Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 10, No. 3, 2022, pp. 774-788
DOI: 10.22034/CMDE.2021.46473.1953

A Study on homotopy analysis method and clique polynomial method

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Abstract

This paper generated the novel approach called the Clique polynomial method (CPM) using the clique polynomials raised in graph theory. Nonlinear initial value problems are converted into nonlinear algebraic equations by discretion with suitable grid points in the current approach. We solved highly nonlinear initial value problems using the Homotopy analysis method (HAM) and Clique polynomial method (CPM). Obtained results reveal that the present technique is better than HAM that is discussed through tables and simulations. Convergence analysis is reflected in terms of theorems.

Keywords. Nonlinear differential equation, Clique polynomial, Complete graph, Homotopy analysis method.
2010 Mathematics Subject Classification. 34A12, 41A10, 65L60.

1. Introduction

Many real-life situations are modelled into non-linear ordinary differential equations (ODEs) in scientific engineering and their applications. Characterization of these equations is done by their exact solutions, but we cannot anticipate the precise solution for all modelled equations with nonlinear terms. Also, we won't find the general scheme that works on the above said equations and regularly each type of equation has to be considered a separate problem. Then we take a look at fascinating semi-analytical and numerical methods. Here we are studying the nonlinear equations through HAM and CPM (newly generated method). Numerical techniques can learn these equations. Also, we can validate the proposed numerical techniques through the analytic solution of non-linear ODEs. Consequently, many mathematicians are working on numerical techniques to solve such non-linear ODEs. One such equation is of the form:

$$\frac{d^n\theta(t)}{dt^n} = f\Big(t,\, \theta,\, \frac{d\theta(t)}{dt}, \cdots, \frac{d^{n-1}\theta(t)}{dt^{n-1}}\Big) \tag{1.1}$$

with the following physical conditions,

$$\frac{d^{k-1}\theta(a)}{dt^{k-1}} = b_k, \quad k = 1,\, 2,\, \cdots, n. \tag{1.2}$$

Where, $f\Big(t,\, \theta,\, \frac{d\theta(t)}{dt}, \cdots, \frac{d^{n-1}\theta(t)}{dt^{n-1}}\Big)$ is the non-linear term. a and b_k are any real constants. Many mathematicians contributed numerous techniques as follows: HAM on different problems [1-5,7,9,12-17], Hermite wavelets method [19], wavelet series collocation method [18], a new approach for KG-equation [10,11]. This study generated a new CPM technique through clique polynomials of the complete graph and solved (1.1) by HAM and CPM. HAM is a semi-analytical method, it is the most acceptable method to solve linear and nonlinear ODEs. We have solved some of the linear and nonlinear ordinary differential equations using HAM in the present study. The obtained results are compared with the CPM. The proposed technique is simple, no modifications are required in the given problems, and

Received: 10 June 2021 ; Accepted: 16 October 2021.
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numerical implementation is straightforward. Our literature survey has not found any research article on nonlinear ODEs through a complete graph of clique polynomials. This impetus us to propose the clique polynomials method for the higher-order ODEs and the current technique's proficiency through tables and graph simulation.

2. Preliminaries of Clique polynomials and some results

Let G be a graph that is free from multi edges and loops. Clique polynomials and related work of graph is introduced by Hoede et al. [6,8]. Clique polynomial of a graph G, denoted by $h(G;t)$, is characterized by,

$$h(G;t) = \sum_{k=0}^{n} a_k \, t^k,$$

where a_k represents the total distinct k-cliques in graph of size k, with $a_0 = 1$. For example, Here we considered a complete graph with four vertices as follows:

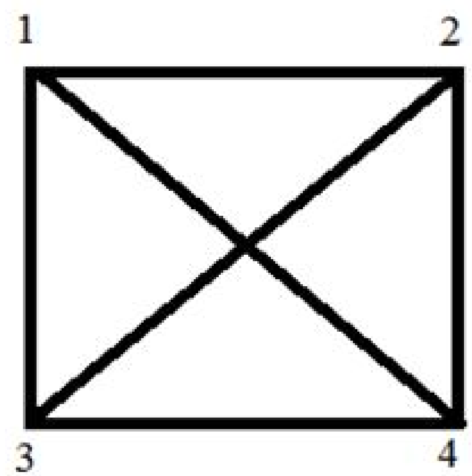

Figure 1. Complete Graph with 4 vertices (K_4)

By the definition of clique polynomial of G concerning Figure.1 we get,

$$h(K_4;t) = a_0 + a_1 \, t + a_2 \, t^2 + a_3 \, t^3 + a_4 \, t^4.$$

a_1 indicates the total number of distinct 1-cliques in K_4 of size 1, therefore, $a_1 = 4$. $a_2 = 6$ indicates the total number of distinct 2-cliques in K_4. $a_3 = 4$ indicates the total number of distinct 3-cliques in K_4. $a_4 = 1$ indicates the total number of distinct 4-cliques in K_4. Hence the required clique polynomial for K_4 is,

$$h(K_4;\, t) = (1 + t)^4.$$

In general, the clique polynomial of a complete graph K_n with n vertices is given by,

$$h(K_n;\, t) = (1 + t)^n.$$

Theorem 2.1. *Let $h(K_n;\, t)$ be the clique polynomials of the $n-1$ regular Graph. Then $h(K_n;\, t)$ are continuous uniformly on $[0, 1]$.*

