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An effective technique for the conformable space-time fractional cubic-quartic nonlinear Schrödinger equation with different laws of nonlinearity

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Abstract

In the present study, we investigate the conformable space-time fractional cubic-quartic nonlinear Schrödinger equation with three different laws of nonlinearity namely, parabolic law, quadratic-cubic law, and weak non-local law. This model governs the propagation of solitons through nonlinear optical fibers. An effective approach namely, the $\exp(-\Phi(\xi))$-expansion method is applied to construct some new soliton solutions of the governing model. Consequently, the dark, singular, rational and periodic solitary wave solutions are successfully revealed. The comparisons with other results are also presented. In addition, the dynamical structures of obtained solutions are presented through 3D and 2D plots.

Keywords. Conformable derivative, Fractional cubic-quartic nonlinear Schrödinger equation, Soliton solutions, Exp$(-\Phi(\xi))$-expansion method.
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1. Introduction

In recent years, many studies have been performed to investigate the non-linear Schrödinger equation (NLS) in optical fibers to understand the dynamical behavior of optical soliton [17, 19, 28, 32, 44]. For example, consider the cubic-quartic nonlinear Schrödinger equation (CQ-NLSE) of the form, as follows [27, 31]

$$iu_t + i\beta u_{xxx} - \gamma u_{xxxx} + F(|u|^2)u = 0, \tag{1.1}$$

where $u = u(x,t)$ is a complex-valued wave profile of time and space. The first term is temporal evolution while β and γ are the coefficients of third order dispersion (3OD) and fourth order dispersion (4OD) respectively. The source of nonlinearity $F(|u|^2) : C \to C$ is a k-times continuously differentiable real-valued algebraic function, so that:

$$F(|u|^2) \in \bigcup_{m,n=1}^{\infty} c^k((-n,n) \times (-m,m) : R^2).$$

Dispersion and nonlinearity are extremely important for the propagation of solitons in nonlinear media. Generally, group velocity dispersion (GVD) leveling with self-phase modulation in a sensitive way allows such solitons to maintain long-distance travel. However, it could occur that GVD is very small and therefore completely ignored, so in this condition, the dispersion impact is rewarded by 3OD and 4OD dispersion impacts. This is generally referred to as solitons that are cubic-quartic (CQ). This model governs the dynamics of pulse transfer through optical fibers and other forms of waveguides. There are many effective techniques have been applied to study the dynamics of optical soliton propagation [18, 49–54].

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In recent years, the studies of physical models with fractional derivatives have attracted a great attention since some materials are well described in fractal media. Consider the space-time fractional CQ-NLSE of the form

$$iD_t^{\alpha}u + \beta D_x^{3\alpha}u - \gamma D_x^{4\alpha}u + F(|u|^2)u = 0, \tag{1.2}$$

where the operator D^{α} of order $\alpha \in (0,1]$ is the conformable fractional derivative. Recently, Eq. (1.2) with parabolic law nonlinearity was studied in [24] to find the exact soliton solutions and the other solutions.

Finding the exact solutions to nonlinear PDEs play an important role in many phenomena in Mathematical physics. In recent years, many new approaches for finding these exact and analytical solutions have been proposed, for example, the sine - cosine method [45, 47], the tanh - coth method [40, 46], the Jacobi elliptic function method [23, 25], the first integral method [26, 41], the F-expansion method [2, 58], the exp-function method [16, 30], the (G'/G) expansion method [4, 7–9, 14], the novel (G'/G) expansion method [5, 10–13, 29], the new mapping method [55, 57], the modified simple equation method [33, 39, 56], the solitary wave ansatz method [38, 42, 43], the Decomposition method [3, 36, 37], the $\exp(-\Phi(\xi))$-expansion method [6, 15, 35, 48], the Lie symmetry analysis [20–22], and so on.

The main objective of this study is to obtain the exact soliton solutions to the Eq. (1.2) having three different laws of nonlinearity namely, parabolic law, quadratic-cubic law, and weak non-local law by using an effective approach called, the $\exp(-\Phi(\xi))$-expansion method. This method is a powerful tool for finding the exact solutions of nonlinear differential equations, and gained considerable attention in recent years. To our best of knowledge, Eq. (1.2) having different laws of nonlinearity is not investigated in the literature by using the proposed method.

This manuscript is organized as follows: The Basic definition and properties of the conformable fractional derivative are presented in the next section. In section 3, we give brief descriptions of the $\exp(-\Phi(\xi))$-expansion method. In section 4, we express the mathematical analysis of Eq. (1.2). In sections 5, the proposed method is applied to solve the Eq. (1.2) for three different laws of nonlinearity. The graphical representations are given in the section 6. Finally, the conclusion of our study is presented in section 7.

2. Conformable Fractional Derivative

A new form of conformable fractional derivative was introduced by Khalil et al.[34]. This new definition of fractional calculus is based on a limit operator which is more natural and effective in satisfying some conventional properties than the existing fractional derivatives. Consider the following basic definition and properties on the conformable derivatives of order α [34]:

Definition 1. Let $f : (0, \infty) \to \mathbb{R}$, then the conformable fractional derivative of f of order α is defined as

$$D_t^{\alpha} f(t) = \lim_{\epsilon \to 0} \frac{f\left(t + \epsilon t^{1-\alpha}\right) - f(t)}{\epsilon}, \tag{2.1}$$

for all $t > 0$, $\alpha \in (0, 1)$.

Note that if the conformable fractional derivative of f of order α exists, then f is α - differentiable.

The most important properties of the conformable fractional derivative are given as the following theorems:

Theorem 1. Let g and f be α-conformable differentiable at $t > 0$, then:

(1) $D_t^{\alpha}(af + bg) = aD_t^{\alpha} f + bD_t^{\alpha} g,\ for\ all\ a, b \in \mathbb{R}.$

(2) $D_t^{\alpha}(t^{\mu}) = \mu t^{\mu-\alpha},\ for\ all\ \mu \in \mathbb{R}.$

(3) $D_t^{\alpha}(fg) = fD_t^{\alpha} g + gD_t^{\alpha} f.$

(4) $D_t^\alpha\left(\dfrac{f}{g}\right)=\dfrac{gD_t^\alpha f-fD_t^\alpha g}{g^2}$.

