Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 10, No. 3, 2022, pp. 799-815
DOI:10.22034/CMDE.2021.45950.1927

Two explicit and implicit finite difference schemes for time fractional Riesz space diffusion equation

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Abstract

In this study, one explicit and one implicit finite difference scheme is introduced for the numerical solution of time-fractional Riesz space diffusion equation. The time derivative is approximated by the standard Grünwald Letnikov formula of order one, while the Riesz space derivative is discretized by Fourier transform-based algorithm of order four. The stability and convergence of the proposed methods are studied. It is proved that the implicit scheme is unconditionally stable, while the explicit scheme is stable conditionally. Some examples are solved to illustrate the efficiency and accuracy of the proposed methods. Numerical results confirm that the accuracy of present schemes is of order one.

Keywords. Fractional derivatives, Fractional diffusion equation, Riesz fractional derivative, Finite differences.
2010 Mathematics Subject Classification. 34A08, 44A10.

1. Introduction

During the last decades, the theory of fractional differential equations has been introduced and developed to solve enormous problems in all branches of science and technology such as physics, mathematics, biology, economics, engineering, and other fields [7, 14, 16]. The analytical solution can not be obtained for most fractional differential equations. So it is important to develop numerical algorithms for solving these equations. Du to the importance of fractional differential equations, fractional finite difference methods were introduced firstly in [12] and latter was developed by others [6, 8, 15].

In this study, we consider the following diffusion equation of fractional type involving Riesz space derivative

$$\begin{aligned} &{}^{C}D_{0,t}^{\beta}u\left(x,t\right)=\eta\frac{\partial^{\alpha}u\left(x,t\right)}{\partial\left|x\right|^{\alpha}}+g\left(x,t\right), \\ &0<\beta\leq1,\ 1<\alpha<2,\ a\leq x\leq b,\ \ 0\leq t\leq T, \end{aligned} \tag{1.1}$$

subject to the initial and boundary conditions

$$\begin{cases} u\left(x,0\right)=\phi\left(x\right), & a\leq x\leq b, \\ u\left(a,t\right)=u\left(b,t\right)=0, & 0\leq t\leq T, \end{cases} \tag{1.2}$$

where $\eta>0$ is a constant coefficient, $g(x,t)$ and $\phi(x)$ are sufficiently smooth functions, and $\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}$ represents the Riesz fractional derivative $(1<\alpha<2)$, which was defined in [19] as follows

$$\frac{\partial^{\alpha}}{\partial\left|x\right|^{\alpha}}u\left(x,t\right)=\ c_{\alpha}\left({}^{RL}D_{a,x}^{\alpha}+{}^{RL}D_{x,b}^{\alpha}\right)u(x,t),$$

Received: 08 May 2021 ; Accepted: 11 October 2021.
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where $c_\alpha = -\frac{1}{2\cos\left(\frac{\alpha\pi}{2}\right)}$. ${}_{a}^{RL}D_x^\alpha u(x,t)$ and ${}_{x}^{RL}D_b^\alpha u(x,t)$ denote the right and left Riemann-Liouville fractional derivatives of order α for $u(x,t)$ respectively.

The analytical solution of Riesz space, time-fractional differential equation (1.1), was studied by Zhang and Liu [20]. Chen et al. [1] used the Laplace and Fourier transform methods to obtain a solution for the space Riesz fractional reaction dispersion equation. Later authors in [17] introduced two implicit and explicit methods for the Riesz time Caputo fractional equation. They also proposed a novel numerical scheme for the Riesz-space fractional advection-dispersion equation in [18]. Din et al. [4] developed a fourth-order numerical algorithm for Riesz fractional derivative and applied it for Riesz space diffusion equation with fractional derivative. A novel meshless space-time backward substitution method was introduced by authors in [11]. Lin et al. [10] used a semi-analytical method for solving a class of time-fractional partial differential equations with variable coefficients. A homogenization function method was studied by [9] for the inverse source problem of the nonlinear time-fractional wave equation.