Theorem 2.2. *Let clique polynomials $h(K_n;\, t)$ of the $n-1$ regular graph defined on $[0, 1]$ are integrable continuous functions. Then integral of these polynomials is continuous on $[0, 1]$ and bounded variation on $[0, 1]$.*

Let $C = \{h_n(t) = h(K_n;\, t) \mid h(K_n;\, t)$ be the clique polynomials for $K_n, \quad n \in N$ and $h(K_0;\, t) = 1\}$. C is Banach space on closed subset A of $\mathbb{R}$ with norm given by,

$$\|h_n\| = \underset{\forall\, t \in A}{Sup}\, |h_n(t)|,\ \forall h_n \in C(A).$$

Hence the Banach space is mathematically denoted as $C(A)$.

Remark 2.3. Clique polynomials are continuous functions in $[a, b]$. Therefore, clique polynomials are members of $C([a, b])$.

Theorem 2.4. *Let h_n be the sequence of continuous functions in $C([a, b])$ and converge to the function h in $C([a, b])$ uniformly on t in $[a, b]$. Then h is a continuous function in $C([a, b])$.*

Proof. By data $h_n(t)$ uniformly converges to $h(t)$. Take $\epsilon > 0$ be an arbitrary real number then,

$$\|h_n(t) - h(t)\| < \frac{\epsilon}{3},\ \forall\, t \in\ [a, b].$$

Since each $h_m(t)$ is continuous in $C([a, b])$ in $t \in\ [a, b]$, then there exists $\delta > 0$, such that $\|h_n(t_0) - h_n(t)\| < \frac{\epsilon}{3}$, whenever $\|t_0 - t\| < \delta$,$\forall t_0\,, t \in [a, b]$. By Minkowski Inequality, we have

$$\begin{aligned}
\|h(t_0) - h(t)\| &= \|h(t_0) - h_m(t_0) + h_m(t_0) + h_m(t) - h_m(t) - h(t)\| \\
&\leq \|h(t_0) - h_m(t_0)\| + \|h_m(t) - h(t)\| + \|h_m(t_0) - h_m(t)\| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \text{ where } \|t_0 - t\| < \delta, \text{ with } t_0\,, t \in [a, b].
\end{aligned}$$

Hence h is continuous in $C([a, b])$. □

Theorem 2.5. *Let the sequence of functions h_n, $\forall\, n \in \mathbb{N}$ converges in $C([a, b])$ uniformly in $t \in [a, b]$. Then there is a function h continuous in $C([a, b])$.*

Proof. Riesz-Fischer theorem says that "If a sequence of functions f_k, $k \in \mathbb{N}$ in $L_2(\mathbb{R})$ converges in itself in $L_2(\mathbb{R})$, then there is a function $f \in L_2(\mathbb{R})$ such that

$$\|f_k - f\| \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty \tag{2.1}$$

the function f is unique within the set whose measure is zero". Next, it is possible to choose a subsequence $\{h_{n_i}(t)\}$, $i = 1, 2, 3, \cdots$ such that,

$$\left\|h_{n_{i+1}}(t) - h_{n_i}(t)\right\| < \frac{1}{2^i}, \forall\, t \in [a, b]. \tag{2.2}$$

But from (2.1),

$$h(t) = \lim_{i \longrightarrow \infty} h_{n_i} = h_{n_i} + (h_{n_{i+1}} - h_{n_i}) + (h_{n_{i+2}} - h_{n_{i+1}}) + \cdots \tag{2.3}$$

From (2.2) and (2.3),consider

$$\begin{aligned}
\|h - h_{n_i}\| &\leq \left\|h_{n_{i+1}} - h_{n_i}\right\| + \left\|h_{n_{i+2}} - h_{n_{i+1}}\right\| + \cdots \\
&< \frac{1}{2^i} + \frac{1}{2^{i+1}} + \cdots = \frac{1}{2^{i-1}}, \quad i = 1, 2, 3, \cdots .
\end{aligned}$$

This implies that subsequence $\{h_{n_i}\}$ converges to h in $C([a, b])$ uniformly on t in $[a, b]$. By theorem (2.4), the function h is continuous in $C([a, b])$ in t on $[a, b]$. Hence proof is completed □

3. Methods

3.1. **Clique Polynomial Method.** Consider the nonlinear higher-order ordinary differential equation is of the form:

$$\frac{d^n\theta(t)}{dt^n} = f\Big(t,\, \theta,\, \frac{d\theta(t)}{dt}, \cdots, \frac{d^{n-1}\theta(t)}{dt^{n-1}}\Big) \tag{3.1}$$

with the following conditions,

$$\frac{d^{k-1}\theta(a)}{dt^{k-1}} = b_k, \quad k = 1,\, 2,\, \cdots, n. \tag{3.2}$$