Moreover, if f is differentiable, then $D_t^\alpha\left(f(t)\right)=t^{1-\alpha}\dfrac{\mathrm{d}f}{\mathrm{d}t}$. The chain rule for conformable fractional derivatives is reported in [1] as the following theorem.

Theorem 2. Let $f:(0,\infty)\to\mathbb{R}$, be a differentiable function and α is order of the conformable derivative. Let g be a function defined in the range of f and also differentiable, then

$$D_t^\alpha\left(fg\right)(t)=t^{1-\alpha}g^{'}(t)f^{'}(g(t)), \tag{2.2}$$

here "prime" is the classical derivative for t.

3. Description of the exp$(-\Phi(\xi))$-expansion method

Consider the nonlinear fractional PDE of the form, as

$$P\left(u,D_t^\alpha u,D_x^\beta u,D_x^{2\alpha}u,D_t^{2\beta}u,D_x^\alpha D_x^\beta u,\cdots\right)=0,\ \ 0<\alpha,\beta<1, \tag{3.1}$$

where u is unknown function and $D_t^\alpha u$ and $D_x^\beta u$ are conformable fractional derivatives.

Consider the transformation

$$u(x,t)=U\left(\xi\right),\ \text{where}\ \xi=\frac{x^\beta}{\beta}-\nu\frac{t^\alpha}{\alpha}, \tag{3.2}$$

permits us to reduce the Eq. (3.1) into an ODE of the form, as

$$H\left(U,U^{'},U^{''},U^{'''},\cdots\right)=0. \tag{3.3}$$

The proposed method can be summarized in the following three steps [35]:

Step 1 : According to the proposed method, the wave solution of Eq. (3.3) can be expressed, as

$$U\left(\xi\right)=\sum_{i=0}^{n}\alpha_i\,\exp\left(-\Phi\left(\xi\right)\right)^i, \tag{3.4}$$

where α_i $(\alpha_n\neq 0)$ are constants, such that

$$\Phi^{'}\left(\xi\right)=\exp\left(-\Phi\left(\xi\right)\right)+\mu\,\exp\left(\Phi\left(\xi\right)\right)+\lambda. \tag{3.5}$$

The above Eq. (3.5) has the formal solutions as follows:

Case 1 : For $\lambda^2-4\mu>0$ and $\mu\neq 0$,

$$\Phi_1\left(\xi\right)=\ln\left[\frac{-\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \left(\xi+C\right)\right)-\lambda}{2\mu}\right]. \tag{3.6}$$

Case 2 : For $\lambda^2-4\mu<0$ and $\mu\neq 0$,

$$\Phi_2\left(\xi\right)=\ln\left[\frac{\sqrt{4\mu-\lambda^2}\tan\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \left(\xi+C\right)\right)-\lambda}{2\mu}\right]. \tag{3.7}$$

Case 3 : For $\lambda^2-4\mu>0$ and $\lambda\neq 0,\ \mu=0$,

$$\Phi_3\left(\xi\right)=-\ln\left[\frac{\lambda}{\cosh\left(\lambda\left(\xi+C\right)\right)+\sinh\left(\lambda\left(\xi+C\right)\right)-1}\right]. \tag{3.8}$$

Case 4 : For $\lambda^2 - 4\mu = 0$ and $\lambda \neq 0,\ \mu \neq 0$,

$$\Phi_4\left(\xi\right) = \ln\left[-\frac{2\left(\lambda\left(\xi + C\right)\right) + 2}{\lambda^2\left(\xi + C\right)}\right]. \tag{3.9}$$

Case 5 : For $\lambda^2 - 4\mu = 0$ and $\lambda = 0,\ \mu = 0$, the solution of the form:

$$\Phi_5\left(\xi\right) = \ln\left(\xi + C\right), \tag{3.10}$$

here C is constant.

Step 2 : The integer n can be found by considering the homogeneous balance. Substituting Eq. (3.4) into Eq. (3.3) and setting the coefficient of each power of $(-\Phi\left(\xi\right))$ to zero provides a system of algebraic equations, which can be solved by Mathematical software to find the values of $\nu,\ \lambda, \mu,$ and α_i.

Step 3 : By substituting $\alpha_i, \lambda, \mu,$ and ν into Eq. (3.4) various types of exact solutions of Eq. (3.1) can be constructed.

4. Mathematical analysis of the model

Recall the conformable space-time fractional CQ-NLSE of the form

$$iD_t^{\alpha}u + \beta D_x^{3\alpha}u - \gamma D_x^{4\alpha}u + F(|u|^2)u = 0. \tag{4.1}$$

Consider the transformation

$$u(x,t) = \phi(\xi)e^{i\theta(x,t)};\ \xi = \frac{x^{\alpha}}{\alpha} - \nu\frac{t^{\alpha}}{\alpha},\ \theta = -k\frac{x^{\alpha}}{\alpha} + \omega\frac{t^{\alpha}}{\alpha}, \tag{4.2}$$

where ω, k, ν are the constants. Inserting Eq. (4.2) into Eq. (4.1) and splitting the real and imaginary parts, we get

$$\left(-k^3\beta + k^4\gamma - \omega\right)\phi + F(\phi^2)\phi + 3k(\beta - 2k\gamma)\phi'' + \gamma\phi^{(4)} = 0, \tag{4.3}$$

$$\left(-3k^2\beta + 4k^3\gamma - \nu\right)\phi' + (\beta - 4k\gamma)\phi^{(3)} = 0. \tag{4.4}$$

From Equation (4.4), we get constraint conditions:

$$\beta = 4k\gamma \text{ and } \nu = -8k^3\gamma. \tag{4.5}$$

By inserting Eq. (4.5) into Eq. (4.3) we get

$$\left(-3k^4\gamma - \omega\right)\phi + F(\phi^2)\phi + 6k^2\gamma\phi'' + \gamma\phi^{(4)} = 0. \tag{4.6}$$

Now, our aim is to solve Eq. (4.6) using the above proposed method for three different law of non-linearity.

