac{1}{p_h(.,0)} \in L(I)$ ={f: f is Lebesgue integrable function}.
Under the above conditions, if f, g are $(\alpha - p)$ - differential at $t \in I$, then
(i) (The Sum Rule)

$$D_p^{\alpha}\big(f + g\big)(t) = D_p^{\alpha} f(t) + D_p^{\alpha} g(t).$$

(ii) (The Product Rule)

$$D_p^{\alpha}\big(f.g\big)(t) = g(t).D_p^{\alpha} f(t) + f(t).D_p^{\alpha} g(t).$$

(iii) (The Quotient Rule)

$$D_p^{\alpha}\big(\frac{f}{g}\big)(t) = \frac{g(t).D_p^{\alpha} f(t) - f(t).D_p^{\alpha} g(t)}{g^2(t)}.$$

(iv) (The Chain Rule)

$$D_p^{\alpha}\big(f \circ g\big)(t) = g'\big(f(t)\big) D_p^{\alpha} f(t).$$

(v)(The relationship between differentiability and $(\alpha - p)$-differentiability) For $\frac{\partial p(t,0,\alpha)}{\partial h} = p_h(t, 0, \alpha) \neq 0$ we have

$$D_p^{\alpha} f(t) = p_h(t, 0, \alpha) f'(t), \tag{2.4}$$

that $0 < \alpha < 1$. For more details, the reader can refer to [22].
In the sequel, we give some conformable fractional derivative of certain functions:

Example 2.4.

$$a) \quad D_p^\alpha\Big(\int_a^t \frac{ds}{p_h(s,0,\alpha)}\Big) = 1,$$

$$b) \quad D_p^\alpha\Big(\int_a^t \frac{ds}{p_h(s,0,\alpha)}\Big)^n = n\Big(\int_a^t \frac{ds}{p_h(s,0,\alpha)}\Big)^{n-1},$$

$$c) \quad D_p^\alpha\Big(e^{c\int_a^t \frac{ds}{p_h(s,0,\alpha)}}\Big) = ce^{c\int_a^t \frac{ds}{p_h(s,0,\alpha)}},$$

$$d) \quad D_p^\alpha\Big(\sin\big(\int_a^t \frac{ds}{p_h(s,0,\alpha)}\big)\Big) = a\cos\big(\int_a^t \frac{ds}{p_h(s,0,\alpha)}\big),$$

$$e) \quad D_p^\alpha\Big(\cos\big(\int_a^t \frac{ds}{p_h(s,0,\alpha)}\big)\Big) = -a\sin\big(\int_a^t \frac{ds}{p_h(s,0,\alpha)}\big).$$

3. Generalized Fractional Integral and Laplace Transform

In this section we present the definition (α,p)-fractional integral of f as bellow:

Definition 3.1. *The Generalized fractional integral of order $0<\alpha<1$ is defined by*

$$I_p^\alpha(f(t)) = \int_0^t f(s)d_\alpha(s) = \int_0^t \frac{f(s)}{p_h(s,0,\alpha)}ds = I_p^1\Big(\frac{f(t)}{p_h(t,0,\alpha)}\Big).$$

It can be easily proved that $D_p^\alpha(I_p^\alpha(f(t))) = f(t)$ and $I_p^\alpha(D_p^\alpha(f(t))) = f(t) - f(a)$.
The following definition gives us the adapted Laplace transform to the (α,p)- fractional derivative.

Definition 3.2. *Let $0<\alpha\le 1$ and $f:[0,\infty]\to\mathbb{R}$ be a real valued function. Then, The Generalized fractional (α,p)-Laplace transform is defined by*

$$\mathcal{L}_p^\alpha(f(t)) = \int_0^\infty \frac{e^{-s\int_a^t \frac{ds}{p_h(t,0,\alpha)}}}{p_h(t,0,\alpha)} f(t)dt = F_p^\alpha(s).$$

Theorem 3.3. *(α,p)-Laplace transform of $D_p^\alpha f(t)$ with $\int_a^t \frac{ds}{p_h(s,0,\alpha)} > 0$, is defined as follows:*

$$\mathcal{L}_p^\alpha\{D_p^\alpha f(t)\} = sF_p^\alpha(s) - y(0),$$

and

$$\mathcal{L}_p^\alpha(D_p^{4\alpha} f(t)) = s^4F_p^\alpha(s) - s^3f(0) - p_h(0,0,\alpha)s^2f'(0) - p_h^2(0,0,\alpha)sf''(0) - p_h^3(0,0,\alpha)f'''(0).$$