Here, a and b_k are any arbitrary constants. Assume higher-order derivative by Clique polynomials is as follows:

$$\frac{d^n\theta(t)}{dt^n} = \sum_{i=1}^{k} a_i\, h(K_i;\, t), \tag{3.3}$$

integrate (3.3) with variable t and limit from a to t.

$$\frac{d^{n-1}\theta(t)}{dt^{n-1}} = \frac{d^{n-1}\theta(a)}{dt^{n-1}} + \int_a^t \sum_{i=1}^{k} a_i\, h(K_i;\, t)\, dt \tag{3.4}$$

$$\frac{d^{n-1}\theta(t)}{dt^{n-1}} = b_n + \int_a^t \sum_{i=1}^{k} a_i\, h(K_i;\, t)\, dt. \tag{3.5}$$

Continuing this procedure up to $(n-1)$ times we get,

$$\begin{cases} \frac{d^{n-2}\theta(t)}{dt^{n-2}} & = b_{n-1} + t\, b_n + \int_a^t \int_a^t \sum\limits_{i=1}^{k} a_i\, h(K_i;\, t)\, dt\, dt \\ & \vdots \\ \theta(t) & = b_1 + t\, b_2 + t^2\, b_3 + \cdots + \frac{t^{n-1}}{(n-1)!} b_n + \int_a^t \cdots \int_a^t \sum\limits_{i=1}^{k} a_i\, h(K_i;\, t)\, dt \cdots dt \end{cases} \tag{3.6}$$

Fit (3.6), (3.5), (3.3) in (3.1)and discrete the obtained equation by given grid points,

$$t_i = \frac{2i-1}{2n},\, i = 1,\, 2,\, \cdots, n. \tag{3.7}$$

This procedure yields a system containing n-algebraic equations. Solving this system with a suitable solver provides Clique polynomials with unknown coefficient values. Fit these values in $\theta(t)$, that gives clique polynomial numerical solutions.

3.2. **Homotopy Analysis Method.** HAM is a semi-analytical technique to solve differential equations (ODE's, PDE's and differential equations having fractional orders). Unlike Perturbation methods that depend on small or large parameters, HAM is independent of these parameters. We have great liberty to opt for the auxiliary function, linear operator, and control convergence parameter. Usually, this method solves the differential equations having high nonlinearity (without large or small parameters). Explanations of HAM and convergence can be seen in the following references [15-17].

4. Applications

Example 4.1. Consider the first-order non-linear problem;

$$\theta\theta' - t = 0,\, \theta(0) = 1. \tag{4.1}$$

The exact solution is $\theta = \sqrt{t^2+1}$. HAM and CPM are applied to equation (4.1), the obtained results are expressed in tables and graphs. Here, the absolute error (AE) of CPM is better than the absolute error of HAM. Figure 3 and Table 1 represent the geometrical and numerical analysis of example 4.1, respectively. Figure 4 reflects the error analysis through graphs.

To solve the equation (4.1) by HAM, assume that,

$$\theta_0(t) = 1,\ \mathcal{L} = \frac{d}{dt} \text{ with the property that } \mathcal{L}[c_1] = 0 \text{ and } H(t) = 1 \tag{4.2}$$

we construct a zero-order deformation equation as,

$$(1-q)\,\mathcal{L}[\Phi(t;q) - \theta_0(t)] = q\,h\,H(t)\,\mathcal{N}[\Phi(t;q)] \tag{4.3}$$

with the following condition,

$$\Phi(0;q) = 1. \tag{4.4}$$

Here, $\Phi(t;q)$ is the solution. Differentiating m times (4.3) and (4.4) concerning the embedding parameter q and then dividing by $m!$ finally set $q = 0$, we get mth-order deformation equation,

$$\mathcal{L}[\theta_m(t) - \chi_m\theta_{m-1}(t)] = h\,H(t)\,\overrightarrow{R_m}(\theta_{m-1}) \tag{4.5}$$

with the following condition,

$$\theta_m(0) = 0. \tag{4.6}$$

Here,

$$\chi_m = \begin{cases} 0 & \text{when } m \le 1, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$\overrightarrow{R_m}(\theta_{m-1}) = \sum_{j=1}^{m-1} \theta_j\,\theta'_{m-1-j} - t(1-\chi_m). \tag{4.7}$$

Using (4.2) and (4.7) in (4.5), we get,

$$\theta_m(t) = \chi_m\theta_{m-1}(t) + h\int_0^t \Big(\sum_{j=1}^{m-1} \theta_j\,\theta'_{m-1-j} - t(1-\chi_m)\Big)dt + c_1,\ \ m \ge 1. \tag{4.8}$$

The integral constant c_1 is determined by condition (4.6). Thus, we successively obtain,

$$\begin{aligned}
\theta_1(t) &= -\frac{ht^2}{2} \\
\theta_2(t) &= -\frac{ht^2}{2} - \frac{h^2t^2}{2} \\
\theta_3(t) &= -\frac{ht^2}{2} - \frac{h^2t^2}{2} + h\Big(\frac{1}{2}(-h-h^2)t^2 + \frac{h^2t^4}{8}\Big) \\
&\ \ \vdots
\end{aligned}$$