5. Implementation of the proposed method

In the following subsections, we implement the $\exp(-\Phi(\xi))$-expansion method to solve the space-time fractional CQ-NLSE for the source of nonlinearity $F(\phi)$ takes the following three different forms:

5.1. **Parabolic Law non-linearity.** First, we consider the parabolic law non-linearity

$$F(\phi) = c_1\phi + c_2\phi^2, \tag{5.1}$$

where c_1 and c_2 are constants such that $c_2 \neq 0$. For the parabolic law non-linearity, the Eq. (4.6) is given by:

$$\left(-3k^4\gamma - \omega\right)\phi + c_1\phi^3 + c_2\phi^5 + 6k^2\gamma\phi'' + \gamma\phi^{(4)}. \tag{5.2}$$

Considering the balance of $\phi^{(4)}$ and ϕ^5 in Eq. (5.2), we get $N = 1$. According to the proposed method the solution of Eq.(5.2) takes the form:

$$\phi = \alpha_0 + \alpha_1 \exp(-\Phi(\xi)), \tag{5.3}$$

where α_0, and $\alpha_1(\neq 0)$ are constants.

Adding Eq. (5.3) together with Eq. (3.5) into Eq. (5.2), taking the coefficients of $\exp(-\Phi(\xi))$ and setting them equal to zero, we find the equations as follows:

$$\begin{aligned}
&\exp(-\Phi(\xi))^5 : 24\gamma\alpha_1 + c_2\alpha_1^5 = 0,\\
&\exp(-\Phi(\xi))^4 : 60\gamma\lambda\alpha_1 + 5c_2\alpha_0\alpha_1^4 = 0,\\
&\exp(-\Phi(\xi))^3 : \alpha_1\left(12k^2\gamma + 50\gamma\lambda^2 + 40\gamma\mu + c_1\alpha_1^2 + 10c_2\alpha_0^2\alpha_1^2\right) = 0,\\
&\exp(-\Phi(\xi))^2 : \alpha_1\left(3\gamma\lambda\left(6k^2 + 5\lambda^2 + 20\mu\right) + 3c_1\alpha_0\alpha_1 + 10c_2\alpha_0^3\alpha_1\right) = 0,\\
&\exp(-\Phi(\xi))^1 : -3k^4\gamma\alpha_1 + 6k^2\gamma\lambda^2\alpha_1 + \gamma\lambda^4\alpha_1 + 12k^2\gamma\mu\alpha_1 + 22\gamma\lambda^2\mu\alpha_1 + 16\gamma\mu^2\alpha_1 - \omega\alpha_1 + 3c_1\alpha_0^2\alpha_1 + 5c_2\alpha_0^4\alpha_1 = 0,\\
&\exp(-\Phi(\xi))^0 : -\left(3k^4\gamma + \omega\right)\alpha_0 + c_1\alpha_0^3 + c_2\alpha_0^5 + \gamma\lambda\mu\left(6k^2 + \lambda^2 + 8\mu\right)\alpha_1 = 0.
\end{aligned} \tag{5.4}$$

Solving the above system by using Mathematica, we find,

$$\begin{aligned}
&\alpha_0 = \pm\lambda\left(-\frac{3\gamma}{2c_2}\right)^{1/4}, \ \alpha_1 = \pm 2\left(-\frac{3\gamma}{2c_2}\right)^{1/4},\\
&c_1 = \left(6k^2 - 5\lambda^2 + 20\mu\right)c_2\sqrt{-\frac{\gamma}{6c_2}},\\
&\omega = \gamma\left(-3k^4 - 3k^2\left(\lambda^2 - 4\mu\right) + \left(\lambda^2 - 4\mu\right)^2\right),
\end{aligned} \tag{5.5}$$

provided that $(\gamma\ c_2) < 0$. Thus, the exact solutions to the conformable space-time fractional CQ-NLSE with parabolic law non-linearity can be found as follows:

Case 1. For $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$, we have the hyperbolic function solution:

$$u_1(x,t) = \pm\left(-\frac{3\gamma}{2c_2}\right)^{1/4}\left[\lambda - \frac{4\mu}{\lambda + \sqrt{\lambda^2 - 4\mu}\,\tanh\left[\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\ (c+\xi)\right]}\right] e^{i\left(-k\frac{x^\alpha}{\alpha} + \omega\frac{t^\alpha}{\alpha}\right)}. \tag{5.6}$$

This is the dark soliton solution. Corresponding $3D$ and $2D$ graphics of $u_1(x,t)$ are shown in Figs. 1 (a) and 1 (b).

Case 2. For $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$, we have the trigonometric function solution:

$$u_2(x,t) = \pm\left(-\frac{3\gamma}{2c_2}\right)^{1/4}\left[\lambda - \frac{4\mu}{\lambda - \sqrt{-\lambda^2 + 4\mu}\,\tan\left[\frac{1}{2}\sqrt{-\lambda^2 + 4\mu}\ (c+\xi)\right]}\right] e^{i\left(-k\frac{x^\alpha}{\alpha} + \omega\frac{t^\alpha}{\alpha}\right)}. \tag{5.7}$$

This is the periodic singular soliton solution. Corresponding $3D$ and $2D$ graphics of $u_2(x,t)$ are shown in Figs. 2 (a) and 2 (b).

Case 3. For $\lambda^2 - 4\mu > 0$ and $\lambda \neq 0,\ \mu = 0$,

$$u_3(x,t) = \pm \left(-\frac{3\gamma}{2c_2}\right)^{1/4} \lambda \coth\left[\frac{1}{2}\lambda(c+\xi)\right] e^{i\left(-k\frac{x^\alpha}{\alpha}+\omega\frac{t^\alpha}{\alpha}\right)}, \tag{5.8}$$

which is the singular soliton solution as shown in Figure 3.