Proof. We prove the first part by the definition (3.2) and item (v), proof of the second part is similar, so we have

$$\mathcal{L}_p^\alpha(D_p^\alpha(f(t))) = \int_0^\infty \frac{e^{-s\int_a^t \frac{ds}{p_h(t,0,\alpha)}}}{p_h(t,0,\alpha)} p_h(t,0,\alpha)f'(t)dt,$$

now by means of integration by parts

$$= e^{-s\int_a^t \frac{ds}{p_h(t,0,\alpha)}} f(t)|_0^\infty + s\int_0^\infty \frac{e^{-s\int_a^t \frac{ds}{p_h(t,0,\alpha)}}}{p_h(t,0,\alpha)} f(t)dt$$
$$= sF_p^\alpha(s) - y(0).$$

□

Also, We refer the interested readers to the references in [20].

Mittag-Leffler function and theirs properties

Definition 3.4. *(Ref.[23]) The 2-parameter Mittag-Leffler is defined for $z, \beta \in \mathbb{C}, \Re(\alpha) > 0$,*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}. \tag{3.1}$$

Lemma 3.5. *(Ref.[23]) $\Re(\alpha > 0)$, the inverse Laplace transform of some spacial functions are as below:*

$$\mathcal{L}^{-1}\{\frac{s^{\alpha}}{s(s^{\alpha} - \lambda)}\} = E_{\alpha}(\lambda t^{\alpha}), \tag{3.2}$$

$$\mathcal{L}^{-1}\{\frac{s^{\alpha-\beta}}{s^{\alpha} - \lambda}\} = t^{\beta-1} E_{\alpha,\beta}(\lambda t^{\alpha}), \tag{3.3}$$

$$\mathcal{L}^{-1}\{\frac{k! s^{\alpha-\beta}}{(s^{\alpha} - \lambda)^{k+1}}\} = t^{k\alpha+\beta-1} E^{(k)}_{\alpha,\beta}(\lambda t^{\alpha}), \tag{3.4}$$

where $\Re(s) > |\lambda|^{\frac{1}{\alpha}}$.

4. Sturm-Liouville problem

In this section, we consider GFSL problems in three different theorems:

Feature 4.1: Let us consider the GFSL problem as follows:

$$D^{4\alpha} y(t) = 0, \qquad 0 < \alpha \leq 1, \tag{4.1}$$

where $y \in AC^n[a, b], q(t)$ is a real-valued and continuous function on $[0, n], n \in \mathbb{R}$.

Theorem 4.1. *The solution of GFSL problem (4.1) is*

$$y(t) = A\left(\int_0^t \frac{ds}{p_h(s, 0, \alpha)} \right)^3 + B\left(\int_0^t \frac{ds}{p_h(s, 0, \alpha)} \right)^2 + C\left(\int_0^t \frac{ds}{p_h(s, 0, \alpha)} \right) + D \tag{4.2}$$

Proof. Applying theorem (3.3) to (4.1), we have

$$s^4 Y_p^{\alpha}(s) - s^3 y(0) - p_h(0, 0, \alpha) s^2 y'(0) - p_h^2(0, 0, \alpha) s y''(0) - p_h^3(0, 0, \alpha) y'''(0) = 0,$$

or

$$Y_p^{\alpha}(s) = y(0)\frac{1}{s} + p_h(0, 0, \alpha) y'(0) \frac{1}{s^2} + p_h^2(0, 0, \alpha) y''(0) \frac{1}{s^3} + p_h^3(0, 0, \alpha) y'''(0) \frac{1}{s^4}. \tag{4.3}$$

Now by inverse Laplace transform, (4.2) is obtained. □

Feature 4.2: Let us consider the GFSL problem as follows:

$$D^{4\alpha} y(t) = \lambda y, \qquad 0 < \alpha \leq 1, \tag{4.4}$$

$$y(0) = y(1) = y^{2\alpha}(0) = y^{2\alpha}(1) = 0, \tag{4.5}$$

where $y \in AC^n[a, b], q(t)$ is a real-valued and continuous function on $[0, n], n \in \mathbb{R}$.