The mth-order approximation can be expressed as,

$$\theta(t) \approx \sum_{j=0}^{m} \theta_j(t). \tag{4.9}$$

From Figure 2, we obtained a value of $h = -1$. Thus, the solution series for (4.1) is,

$$\theta(t) = 1 + \frac{t^2}{2} - \frac{t^4}{8} + \frac{t^6}{16} - \frac{5t^8}{128} + \frac{7t^{10}}{256} - \frac{21t^{12}}{1024} + \frac{33t^{14}}{2048} - \frac{429t^{16}}{32768} + \cdots \tag{4.10}$$

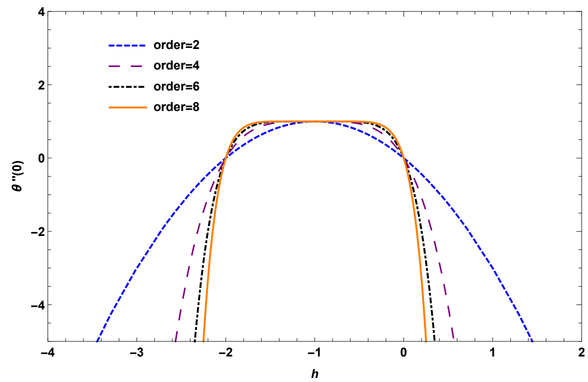

FIGURE 2. Graph of $\theta''(0)$ verses h

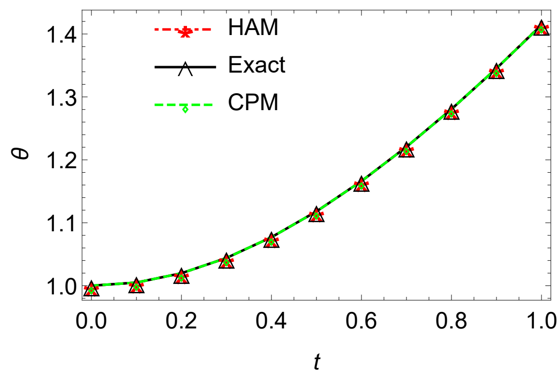

FIGURE 3. Graphical representation of Exact, HAM, and CPM solution, for example 4.1.

TABLE 1. Comparison between HAM and CPM, absolute errors (AE) with Exact Solution, for example 4.1.

x	Exact Solution	AE by HAM (65th approximation)	AE by CPM at n=20
0	1	0	0
0.1	1.004988	0	3.1623×10^{-21}
0.2	1.019804	0	3.5777×10^{-20}
0.3	1.0440307	4.440892×10^{-16}	1.4789×10^{-19}
0.4	1.077033	2.220446×10^{-16}	4.0477×10^{-19}
0.5	1.118034	0	8.8388×10^{-19}
0.6	1.166190	2.220446×10^{-16}	1.6731×10^{-18}
0.7	1.220656	2.886579×10^{-14}	2.8697×10^{-18}
0.8	1.280625	2.294531×10^{-10}	4.5795×10^{-18}
0.9	1.345362	6.271076×10^{-7}	6.9159×10^{-18}
1.0	1.414214	0.0007354	1.0065×10^{-17}

Example 4.2. Consider the first-order problem of the form;

$$\theta' + \theta\theta' - t = 0, \;\; \theta(0) = 1.$$

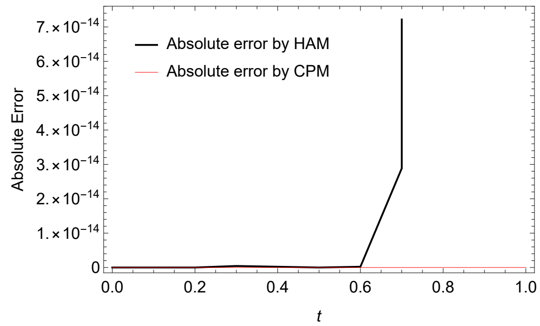

FIGURE 4. Error analysis for the HAM and CPM, for example 4.1.

TABLE 2. Comparison between HAM and CPM absolute errors (AE) with Exact Solution, for example 4.2.

x	Exact Solution	AE by HAM (25^{th} approximation)	AE by CPM at n=20
0	1	0	0
0.1	1.002498	2.220446×10^{-16}	4.2320×10^{-18}
0.2	1.009975	0	4.2000×10^{-18}
0.3	1.022375	0	4.1461×10^{-18}
0.4	1.039608	0	4.0694×10^{-18}
0.5	1.061553	0	3.9686×10^{-17}
0.6	1.088061	0	3.8419×10^{-17}
0.7	1.118962	1.998401×10^{-15}	3.6865×10^{-17}
0.8	1.154066	6.949996×10^{-14}	3.4986×10^{-16}
0.9	1.193171	1.825651×10^{-12}	3.2726×10^{-16}
1.0	1.236068	3.367262×10^{-11}	4.5300×10^{-15}

The exact solution is $\theta = -1 + \sqrt{t^2 + 4}$. From Figure 6, we can observe that the absolute error (AE) of CPM is better when compared with the AE of HAM. Figure 5 shows the graphical representation of the CPM, HAM, and exact solution. To solve example 4.2 by HAM, let us consider the following initial guess, linear operator and auxiliary function: $\theta_0(t) = 1$, $\mathcal{L} = \frac{d}{dt}$ with the property that $\mathcal{L}[c_1] = 0$ and $H(t) = 1$.