Case 4. For $\lambda^2 - 4\mu = 0$ and $\lambda \neq 0,\ \mu \neq 0$,

$$u_4(x,t) = \pm \left(-\frac{3\gamma}{2c_2}\right)^{1/4} \left[\frac{2\lambda}{2+\lambda(c+\xi)}\right] e^{i\left(-k\frac{x^\alpha}{\alpha}+\omega\frac{t^\alpha}{\alpha}\right)}, \tag{5.9}$$

which is the rational function solution.

Case 5. For $\lambda^2 - 4\mu = 0$ and $\lambda = 0,\ \mu = 0$,

$$u_5(x,t) = \pm \left(-\frac{3\gamma}{2c_2}\right)^{1/4} \left[\frac{2}{c+\xi}\right] e^{i\left(-k\frac{x^\alpha}{\alpha}+\omega\frac{t^\alpha}{\alpha}\right)}, \tag{5.10}$$

where $\xi = \frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha}$.

Note that the obtained solutions are in agreement with the solution obtained in [24, 54] for specific choice of parameters.

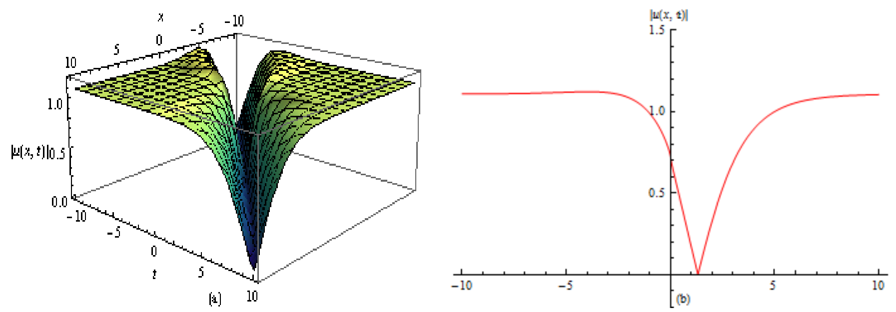

FIGURE 1. (a) Three dimensional plot for dark soliton solution of (5.6) where $\lambda = 3, \mu = 2, \gamma = 1, c = -1, c_2 = -1, \alpha = 0.8, \nu = 1$ (b) Two dimensional line plot of (5.6) for $t = 1$.

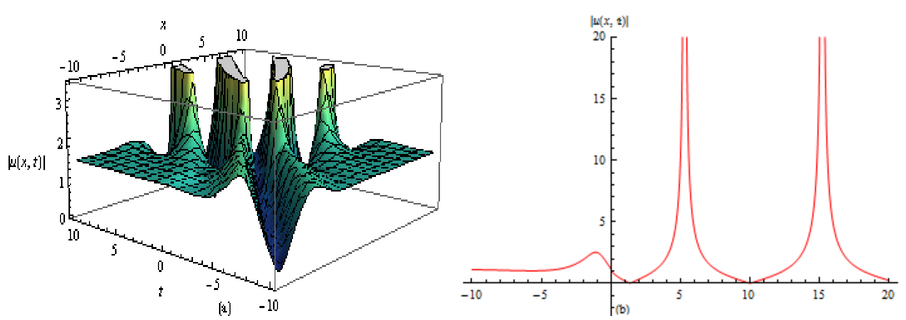

FIGURE 2. (a) Three dimensional plot for periodic singular soliton solution of (5.7) where $\lambda = 3, \mu = 2.7, \gamma = 1, c = -1, c_2 = -1, \alpha = 0.8, \nu = 1$ (b) two dimensional line plot of (5.7) for $t = 1$.

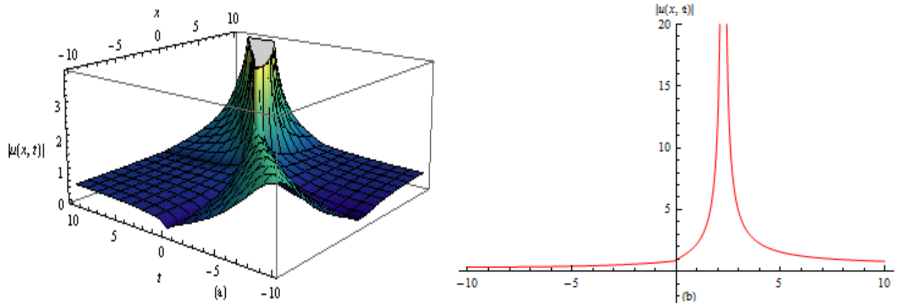

FIGURE 3. (a) Three dimensional plot for singular soliton solution of (5.8) where $\lambda = 5, \mu = 0, \gamma = 1, c = -1, c_2 = -1, \alpha = 0.5, \nu = 1$ (b) Two dimensional line plot of (5.8) with $t = 1$.

5.2. **Quadratic-Cubic law non-linearity.** Consider the functional,

$$F(\phi) = c_1\sqrt{\phi} + c_2\ \phi, \tag{5.11}$$

where c_1 and c_2 are constants. For quadratic-cubic law non-linearity, the Eq. (4.6) is given by:

$$\left(-3k^4\gamma - \omega\right)\phi + c_1\phi^2 + c_2\phi^3 + 6k^2\gamma\phi'' + \gamma\phi^{(4)}. \tag{5.12}$$

Considering the balance of $\phi^{(4)}$ and ϕ^3 in Eq. (5.12), we get $N = 2$. The solution of Eq. (5.12) takes the form:

$$\phi = \alpha_0 + \alpha_1\exp(-\Phi(\xi)) + \alpha_2\exp(-\Phi(\xi))^2, \tag{5.13}$$

where α_0, α_1 and $\alpha_2(\neq 0)$ are constants.