Most of the available methods have a lot of computational complexity. In this study, we introduce a simple method to solve the time-fractional Riesz space diffusion equation (1.1). We apply the first-order standard Grünwald Letnikov formula for Caputo fractional derivative to discretize (1.1) along the time axis. Then, the fourth-order numerical algorithm based on Fourier transform method [4] is used for approximating the Riesz derivative on each time step. We introduce two finite difference schemes to the numerical solution of time-fractional Riesz space diffusion equation (1.1) and (1.2). We provide the stability analysis and convergence of the proposed methods. We show that the order of convergence is $O(\tau)$, while τ is the uniform time step size. Due to the high accuracy of the Riesz derivative approximation, it is not necessary to select a very small spatial step length, and this saves computational costs.

The rest of the paper is organized as follows: Some basic definitions of fractional calculus are presented in section 2. In section 3 the numerical algorithms for equation (1.1) and (1.2) are introduced. The stability and convergence of the proposed methods are given in section 4. Section 5 is devoted to numerical illustrations. Finally, the conclusion is provided in section 6.

2. Basic definitions

Some necessary definitions of fractional calculus are introduced in this section. Since the Riemann-Liouville and the Caputo derivatives are often used, as well as the Riesz fractional derivative is defined based on left and right Riemann-Liouville and Caputo derivatives, we focus on these definitions of fractional calculus. Furthermore in the modeling of most physical problems, the initial conditions are given in integer order derivatives and the integer-order derivatives coincide with Caputo initial conditions definition; therefore the Caputo derivative is often used in numerical algorithms.

Definition 2.1. The left and right α order Riemann-Liouville integrals ($\alpha > 0$) of a function $f(x)$, on the interval (a,b) are defined as follows [16]:

$$\begin{cases} {}_aJ_x^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int\limits_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \\ {}_tJ_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int\limits_t^b \frac{f(s)}{(s-t)^{1-\alpha}} ds, \end{cases}$$

where $\Gamma(\mathrm{z}) = \int\limits_0^\infty \mathrm{e}^{-\mathrm{t}} \mathrm{t}^{\mathrm{z}-1} \mathrm{dt}$, $\mathrm{z} \in \mathrm{C}$ is the Gamma function.

Definition 2.2. The left and right Riemann-Liouville derivatives of order $\alpha > 0$ for a function $f(x)$, defined on the interval (a,b) are given as follows [16]:

$$\begin{cases} {}^{RL}D_{a,t}^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int\limits_a^t (t-s)^{m-\alpha-1} f(s)\, ds, \\ {}^{RL}D_{t,b}^\alpha f(t) = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int\limits_t^b (s-t)^{m-\alpha-1} f(s)\, ds, \end{cases}$$

where $m-1<\alpha\leq m$.

Definition 2.3. The left and right Caputo derivatives of order α, are defined as follows[16]:

$$\begin{cases} {}^{C}D_{a,t}^{\alpha}f\left(t\right)=\frac{1}{\Gamma(m-\alpha)}\int\limits_{a}^{t}\left(t-s\right)^{m-\alpha-1}\frac{d^{m}}{ds^{m}}f\left(s\right)ds, & m-1<\alpha\leq m\ , \\ {}^{C}D_{t,b}^{\alpha}f\left(t\right)=\frac{(-1)^{m}}{\Gamma(m-\alpha)}\int\limits_{t}^{b}\left(s-t\right)^{m-\alpha-1}\frac{d^{m}}{ds^{m}}f\left(s\right)ds, & m-1<\alpha\leq m. \end{cases}$$

From the Caputo derivative definitions we have

$$\begin{cases} {}^{C}D_{a,t}^{\alpha}c=0, \\ {}^{C}D_{t,b}^{\alpha}c=0, \end{cases} \quad (c \text{ is a constant}) \tag{2.1}$$

$${}^{C}D_{a,t}^{\alpha}(t-a)^{\beta}=\begin{cases} 0 & \beta\in N_0 \text{ and } \beta<\alpha \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(t-a)^{\beta-\alpha} & \beta\in N_0 \text{ and } \beta\geq\alpha \\ & \text{or } \beta\notin N_0 \end{cases} \tag{2.2}$$

$${}^{C}D_{t,b}^{\alpha}(t-b)^{\beta}=\begin{cases} 0, & \beta\in N_0 \text{ and } \beta<\alpha \\ (-1)^{\beta}\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(b-t)^{\beta-\alpha} & \beta\in N_0 \text{ and } \beta\geq\alpha \\ & \text{or } \beta\notin N_0 \end{cases} \tag{2.3}$$