The approximations are:

$$\theta_1(t) = -\frac{ht^2}{2},$$
$$\theta_2(t) = -\frac{ht^2}{2} - h^2t^2,$$
$$\theta_3(t) = -\frac{ht^2}{2} - 2h^2t^2 - 2h^3t^2 + \frac{h^3t^4}{8},$$
$$\vdots$$

By taking $h = -\frac{1}{2}$. Thus, the solution series for example 4.2 is,

$$\theta(t) = 1 + \frac{t^2}{4} - \frac{t^4}{64} + \frac{t^6}{512} - \frac{5t^8}{16384} + \frac{7t^{10}}{131072} - \frac{21t^{12}}{2097152} + \frac{33t^{14}}{16777216} - \frac{429t^{16}}{1073741824} + \cdots$$

Example 4.3. Consider the first-order non-linear initial value problem of the form;

$$\theta' = \theta^2, \ \theta(0) = 1.$$

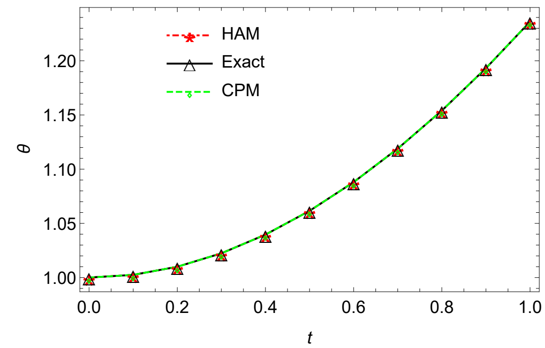

FIGURE 5. Graphical representation of Exact, HAM, and CPM solution, for example 4.2.

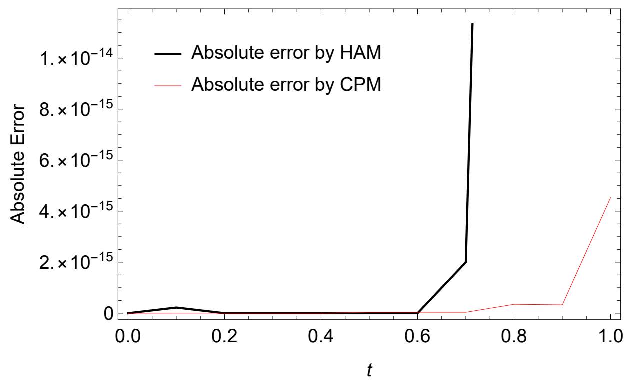

FIGURE 6. Error analysis for the HAM and CPM, for example 4.2.

The exact solution is $\theta = \frac{1}{1-t}$. A numerical comparison of these two different methods is drawn in Table 3. Figures 7 and 8 represents the graphical comparison and absolute errors of these two methods, respectively. To solve example 4.3 by HAM, let us consider the following initial guess, linear operator and auxiliary function: $\theta_0(t) = 1$, $\mathcal{L} = \frac{d}{dt}$ with the property that $\mathcal{L}[c_1] = 0$ and $H(t) = 1$.

The approximations of the solution,

$$\begin{aligned}
\theta_1(t) &= -ht, \\
\theta_2(t) &= -ht - h^2t + h^2t^2, \\
\theta_3(t) &= -ht - 2h^2t - h^3t + 2h^2t^2 + 2h^3t^2 - h^3t^3, \\
&\vdots
\end{aligned}$$

By taking $h = -1$. Thus, the solution series for example 4.3 is,

$$\theta(t) = 1 + t + t^2 + t^3 + t^4 + t^5 + t^6 + t^7 + t^8 + t^9 + t^{10} + t^{11} + t^{12} + t^{13} + \cdots$$

From Table 3, it is clear that the absolute error in CPM is reasonable compared to the absolute error in HAM.

Example 4.4. Consider the first-order non-linear initial value problem of the form;

$$\theta' = -\theta + \theta^2, \ \theta(0) = 4.$$

The exact solution is $\theta = -\frac{1}{\exp{(t+\ln\frac{3}{4})}-1}, t < \ln\frac{4}{3}$. We solved this problem by HAM and CPM and discussed the obtained results in the table and graphs. For HAM solution, assume $\theta_0(t) = 4$, $\mathcal{L} = \frac{d}{dt}$ with the property that $\mathcal{L}[c_1] = 0$