Substituting Eq. (5.13) together with Eq. (3.5) into Eq. (5.12), considering the coefficients of $\exp(-\Phi(\xi))$ and setting them equal to zero, we find the equations as follows:

$$\begin{aligned}
&\exp(-\Phi(\xi))^6 : 120\gamma\alpha_2 + c_2\alpha_2^3 = 0,\\
&\exp(-\Phi(\xi))^5 : 112\gamma\lambda\alpha_2 + \alpha_1\left(8\gamma + c_2\alpha_2^2\right) = 0,\\
&\exp(-\Phi(\xi))^4 : 60\gamma\lambda\alpha_1 + 3c_2\alpha_1^2\alpha_2 + \alpha_2\left(36k^2\gamma + 330\gamma\lambda^2 + 240\gamma\mu + c_1\alpha_2\right) + 3c_2\alpha_0\alpha_2^2 = 0,\\
&\exp(-\Phi(\xi))^3 : 2\gamma\left(6k^2 + 25\lambda^2 + 20\mu\right)\alpha_1 + c_2\alpha_1^3 + 2\left(5\gamma\lambda\left(6k^2 + 13\lambda^2 + 44\mu\right) + \left(c_1 + 3c_2\alpha_0\right)\alpha_1\right)\alpha_2 = 0,\\
&\exp(-\Phi(\xi))^2 : \alpha_1\left(3\gamma\lambda\left(6k^2 + 5\left(\lambda^2 + 4\mu\right)\right) + \left(c_1 + 3c_2\alpha_0\right)\alpha_1\right)\\
&\qquad + \gamma\left(-3k^4 + 24k^2\left(\lambda^2 + 2\mu\right) + 8\left(2\lambda^4 + 29\lambda^2\mu + 17\mu^2\right)\right)\alpha_2\\
&\qquad + \left(-\omega + 2c_1\alpha_0 + 3c_2\alpha_0^2\right)\alpha_2 = 0,\\
&\exp(-\Phi(\xi))^1 : \gamma\left(-3k^4 + \lambda^4 + 22\lambda^2\mu + 16\mu^2 + 6k^2\left(\lambda^2 + 2\mu\right)\right)\alpha_1\\
&\qquad \left(-\omega + 2c_1\alpha_0 + 3c_2\alpha_0^2\right)\alpha_1 + 6\gamma\lambda\mu\left(6k^2 + 5\left(\lambda^2 + 4\mu\right)\right)\alpha_2 = 0,\\
&\exp(-\Phi(\xi))^0 : \alpha_0\left(-3k^4\gamma - \omega + \alpha_0\left(c_1 + c_2\alpha_0\right)\right) + \gamma\lambda\mu\left(6k^2 + \lambda^2 + 8\mu\right)\alpha_1 + 2\gamma\mu^2\left(6k^2 + 7\lambda^2 + 8\mu\right)\alpha_2 = 0.
\end{aligned} \tag{5.14}$$

Solving the above system by using Mathematica, we find,

$$\begin{aligned}
&\alpha_0 = \pm 2\mu\sqrt{-\frac{30\gamma}{c_2}},\ \ \alpha_1 = \pm 2\lambda\sqrt{-\frac{30\gamma}{c_2}},\\
&\alpha_2 = \pm 2\sqrt{-\frac{30\gamma}{c_2}},\ \ c_1 = \pm\left(6k^2 + 5\left(\lambda^2 - 4\mu\right)\right)c_2\sqrt{-\frac{3\gamma}{10c_2}},\\
&\omega = \gamma\left(-3k^4 + 6k^2\left(\lambda^2 - 4\mu\right) + \left(\lambda^2 - 4\mu\right)^2\right),
\end{aligned} \tag{5.15}$$

provided that $(\gamma\ c_2) < 0$. Thus, the exact solutions to the conformable space-time fractional CQ-NLSE with quadratic-cubic law non-linearity can be constructed as follows:

Case 1. For $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$, we have the hyperbolic function solution:

$$u_1(x,t) = \pm\, 2\mu\sqrt{-\frac{30\gamma}{c_2}}\left[1 - \frac{2\left(\lambda^2 - 2\mu + \lambda\sqrt{\lambda^2-4\mu}\ \tanh\left[\frac{1}{2}\sqrt{\lambda^2-4\mu}(c+\xi)\right]\right)}{\left(\lambda + \sqrt{\lambda^2-4\mu}\ \tanh\left[\frac{1}{2}\sqrt{\lambda^2-4\mu}(c+\xi)\right]\right)^2}\right] e^{i\left(-k\frac{x^\alpha}{\alpha}+\omega\frac{t^\alpha}{\alpha}\right)}. \tag{5.16}$$

This is the dark soliton solution. Corresponding $3D$ and $2D$ graphics of $u_1(x,t)$ are shown in Figs. 4 (a) and 4 (b).

Case 2. For $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$, we have the trigonometric function solution:

$$u_2(x,t) = \pm\, 2\ \mu\sqrt{-\frac{30\gamma}{c_2}}\left[1 + \frac{2\left(-\lambda^2 + 2\mu + \lambda\sqrt{-\lambda^2+4\mu}\ \tan\left[\frac{1}{2}\sqrt{-\lambda^2+4\mu}(c+\xi)\right]\right)}{\left(\lambda - \sqrt{-\lambda^2+4\mu}\ \tan\left[\frac{1}{2}\sqrt{-\lambda^2+4\mu}(c+\xi)\right]\right)^2}\right] e^{i\left(-k\frac{x^\alpha}{\alpha}+\omega\frac{t^\alpha}{\alpha}\right)}. \tag{5.17}$$

This is the periodic singular soliton solution. Corresponding $3D$ and $2D$ graphics of $u_2(x,t)$ are shown in Figs. 5 (a) and 5 (b).

Case 3. For $\lambda^2 - 4\mu > 0$ and $\lambda \neq 0,\ \mu = 0$,

$$u_3(x,t) = \pm 2\sqrt{-\frac{30\gamma}{c_2}}\left[\mu + \frac{1}{4}\lambda^2\ \mathrm{csch}^2\left[\frac{1}{2}\lambda(c+\xi)\right]\right] e^{i\left(-k\frac{x^\alpha}{\alpha}+\omega\frac{t^\alpha}{\alpha}\right)}, \tag{5.18}$$

which is the singular soliton solution. Corresponding $3D$ and $2D$ graphics of $u_3(x,t)$ are shown in Figs. 6 (a) and 6 (b).