Theorem 4.2. *The eigenvalues of GFSL problem (4.4) and (4.5) is*

$$\lambda = \frac{n^4 \pi^4}{(\int_0^1 \frac{ds}{p_h(s,0,\alpha)})^4}, \; n = 1, 2, ...$$

Proof. Apply the Laplace transform in theorem (3.3) on (4.4), we have

$$s^4 Y_p^{\alpha}(s) - s^3 y(0) - p_h(0, 0, \alpha) s^2 y'(0) - p_h^2(0, 0, \alpha) s y''(0) - p_h^3(0, 0, \alpha) y'''(0) = \lambda Y_p^{\alpha}(s),$$

or

$$Y_p^{\alpha}(s) = A\frac{s^3}{s^4 - \lambda} + B\frac{s^2}{s^4 - \lambda} + C\frac{s}{s^4 - \lambda} + D\frac{1}{s^4 - \lambda}, \tag{4.6}$$

where $A=y(0)$, $B=p_h(0,0,\alpha)y'(0)$, $C=p_h^2(0,0,\alpha)y''(0)$ and $D=p_h^3(0,0,\alpha)y'''(0)$. Now by lemma (3.5) to (4.6) we have

$$\begin{aligned} y(t)=&AE_{4,1}\Big(\lambda\big(\int_0^t\frac{ds}{p_h(s,0,\alpha)}\big)^4\Big)\\ &+B\Big(\int_0^t\frac{ds}{p_h(s,0,\alpha)}\Big)E_{4,2}\Big(\lambda\big(\int_0^t\frac{ds}{p_h(s,0,\alpha)}\big)^4\Big)\\ &+C\Big(\int_0^t\frac{ds}{p_h(s,0,\alpha)}\Big)^2E_{4,3}\Big(\lambda\big(\int_0^t\frac{ds}{p_h(s,0,\alpha)}\big)^4\Big)\\ &+D\Big(\int_0^t\frac{ds}{p_h(s,0,\alpha)}\Big)^3E_{4,4}\Big(\lambda\big(\int_0^t\frac{ds}{p_h(s,0,\alpha)}\big)^4\Big). \end{aligned} \tag{4.7}$$

on the other hand

$$\begin{aligned} y^{(2\alpha)}(t)=&A\Big(\lambda\big(\int_0^t\frac{ds}{p_h(s,0,\alpha)}\big)^2\Big)E_{4,3}\Big(\lambda\big(\int_0^t\frac{ds}{p_h(s,0,\alpha)}\big)^4\Big)\\ &+B\Big(\lambda\big(\int_0^t\frac{ds}{p_h(s,0,\alpha)}\big)^3\Big)E_{4,4}\Big(\lambda\big(\int_0^t\frac{ds}{p_h(s,0,\alpha)}\big)^4\Big)\\ &+CE_{4,1}\Big(\lambda\big(\int_0^t\frac{ds}{p_h(s,0,\alpha)}\big)^4\Big)\\ &+D\Big(\int_0^t\frac{ds}{p_h(s,0,\alpha)}\Big)^3E_{4,2}\Big(\lambda\big(\int_0^t\frac{ds}{p_h(s,0,\alpha)}\big)^4\Big). \end{aligned}$$

Finally, by imposing the boundary conditions (4.5), yields

$$\begin{cases} B\Big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\Big)E_{4,2}\Big(\lambda\big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\big)^4\Big)\\ +D\Big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\Big)^3E_{4,4}\Big(\lambda\big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\big)^4\Big)=0,\\ B\lambda\Big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\Big)^3E_{4,4}\Big(\lambda\big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\big)^4\Big)\\ +D\Big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\Big)^2E_{4,2}\Big(\lambda\big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\big)^4\Big)=0. \end{cases} \tag{4.8}$$

we obtain

$$E_{4,2}^2\Big(\lambda\big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\big)^4\Big)-\frac{\lambda\big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\big)^4}{\int_0^1\frac{ds}{p_h(s,0,\alpha)}}E_{4,4}^2\Big(\lambda\big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\big)^4\Big)=0, \tag{4.9}$$

From the Mittag-Leffler integral representation [24], we have the following relation

$$E_{\alpha,\beta}(z)=\frac{1}{2\pi i}\int_C\frac{s^{\alpha-\beta}}{s^\alpha-z}e^s ds, \tag{4.10}$$

where C is a loop which starts and ends at $-\infty$ and encircles the circular disc $|t|\le|z|^{\frac{1}{\alpha}}$ in the positive sense: $-\pi\le\arg s\le\pi$.