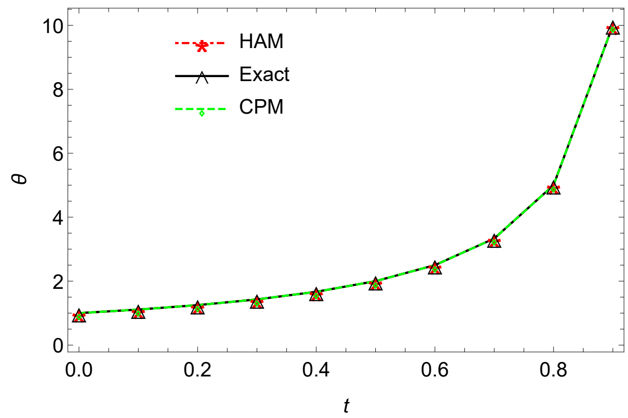

FIGURE 7. Graphical representation of Exact, HAM, and CPM solution, for example 4.3

TABLE 3. Comparison between HAM and CPM absolute errors (AE) with Exact Solution, for example 4.3.

x	Exact Solution	AE by HAM (65^{th} approximation)	AE by CPM n=20
0	1	0	4.4409×10^{-16}
0.1	1.111111	2.220446×10^{-16}	6.6613×10^{-16}
0.2	1.250000	2.220446×10^{-16}	6.6613×10^{-16}
0.3	1.428571	2.220446×10^{-16}	8.8818×10^{-16}
0.4	1.666667	0	1.3323×10^{-15}
0.5	2.000000	0	1.7764×10^{-15}
0.6	2.500000	6.217249×10^{-15}	3.1086×10^{-15}
0.7	3.333333	1.992277×10^{-10}	5.3291×10^{-15}
0.8	5.000000	2.008673×10^{-6}	1.0658×10^{-14}
0.9	10.00000	0.0095500	4.4409×10^{-14}

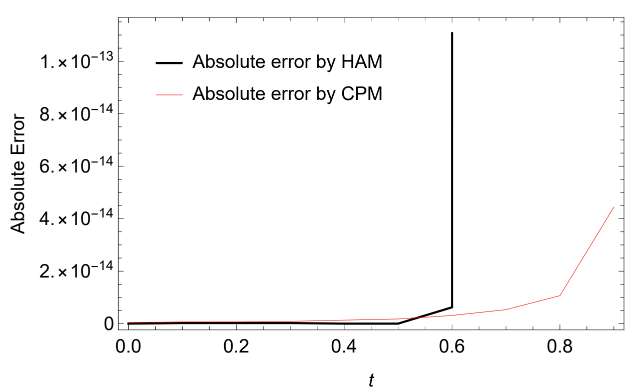

FIGURE 8. Error analysis for the HAM and CPM, for example 4.3.

and $H(t) = 1$. The approximations are:

$$\theta_1(t) = -12ht,$$
$$\theta_2(t) = -12ht - 12h^2t + 42h^2t^2,$$
$$\theta_3(t) = -12ht - 24h^2t - 12h^3t + 84h^2t^2 + 84h^3t^2 - 146h^3t^3,$$
$$\vdots$$

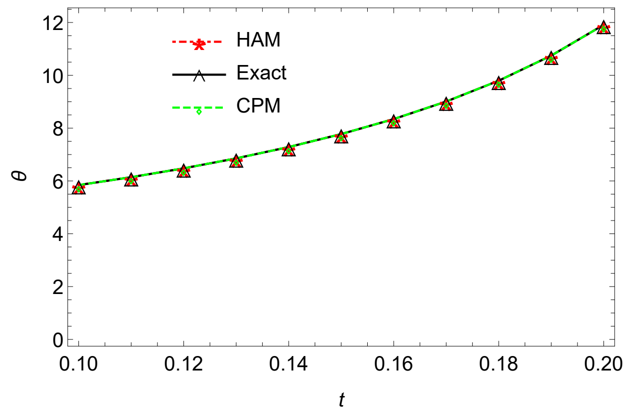

FIGURE 9. Graphical representation of Exact, HAM, and CPM solution for example 4.4

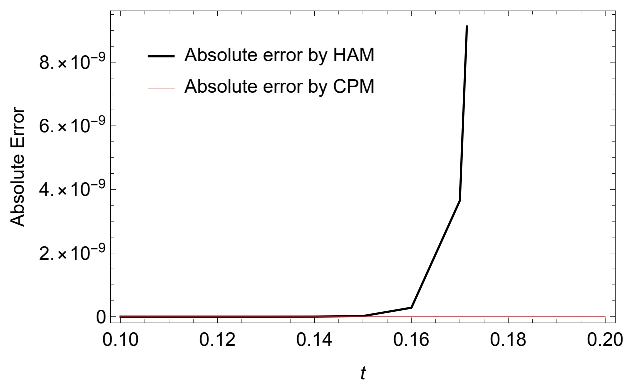

FIGURE 10. Error analysis for the HAM and CPM, for example 4.4.

By taking $h = -1$, thus, HAM solution is,

$$\theta(t) = 4 + 12t + 42t^2 + 146t^3 + \frac{1015t^4}{2} + \frac{17641t^5}{10} + \frac{367927t^6}{60} + \frac{8952553t^7}{420} + \frac{7113053t^8}{96} + \frac{7788499561t^9}{30240} + \cdots$$

From Table 4, one can observe that the absolute error in CPM is better than the absolute error in HAM.