There are relations between Riemann-Liouville derivatives and Caputo derivatives as follows

$${}^{C}D_{a,t}^{\alpha}f\left(t\right)={}^{RL}D_{a,t}^{\alpha}f\left(t\right)-\sum_{k=0}^{m-1}\frac{f^{(k)}\left(a\right)}{\Gamma\left(1+k-\alpha\right)}\left(t-a\right)^{k-\alpha}, \tag{2.4}$$

$${}^{C}D_{t,b}^{\alpha}f\left(x\right)={}^{RL}D_{t,b}^{\alpha}f\left(t\right)-\sum_{k=0}^{m-1}\frac{f^{(k)}\left(b\right)}{\Gamma\left(1+k-\alpha\right)}\left(b-t\right)^{k-\alpha}. \tag{2.5}$$

It is clear that if $f^{(k)}\left(a\right)=0,\quad k=0,1,..,m-1$ then the left Riemann-Liouville derivative and the left Caputo derivative are equivalent, likewise the right Riemann-Liouville derivative and the right Caputo derivative are equivalent when $f^{(k)}\left(b\right)=0,\quad k=0,1,..,m-1$. For comprehensive properties of fractional derivatives and integrals, one can refer to the literature [13, 16].

Lemma 2.4. *[16] If $m-1<\alpha<m,\ \ m\in N$, then*

$${}^{C}D_{a,t}^{\alpha}J_{a,t}^{\alpha}f\left(t\right)=f\left(t\right); \tag{2.6}$$

$$J_{a,t}^{\alpha}\,{}^{C}D_{a,t}^{\alpha}f\left(t\right)=f\left(t\right)-\sum_{k=0}^{m-1}\frac{f^{(k)}\left(a^{+}\right)}{k!}(t-a)^{k},\quad t>0. \tag{2.7}$$

3. Method of Solution

Consider the fractional diffusion equation (1.1) and (1.2) with Riesz space derivative and Caputo fractional time derivative.

Firstly, we set the points $x_i=a+ih, i=0,1,\ldots,M$, and $t_n=n\tau, n=0,1,\ldots,N$, where $h=(b-a)/M$ and $\tau=T/N$, are the uniform spatial and temporal step lengths respectively, and M and N are integers. We use the notation u_i^n for $u(x_i,t_n)$.

3.1. **Time discretization.** If $u(x,t)$ is suitably smooth function with respect to t, the Grünwald-Letnikov and Riemann Liouville's fractional derivatives are equivalent. Denoting $w_k^{(\beta)} = (-1)^k \binom{\beta}{k}$, one gets

$$\left[{}^{RL}D_{a,t}^{\beta} u(x,t)\right]_{(x,t)=(x_i,t_n)} \simeq \frac{1}{\tau^\beta} \sum_{k=0}^{n} w_{n-k}^{(\beta)} u_i^k. \tag{3.1}$$

The formula (3.1) is convergent of order one for any $\beta > 0$ (see [16]), and is referred to as the standard Grünwald Letnikov formula. Formula (3.1) can be modified for approximating the Caputo fractional derivative as follows

$$\left[{}^{C}D_{a,t}^{\beta} u(x,t)\right]_{(x,t)=(x_i,t_n)} \simeq \frac{1}{\tau^\beta} \sum_{k=0}^{n} w_{n-k}^{(\beta)} \left(u_i^k - u_i^0\right), \ 0<\beta<1 \tag{3.2}$$

which is still convergent of order one [16].