We see that

$$E_{4,2}\Big(\lambda\big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\big)^4\Big)=\frac{1}{2\pi i}\int_C\frac{s^2}{s^4-\lambda\big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\big)^4}e^s ds. \tag{4.11}$$

For solving this integral, we use Cauchy's residue theorem.

$$s^4-\lambda\Big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\Big)^4=0\Longrightarrow s_k=\Big(\lambda\Big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\Big)^4\Big)^{\frac14}e^{i(\frac{k\pi}{2})}$$
$$k=\ldots-1,0,1,\ldots.$$

Acceptable poles are

$$s_{-1}=-i\Big(\lambda\Big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\Big)^4\Big)^{\frac14}\ ,\ s_0=\Big(\lambda\Big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\Big)^4\Big)^{\frac14},$$
$$s_1=i\Big(\lambda\Big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\Big)^4\Big)^{\frac14}\ ,\ s_2=-\Big(\lambda\Big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\Big)^4\Big)^{\frac14}.$$

Thus

$$E_{4,2}\Big(\lambda\Big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\Big)^4\Big)=\frac14\sum_{i=-1}^{2}\frac{e^{s_i}}{s_i}.$$

After calculations we have

$$E_{4,2}\Big(\lambda\Big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\Big)^4\Big)=\frac{1}{\Big(\lambda\big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\big)^4\Big)^{\frac14}}\times$$
$$\Big\{\sinh\Big(\Big(\lambda\Big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\Big)^4\Big)^{\frac14}\Big)+\sin\Big(\Big(\lambda\Big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\Big)^4\Big)^{\frac14}\Big)\Big\}. \tag{4.12}$$

In similarly on $E_{4,4}\Big(\lambda\big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\big)^4\Big)$ we obtain

$$E_{4,4}\Big(\lambda\Big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\Big)^4\Big)=\frac{1}{\Big(\lambda\big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\big)^4\Big)^{\frac34}}\times$$
$$\Big\{\sinh\Big(\Big(\lambda\Big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\Big)^4\Big)^{\frac14}\Big)-\sin\Big(\Big(\lambda\Big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\Big)^4\Big)^{\frac14}\Big)\Big\}. \tag{4.13}$$

With substitution (4.12) and (4.13) in (4.9) we get

$$\frac{1}{2\Big(\lambda\big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\big)^4\Big)^{\frac12}}\Big\{\Big(1-\frac{1}{\int_0^1\frac{ds}{p_h(s,0,\alpha)}}\Big)\sin^2h\Big(\Big(\lambda\Big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\Big)^4\Big)^{\frac14}\Big)$$
$$+\Big(1-\frac{1}{\int_0^1\frac{ds}{p_h(s,0,\alpha)}}\Big)\sin^2\Big(\Big(\lambda\Big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\Big)^4\Big)^{\frac14}\Big)+2\Big(1+\frac{1}{\int_0^1\frac{ds}{p_h(s,0,\alpha)}}\Big)\times$$
$$\sinh\Big(\Big(\lambda\Big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\Big)^4\Big)^{\frac14}\Big)\sin\Big(\Big(\lambda\Big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\Big)^4\Big)^{\frac14}\Big)\Big\}=0, \tag{4.14}$$

where $\lambda(\int_0^t\frac{ds}{p_h(s,0,\alpha)})$ non-zero and (4.14) is transcendental equation. Now to get eigenvalues of this equation, we rewrite (4.14).