Example 4.5. Consider the second-order non-linear initial value problem of the form;

$$\theta''(t) + e^{-2\theta(t)} = 0,\ \theta(0) = 0,\ \theta'(0) = 1$$

The exact solution is $\theta(t) = log(1 + t)$. The HAM and CPM are applied to this problem, and the obtained results are discussed in the table and graphs. Using the transformation $u = e^{2\theta}$, example 4.5 becomes,

$$u\,u'' - u'^2 + 2\,u = 0, u(0) = 1, u'(0) = 2. \tag{4.11}$$

TABLE 4. Comparison between HAM and CPM absolute errors (AE) with Exact Solution, for example 4.4.

x	Exact Solution	AE by HAM (40^{th} approximation)	AE by CPM n=20
0.1	5.84379	1.776357×10^{-15}	3.4100×10^{-18}
0.11	6.142829	8.881784×10^{-16}	1.3640×10^{-17}
0.12	6.477634	2.664535×10^{-15}	3.0690×10^{-17}
0.13	6.855009	3.996803×10^{-14}	5.4560×10^{-17}
0.14	7.283605	1.018741×10^{-12}	8.5250×10^{-17}
0.15	7.774579	1.842259×10^{-11}	1.2276×10^{-16}
0.16	8.342590	2.801315×10^{-10}	1.6709×10^{-16}
0.17	9.007276	3.649804×10^{-9}	2.1824×10^{-16}
0.18	9.795568670	4.155321×10^{-8}	2.7621×10^{-16}
0.19	10.74543191	4.203955×10^{-7}	3.4100×10^{-16}
0.20	11.91214582	0.000003836	3.4100×10^{-18}

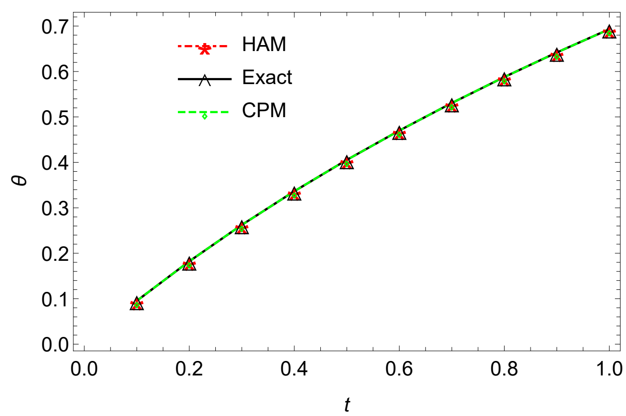

FIGURE 11. Graphical representation of Exact, HAM, and CPM solution, for example 4.5.

Assume, $u_0(t) = 2t + 1$, $\mathcal{L} = \frac{d^2}{dt^2}$ with the property that $\mathcal{L}[c_1 + c_2 t] = 0$ and $H(t) = 1$. The approximations of (4.11) are:

$$\begin{aligned} u_1(t) &= -ht^2 + \frac{2ht^3}{3}, \\ u_2(t) &= -ht^2 - h^2t^2 + \frac{2ht^3}{3} + \frac{4h^2t^3}{3} - \frac{h^2t^4}{6} + \frac{h^2t^5}{15}, \\ u_3(t) &= -ht^2 - 2h^2t^2 - h^3t^2 + \frac{2ht^3}{3} + \frac{8h^2t^3}{3} + 2h^3t^3 - \frac{h^2t^4}{3} - \frac{h^3t^4}{2} + \frac{2h^2t^5}{15} + \frac{4h^3t^5}{15} \\ &\quad - \frac{h^3t^6}{90} + \frac{h^3t^7}{315}, \\ &\ \vdots \end{aligned}$$

Thus the HAM solution is given by,

$$u(t) = 1 + 2t - 20\big(-t^2 + \frac{2t^3}{3}\big) - 19(0 + t^2 - 1.33333t^3 + 0.166667t^4 - 0.06667t^5) - \cdots$$

From Table 5, it is clear that the absolute errors in CPM are reasonable compared to the absolute errors in HAM for $0 \leq t \leq 1$.

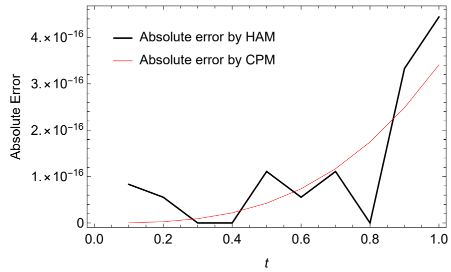

FIGURE 12. Error analysis for the HAM and CPM, for example 4.5.