Case 4. For $\lambda^2 - 4\mu = 0$ and $\lambda \neq 0,\ \mu \neq 0$,

$$u_4(x,t) = \pm 2\sqrt{-\frac{30\gamma}{c_2}}\left[\mu - \frac{\lambda^2}{4} + \frac{\lambda^2}{(2+\lambda(c+\xi))^2}\right] e^{i\left(-k\frac{x^\alpha}{\alpha}+\omega\frac{t^\alpha}{\alpha}\right)}, \tag{5.19}$$

which is the rational function solution.

Case 5. For $\lambda^2 - 4\mu = 0$ and $\lambda = 0,\ \mu = 0$,

$$u_5(x,t) = \pm 2\sqrt{-\frac{30\gamma}{c_2}}\left[\frac{1}{(c+\xi)^2}\right] e^{i\left(-k\frac{x^\alpha}{\alpha}+\omega\frac{t^\alpha}{\alpha}\right)}, \tag{5.20}$$

where $\xi = \frac{x^\alpha}{\alpha} - \nu\frac{t^\alpha}{\alpha}$.

Note that the obtained solutions are in agreement with the solution obtained in ref. [54] for specific choice of parameters.

5.3. **Weak non-local law non-linearity.** Consider the non-linearity,

$$F(\phi) = c_1\phi + c_2\phi^2 + c_3\phi_{xx}. \tag{5.21}$$

where c_1, c_2 and c_3 are constants. Corresponding to the weak non-local law non-linearity, the Eq. (4.6) is given by:

$$c_1\phi^3 + c_2\phi^5 + \phi\left(-k^3\beta + k^4\gamma - \omega + 2c_3\phi'^2\right) + 3k(\beta - 2k\gamma)\phi'' + 2c_3\phi^2\phi'' + \gamma\phi^{(4)}. \tag{5.22}$$

Taking the balance of $\phi^{(4)}$ and ϕ^5 in Eq. (5.22), we get $N = 1$. The solution of Eq. (5.22) takes formal form:

$$\phi = \alpha_0 + \alpha_1\exp(-\Phi(\xi)), \tag{5.23}$$

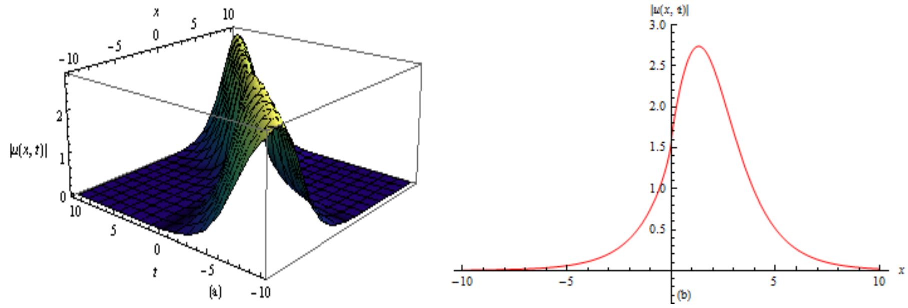

FIGURE 4. (a) Three dimensional plot for dark soliton solution of (5.16) where $\lambda = 3, \mu = 2, \gamma = 1, c = -1, c_2 = -1, \alpha = 0.8, \nu = 1$ (b) Two dimensional line plot of (5.16) with $t = 1$.

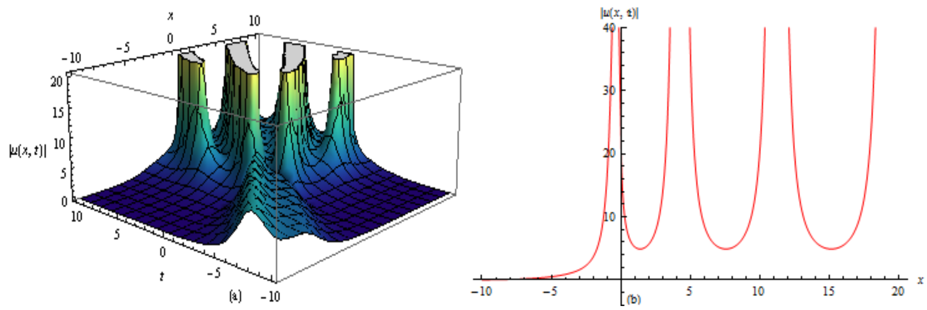

FIGURE 5. (a) Three dimensional plot for periodic singular soliton solution of (5.17) where $\lambda = 3, \mu = 2.7, \gamma = 1, c = -1, c_2 = -1, \alpha = 0.8, \nu = 1$ (b) Two dimensional line plot of (5.17) with $t = 1$.

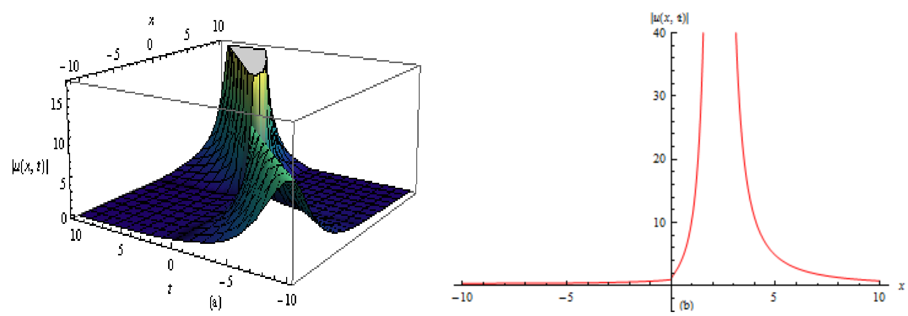

FIGURE 6. (a) Three dimensional plot for singular soliton solution of (5.18) where $\lambda = 0.5, \mu = 2, \gamma = 1, c = -1, c_2 = -1, \alpha = 0.5, \nu = 1$ (b) Two dimensional line plot of (5.18) with $t = 1$.

where α_0, and α_1 are constants.