$$\frac{\int_0^1\frac{ds}{p_h(s,0,\alpha)}-1}{\int_0^1\frac{ds}{p_h(s,0,\alpha)}}\Big(\sin^2h\big(\lambda^{\frac14}\int_0^1\frac{ds}{p_h(s,0,\alpha)}\big)+\sin^2\big(\lambda^{\frac14}\int_0^1\frac{ds}{p_h(s,0,\alpha)}\big)\Big)$$
$$+\frac{2\big(\int_0^1\frac{ds}{p_h(s,0,\alpha)}\big)+1)}{\int_0^1\frac{ds}{p_h(s,0,\alpha)}}\sinh\big(\lambda^{\frac14}\int_0^1\frac{ds}{p_h(s,0,\alpha)}\big)\sin\big(\lambda^{\frac14}\int_0^1\frac{ds}{p_h(s,0,\alpha)}\big)=0. \tag{4.15}$$

TABLE 1. The eigenvalues λ_n

α=0.76	α= 0.86	α=1.0
λ_1=32.498	λ_1=53.284	λ_1=97.410
λ_2=519.96	λ_2=852.54	λ_2= 1558.6
λ_3= 2632.3	λ_3=4316.0	λ_3=7890.2
	λ_4=13640.0	λ_4= 24937.0
		λ_5=60881.0
$\vdots$	$\vdots$	$\vdots$

Now from (4.15) we have

$$\sinh\big(\lambda^{\frac{1}{4}}\int_0^1 \frac{ds}{p_h(s,0,\alpha)}\big) = \sin\big(\lambda^{\frac{1}{4}}\int_0^1 \frac{ds}{p_h(s,0,\alpha)}\big) = 0,$$

it can be easily concluded that

$$\lambda = \frac{n^4\pi^4}{(\int_0^1 \frac{ds}{p_h(s,0,\alpha)})^4},\ n = 1, 2, ...$$

We suppose $p(t, h, \alpha) = t + ht^{1-\alpha}$ (conformable fractional [14]) with $\frac{3}{4} < \alpha < 1$, then $p_h(t, h, \alpha) = t^{1-\alpha}$ and $\int_0^1 \frac{ds}{s^{1-\alpha}} = \frac{1}{\alpha} > 0$, thus

$$\lambda = \frac{n^4\pi^4}{(\int_0^1 \frac{ds}{s^{1-\alpha}})^4} = (n\pi\alpha)^4,\ n = 1, 2, ...$$

□

If $\alpha = 1$, we have the classic form, in this case, from (4.15) we have

$$\sinh(\lambda^{\frac{1}{4}})\sin(\lambda^{\frac{1}{4}}) = 0,$$

that we get

$$\lambda = (k\pi)^4,\ k = 0, 1, 2, ...$$

Table 1 shows the eigenvalues for different values α and in Figures 1 and 2 for $\alpha = 1$ and $\alpha = 0.86$ the eigenfunctions(EFs) is given, respectively.

Feature 4.3: Let us consider the CFSL problem as follows:

$$D^{4\alpha}y(t) + q(t)y(t) = \lambda y, \tag{4.16}$$

$$y(0) = c_1,\ y^{\alpha}(0) = c_2,\ y^{2\alpha}(0) = c_3,\ y^{3\alpha}(0) = c_4, \tag{4.17}$$

where $y \in AC^n[a, b], q(t)$ is a real-valued and continuous function on $[0, n], n \in \mathbb{R}$.

Theorem 4.3. *The solution of CFSL problem (4.16) and (4.17) is*

$$
\begin{aligned}
y(t) &= c_1 E_{4,1}\Big(\lambda\big(\int_0^t \frac{ds}{p_h(s,0,\alpha)}\big)^4\Big)\\
&+ c_2\big(\int_0^t \frac{ds}{p_h(s,0,\alpha)}\big)E_{4,2}\Big(\lambda\big(\int_0^t \frac{ds}{p_h(s,0,\alpha)}\big)^4\Big)\\
&+ c_3\big(\int_0^t \frac{ds}{p_h(s,0,\alpha)}\big)^2E_{4,3}\Big(\lambda\big(\int_0^t \frac{ds}{p_h(s,0,\alpha)}\big)^4\Big)\\
&+ c_4\big(\int_0^t \frac{ds}{p_h(s,0,\alpha)}\big)^3E_{4,4}\Big(\lambda\big(\int_0^t \frac{ds}{p_h(s,0,\alpha)}\big)^4\Big)\\
&- \int_0^{\int_0^t \frac{ds}{p_h(s,0,\alpha)}}\Big(\int_0^t \frac{ds}{p_h(s,0,\alpha)}-\tau\Big)^3 E_{4,4}\Big(\lambda\big(\int_0^t \frac{ds}{p_h(s,0,\alpha)}-\tau\big)^4\Big)\\
&\times q(\tau)y(\tau,\lambda)d\tau.
\end{aligned}
$$