TABLE 5. Comparison between HAM and CPM absolute errors (AE) with Exact Solution, for example 4.5.

x	Exact Solution	AE by HAM (20^{th} approximation)	AE by CPM n=20
0.1	0.095310	8.3266×10^{-17}	3.4100×10^{-19}
0.2	0.182322	5.5511×10^{-17}	2.7280×10^{-18}
0.3	0.262364	0	9.2070×10^{-18}
0.4	0.336472	0	2.1824×10^{-17}
0.5	0.405465	1.1102×10^{-16}	4.2625×10^{-17}
0.6	0.470004	5.5511×10^{-17}	7.3656×10^{-17}
0.7	0.530628	1.1102×10^{-16}	1.1696×10^{-16}
0.8	0.587787	0	1.7459×10^{-16}
0.9	0.641854	3.3306×10^{-16}	2.4859×10^{-16}
1.0	0.693147	4.4408×10^{-16}	3.4100×10^{-16}

Example 4.6. Consider the third-order linear initial value problem of the form;

$$\theta'''(t) + \theta = 0, \ \theta(0) = 1, \ \theta'(0) = -1, \ \theta''(0) = 1.$$

The exact solution is $\theta = e^{-t}$. On increasing values of n, the accuracy in the solution also increases that can be seen in Table 6. Figure 14 shows that absolute error by CPM is better than AE by HAM. Let us consider, $\theta_0(t) = 1 - t + \frac{t^2}{2}$, $\mathcal{L} = \frac{d^3}{dt^3}$ with the property that $\mathcal{L}[c_1 + c_2 t + c_3 t^2] = 0$ and $H(t) = 1$.

The approximations are:

$$\begin{aligned}
\theta_1(t) &= \frac{ht^3}{6} - \frac{ht^4}{24} + \frac{ht^5}{120},\\
\theta_2(t) &= \frac{ht^3}{6} + \frac{h^2t^3}{6} - \frac{ht^4}{24} - \frac{h^2t^4}{24} + \frac{ht^5}{120} + \frac{h^2t^5}{120} + \frac{h^2t^6}{720} - \frac{h^2t^7}{5040} + \frac{h^2t^8}{40320},\\
\theta_3(t) &= \frac{ht^3}{6} + \frac{h^2t^3}{3} + \frac{h^3t^3}{6} - \frac{ht^4}{24} - \frac{h^2t^4}{12} - \frac{h^3t^4}{24} + \frac{ht^5}{120} + \frac{h^2t^5}{60} + \frac{h^3t^5}{120}\\
&+ \frac{h^2t^6}{360} + \frac{h^3t^6}{360} - \frac{h^2t^7}{2520} - \frac{h^3t^7}{2520} + \frac{h^2t^8}{20160} + \frac{h^3t^8}{20160}\\
&+ \frac{h^3t^9}{362880} - \frac{h^3t^{10}}{3628800} + \frac{h^3t^{11}}{39916800},\\
&\vdots
\end{aligned}$$

By taking $h = -1$, the HAM is given by,

$$\theta(t) = 1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} - \frac{t^5}{120} + \cdots$$

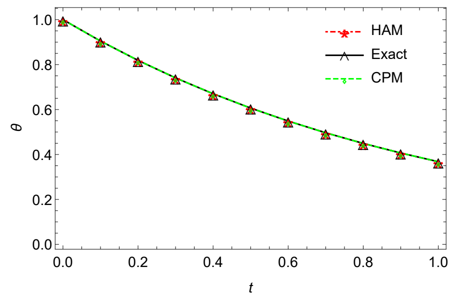

FIGURE 13. Graphical representation of Exact, HAM, and CPM solution, for example 4.6.

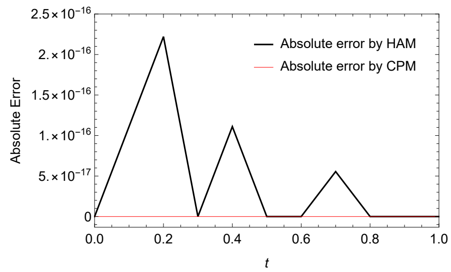

FIGURE 14. Error analysis for the HAM and CPM, for example 4.6.

TABLE 6. Comparison between HAM and CPM absolute errors (AE) with Exact Solution, for example 4.6.

x	Exact Solution	AE by HAM (25^{th} approximation)	AE by CPM n=25
0	1	0	0
0.1	0.904837	1.1102×10^{-16}	0
0.2	0.818731	2.2204×10^{-16}	0
0.3	0.740818	0	0
0.4	0.670320	1.1102×10^{-16}	8.7296×10^{-29}
0.5	0.606531	0	0
0.6	0.548812	0	0
0.7	0.496585	5.5511×10^{-17}	0
0.8	0.449329	0	0
0.9	0.406569	0	0
1.0	0.367879	0	0

5. CONCLUSION

This paper developed a new approach for initial value problems through clique polynomials of the complete graph. In this approach, given ODE is transformed into a system of algebraic equations via discrete grid points. We solved six examples with different orders using HAM and CPM. The obtained results from both methods are compared through tables and graphs. This study reveals that CPM yields better results, consumes significantly less time than HAM, and it is straightforward. Also, we discussed some theorems with proof on clique polynomials.

Acknowledgment

The authors wish to thank the reviewers for useful comments which helped in improving the paper considerably.

Funding information

The authors state no funding involved.

Conflict of interest

The authors state no conflict of interest.

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