3.2. **Spatial discretization.** The authors in [4], constructed a new computational scheme for the Riesz fractional derivative of order α $(1 < \alpha < 2)$, by using the Fourier transpose method as follows

$$\begin{aligned}\left[\frac{\partial^\alpha}{\partial |x|^\alpha} u(x,t)\right]_{(x_i,t_n)} \simeq\ & \frac{\alpha}{24h^\alpha} \sum_{k=-M+m+1}^{m-1} s_k^{(\alpha)} u_{m-(k+1)}^n \\ & + \frac{\alpha}{24h^\alpha} \sum_{k=-M+m+1}^{m-1} s_k^{(\alpha)} u_{m-(k-1)}^n \\ & - \left(1+\frac{\alpha}{12}\right) \frac{1}{h^\alpha} \sum_{k=-M+m+1}^{m-1} s_k^{(\alpha)} u_{m-k}^n,\end{aligned} \tag{3.3}$$

the scheme (3.3) is of order four, where

$$s_k^{(\alpha)} = \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2}-k+1)\Gamma(\frac{\alpha}{2}+k+1)}, \ k \in \mathbb{Z}. \tag{3.4}$$

3.3. **Numerical Methods.** In this part, we develop one explicit and one implicit scheme for numerical solution of time fractional space Riesz diffusion equation (1.1) and (1.2).

Scheme 1: We discretize (1.1) as follows

$$\left[{}^{C}D_{0,t}^{\beta} u(x,t)\right]_{(x_i,t_n)} = \eta \left[\frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha}\right]_{(x_i,t_{n-1})} + g(x_i,t_{n-1}). \tag{3.5}$$

Substituting (3.2) and (3.3) in (3.5) yields

$$\begin{aligned}\frac{1}{\tau^\beta} \sum_{k=0}^{n} w_{n-k}^{(\beta)} (u_i^k - u_i^0) =\ & \eta \left\{ \frac{\alpha}{24h^\alpha} \sum_{k=-M+m+1}^{m-1} s_k^{(\alpha)} u_{m-(k+1)}^{n-1} \right. \\ & + \frac{\alpha}{24h^\alpha} \sum_{k=-M+m+1}^{m-1} s_k^{(\alpha)} u_{m-(k-1)}^{n-1} \\ & \left. - \left(1+\frac{\alpha}{12}\right) \frac{1}{h^\alpha} \sum_{k=-M+m+1}^{m-1} s_k^{(\alpha)} u_{m-k}^{n-1} \right\} \\ & + g_i^{n-1} + O(\tau + h^4),\end{aligned} \tag{3.6}$$

where $g_i^{n-1} = g(x_i, t_{n-1})$. After some simplifications and neglecting the truncation error, we obtain

$$\begin{aligned} U_i^n = \quad & \beta U_i^{n-1} - 2\tau^{\beta-1}\Bigg\{-\mu_1 \sum_{k=-M+m+1}^{m-1} s_k^{(\alpha)} U_{m-(k+1)}^{n-1} \\ & -\mu_1 \sum_{k=-M+m+1}^{m-1} s_k^{(\alpha)} U_{m-(k-1)}^{n-1} + \mu_2 \sum_{k=-M+m+1}^{m-1} s_k^{(\alpha)} U_{m-k}^{n-1}\Bigg\} \\ & - \sum_{k=0}^{n-2} w_{n-k}^{(\beta)} U_i^k + \sum_{k=0}^{n} w_{n-k}^{(\beta)} U_i^0 + \tau^\beta g_i^{n-1}, \end{aligned} \tag{3.7}$$

where $\mu_1 = \frac{\alpha\tau\eta}{48h^\alpha}$, $\mu_2 = \frac{\tau\eta}{2h^\alpha}\left(1 + \frac{\alpha}{12}\right)$ and U_i^n is the approximation value for u_i^n. Denoting

$$\begin{aligned} &\mathbf{U}^k = (U_1^k,\ U_2^k,\ \dots,\ U_{M-1}^k)^T, \\ &\mathbf{F}^k = (g_1^k,\ g_2^k,\ \dots,\ g_{M-1}^k)^T, \end{aligned}$$