Proof. Taking Laplace transform (3.3) of both sides of (4.16), also using the (4.17), we have

$$s^4Y_p^{\alpha}(s)-s^3y(0)-s^2y^{\alpha}(0)-sy^{2\alpha}(0)-y^{3\alpha}(0)+\mathcal{L}_{\alpha}\{q(t)y(t)\}=\lambda Y_p^{\alpha}(s),$$

or

$$Y_p^{\alpha}(s)=c_1\frac{s^3}{s^4-\lambda}+c_2\frac{s^2}{s^4-\lambda}+c_3\frac{s}{s^4-\lambda}+c_4\frac{1}{s^4-\lambda}-\frac{1}{s^4-\lambda}\mathcal{L}_{\alpha}\{q(t)y(t)\}. \tag{4.18}$$

Applying the inverse Laplace transform in lemma (3.5) to both sides of (4.18) and using by convolution theorem (3.3), the corollary of the theorem is obtained. □

In Figures 3 and 4, eigenfunctions are given for two different eigenvalues and different values α.

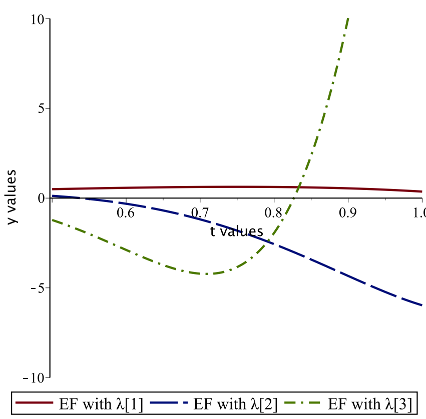

FIGURE 1.

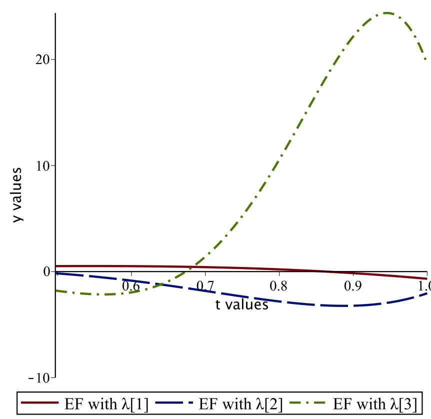

FIGURE 2.

5. Conclusions

In this article, we presented the definition of the general fractional integral operator and Laplace transformation based on the generalized conformable derivative definition. In the following, we have introduced and solved three types of Sturm-Liouville problems.

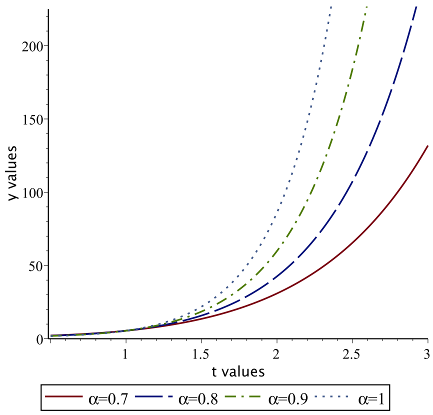

Figure 3.

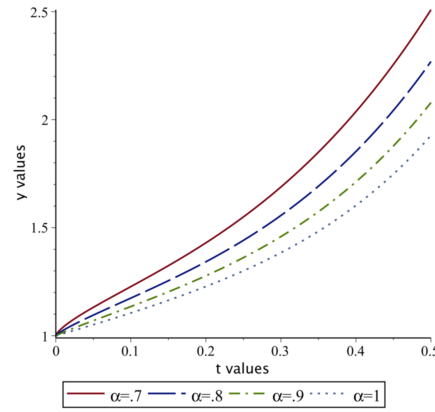

Figure 4.

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