Adding Eq. (5.23) together with Eq. (3.5) into Eq. (5.22), taking the coefficients of $\exp(-\Phi(\xi))$ and setting them equal to zero, we find the equations as follows:

$$
\begin{aligned}
&\exp(-\Phi(\xi))^5 : 24\gamma\alpha_1 + 6c_3\alpha_1^3 + c_2\alpha_1^5 = 0,\\
&\exp(-\Phi(\xi))^4 : 5\alpha_1\left(2\lambda\left(6\gamma + c_3\alpha_1^2\right) + \alpha_0\left(2c_3\alpha_1 + c_2\alpha_1^3\right)\right) = 0,\\
&\exp(-\Phi(\xi))^3 : 2\gamma\left(6k^2 + 25\lambda^2 + 20\mu\right) + 4c_3\alpha_0^2 + 16c_3\lambda\alpha_0\alpha_1 + \left(4c_3\left(\lambda^2 + 2\mu\right) + c_1 + 10c_2\alpha_0^2\right)\alpha_1^2 = 0,\\
&\exp(-\Phi(\xi))^2 : 3\gamma\lambda\left(6k^2 + 5\left(\lambda^2 + 4\mu\right)\right) + 6c_3\lambda\alpha_0^2 + \alpha_0\left(6c_3\left(\lambda^2 + 2\mu\right) + 3c_1 + 10c_2\alpha_0^2\right)\alpha_1 + 6c_3\lambda\mu\alpha_1^2 = 0,\\
&\exp(-\Phi(\xi))^1 : \gamma\left(-3k^4 + \lambda^4 + 22\lambda^2\mu + 16\mu^2 + 6k^2\left(\lambda^2 + 2\mu\right)\right) - \omega + 5c_2\alpha_0^4\\
&\qquad\qquad + \left(2c_3\left(\lambda^2 + 2\mu\right) + 3c_1\right)\alpha_0^2 + 8c_3\lambda\mu\alpha_0\alpha_1 + 2c_3\mu^2\alpha_1^2 = 0,\\
&\exp(-\Phi(\xi))^0 : c_1\alpha_0^3 + c_2\alpha_0^5 + \gamma\lambda\mu\left(6k^2 + \lambda^2 + 8\mu\right)\alpha_1 + 2c_3\lambda\mu\alpha_0^2\alpha_1 - \alpha_0\left(3k^4\gamma + \omega - 2c_3\mu^2\alpha_1^2\right) = 0.
\end{aligned}
\tag{5.24}
$$

Solving the above system by using Mathematica, we find,

$$
\begin{aligned}
&\alpha_0 = \pm\frac{1}{2}\lambda\sqrt{\frac{-3c_3 + \sqrt{9c_3^2 - 24\gamma c_2}}{c_2}},\ \ \alpha_1 = \pm\sqrt{\frac{-3c_3 + \sqrt{9c_3^2 - 24\gamma c_2}}{c_2}},\\
&c_1 = \frac{1}{12}\left[9c_3\left(2k^2 + \lambda^2 - 4\mu\right) + \left(6k^2 - 5\lambda^2 + 20\mu\right)\sqrt{9c_3^2 - 24\gamma c_2}\right],\\
&\omega = \gamma\left(-3k^4 - 3k^2\left(\lambda^2 - 4\mu\right) + \left(\lambda^2 - 4\mu\right)^2\right) + \frac{c_3}{8c_2}\left(\lambda^2 - 4\mu\right)^2\left(-3c_3 + \sqrt{9c^2 - 24\gamma c_2}\right),
\end{aligned}
\tag{5.25}
$$

provided that $\gamma < 0$ and $c_2 > 0$. Thus, the exact solutions to the conformable space-time fractional CQ-NLSE for the weak non-local law non-linearity can be constructed as follows:

Case 1. For $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$, we have the hyperbolic function solution:

$$
u_1(x,t) = \gamma\left(-3k^4 - 3k^2\left(\lambda^2 - 4\mu\right) + \left(\lambda^2 - 4\mu\right)^2\right). \tag{5.26}
$$

This is the dark soliton solution. Corresponding $3D$ and $2D$ graphics of $u_1(x,t)$ are shown in Figures 7 (a) and 7 (b).

Case 2. For $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$, we have the trigonometric function solution:

$$
u_2(x,t) = \pm\frac{1}{2}\sqrt{\frac{-3c_3 + \sqrt{9c_3^2 - 24\gamma c_2}}{c_2}}\left(\lambda - \frac{4\mu}{\lambda - \sqrt{-\lambda^2 + 4\mu}\,\tan\left[\frac{1}{2}\sqrt{-\lambda^2 + 4\mu}(c+\xi)\right]}\right)e^{i\left(-k\frac{x^\alpha}{\alpha} + \omega\frac{t^\alpha}{\alpha}\right)}. \tag{5.27}
$$

This is the periodic singular soliton solution. Corresponding $3D$ and $2D$ graphics of $u_2(x,t)$ are shown in Figures. 8 (a) and 8 (b).

Case 3. For $\lambda^2 - 4\mu > 0$ and $\lambda \neq 0,\ \mu = 0$,

$$
u_3(x,t) = \pm\frac{1}{2}\lambda\sqrt{\frac{-3c_3 + \sqrt{9c_3^2 - 24\gamma c_2}}{c_2}}\coth\left[\frac{1}{2}\lambda(c+\xi)\right]e^{i\left(-k\frac{x^\alpha}{\alpha} + \omega\frac{t^\alpha}{\alpha}\right)}, \tag{5.28}
$$

which is the singular soliton solution. Corresponding $3D$ and $2D$ graphics of $u_3(x,t)$ are shown in Figures 9 (a) and 9 (b).

Case 4. For $\lambda^2 - 4\mu = 0$ and $\lambda \neq 0,\ \mu \neq 0$,

$$
u_4(x,t) = \pm\sqrt{\frac{-3c_3 + \sqrt{9c_3^2 - 24\gamma c_2}}{c_2}}\ \frac{\lambda}{2 + \lambda(c+\xi)}\ e^{i\left(-k\frac{x^\alpha}{\alpha} + \omega\frac{t^\alpha}{\alpha}\right)}, \tag{5.29}
$$

which is the rational function solution.

Case 5. For $\lambda^2 - 4\mu = 0$ and $\lambda = 0,\ \mu = 0,$

$$u_5(x,t) = \pm\sqrt{\frac{-3c_3 + \sqrt{9c_3^2 - 24\gamma c_2}}{c_2}}\ \frac{1}{(c+\xi)}\ e^{i\left(-k\frac{x^\alpha}{\alpha} + \omega\frac{t^\alpha}{\alpha}\right)}. \tag{5.30}$$

Where $\xi = \dfrac{x^\alpha}{\alpha} - \nu\dfrac{t^\alpha}{\alpha}$.

Note that the obtained solutions are consists of the solution obtained in [54] for specific choice of parameters. Moreover, on comparing our extracted solutions with the other existing solutions in the literature, it is found that our results are new and not found in the literature except for the special choice of parameters as mentioned above.

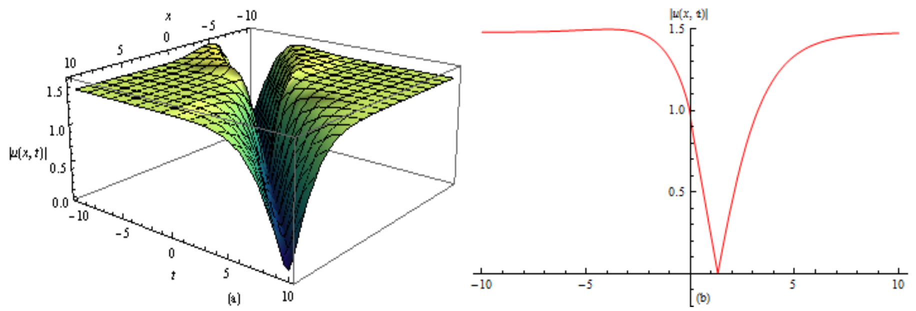

FIGURE 7. (a) Three dimensional plot for dark soliton solution of (5.26) where $\lambda = 3, \mu = 2, \gamma = -1, c = -1, c_2 = 1, c_3 = 1, \alpha = 0.8, \nu = 1$ (b) Two dimensional line plot of (5.26) with $t = 1$.

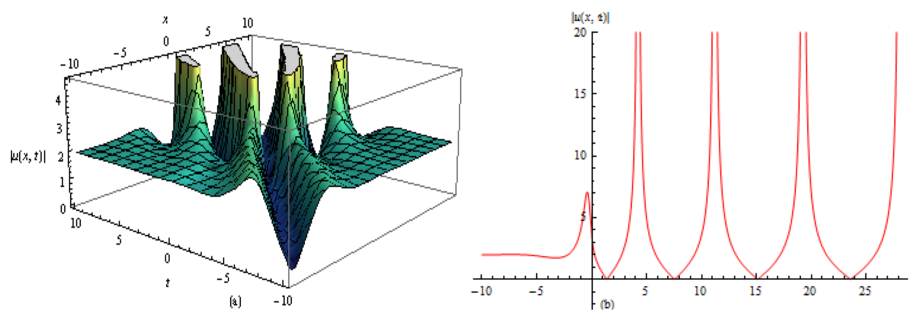

FIGURE 8. (a) Three dimensional plot for periodic singular soliton solution of (5.27) where $\lambda = 3, \mu = 2.7, \gamma = -1, c = -1, c_2 = 1, c_3 = 1, \alpha = 0.8, \nu = 1$ (b) Two dimensional line plot of (5.27) with $t = 1$.

6. Graphical representation

In this section, we present the graphs of some solutions for Eq. (1.2). Let us now examine Figures 1-9, as it illustrates the dynamical structures of obtained solutions through 3D and 2D plots. To this aim, we select some special values of the obtained parameters. From the above figures, one can see that the obtained solutions possess the dark soliton solution, the singular soliton solution and the periodic wave solution of Eq. (1.2). Also, the exact solutions and figures obtained in this paper give us a different physical interpretation for the space-time fractional cubic-quartic nonlinear Schrödinger equation.

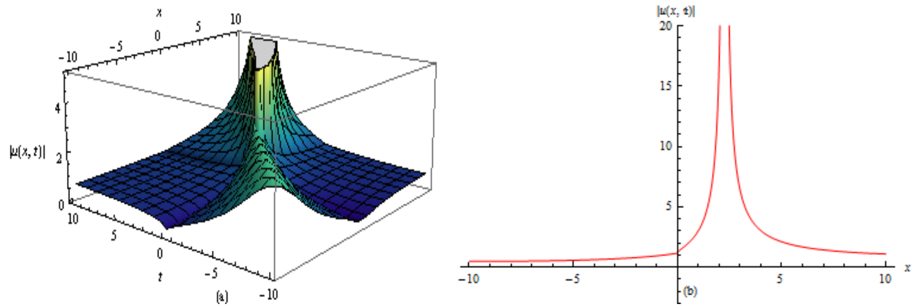

FIGURE 9. (a) Three dimensional plot for singular soliton solution of (5.28) where $\lambda = 0.5, \mu = 0, \gamma = -1, c = -1, c_2 = 1, c_3 = 1, \alpha = 0.5, \nu = 1$ (b) Two dimensional line plot of (5.28) with $t = 1$.

7. Conclusions

In this paper, we have obtained the optical soliton solutions to the space-time fractional cubic-quartic nonlinear Schrödinger equation with different laws of nonlinearity. By utilizing conformable fractional derivative and wave transformation, the fractional cubic-quartic nonlinear Schrödinger equation is converted to an ODE. The resulting ODE is solved by employing the $\exp(-\Phi(x))$ expansion method. Based on the method, explicit solutions such as dark soliton, singular soliton and periodic wave solutions are obtained. By comparing our solutions to the results obtained in literature, our obtained solutions are new and different. Moreover, the obtained solutions are plotted graphically to check the dynamical behaviour of the solutions. It can be observed that the proposed method is an efficient and reliable technique for finding the soliton solutions of the governing equation that plays a significant role in the field of nonlinear optics.

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