the system (3.7) can be rewritten as the following matrix representation

$$\begin{cases} \mathbf{U}^1 = \mathbf{A}\mathbf{U}^0 + \tau^\beta \mathbf{F}^0, \\ \mathbf{U}^n = \mathbf{B}\mathbf{U}^{n-1} - \displaystyle\sum_{k=1}^{n-2} w_{n-k}^{(\beta)} \mathbf{U}^k + \sum_{k=1}^{n} w_{n-k}^{(\beta)} \mathbf{U}^0 + \tau^\beta \mathbf{F}^{n-1},\ n > 1, \end{cases} \tag{3.8}$$

where $\mathbf{A} = (\mathbf{I} - 2\tau^{\beta-1}\mathbf{H})$, $\mathbf{B} = (\beta\ \mathbf{I} - 2\tau^{\beta-1}\mathbf{H})$, $\mathbf{H} = \mu_2\mathbf{G} - \mu_1\mathbf{G}^+ - \mu_1\mathbf{G}^-$, $\mathbf{I}$ is the identity matrix and

$$\mathbf{G} = \begin{pmatrix} s_0^{(\alpha)} & s_{-1}^{(\alpha)} & s_{-2}^{(\alpha)} & \dots & s_{4-M}^{(\alpha)} & s_{3-M}^{(\alpha)} & s_{2-M}^{(\alpha)} \\ s_1^{(\alpha)} & s_0^{(\alpha)} & s_{-1}^{(\alpha)} & s_{-2}^{(\alpha)} & \dots & s_{4-M}^{(\alpha)} & s_{3-M}^{(\alpha)} \\ s_2^{(\alpha)} & s_1^{(\alpha)} & s_0^{(\alpha)} & s_{-1}^{(\alpha)} & s_{-2}^{(\alpha)} & \dots & s_{4-M}^{(\alpha)} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ s_{M-4}^{(\alpha)} & \dots & s_2^{(\alpha)} & s_1^{(\alpha)} & s_0^{(\alpha)} & s_{-1}^{(\alpha)} & s_{-2}^{(\alpha)} \\ s_{M-3}^{(\alpha)} & s_{M-4}^{(\alpha)} & \dots & s_2^{(\alpha)} & s_1^{(\alpha)} & s_0^{(\alpha)} & s_{-1}^{(\alpha)} \\ s_{M-2}^{(\alpha)} & s_{M-3}^{(\alpha)} & s_{M-4}^{(\alpha)} & \dots & s_2^{(\alpha)} & s_1^{(\alpha)} & s_0^{(\alpha)} \end{pmatrix},$$

$$\mathbf{G}^+ = \begin{pmatrix} 0 & s_0^{(\alpha)} & s_{-1}^{(\alpha)} & \dots & s_{5-M}^{(\alpha)} & s_{4-M}^{(\alpha)} & s_{3-M}^{(\alpha)} \\ 0 & s_1^{(\alpha)} & s_0^{(\alpha)} & s_{-1}^{(\alpha)} & \dots & s_{5-M}^{(\alpha)} & s_{4-M}^{(\alpha)} \\ 0 & s_2^{(\alpha)} & s_1^{(\alpha)} & s_0^{(\alpha)} & s_{-1}^{(\alpha)} & \dots & s_{5-M}^{(\alpha)} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & s_3^{(\alpha)} & s_2^{(\alpha)} & s_1^{(\alpha)} & s_0^{(\alpha)} & s_{-1}^{(\alpha)} \\ 0 & s_{M-3}^{(\alpha)} & \dots & s_3^{(\alpha)} & s_2^{(\alpha)} & s_1^{(\alpha)} & s_0^{(\alpha)} \\ 0 & s_{M-2}^{(\alpha)} & s_{M-3}^{(\alpha)} & \dots & s_3^{(\alpha)} & s_2^{(\alpha)} & s_1^{(\alpha)} \end{pmatrix},$$

$$\mathbf{G}^- = \begin{pmatrix} s_{-1}^{(\alpha)} & s_{-2}^{(\alpha)} & s_{-3}^{(\alpha)} & \dots & s_{3-M}^{(\alpha)} & s_{2-M}^{(\alpha)} & 0 \\ s_0^{(\alpha)} & s_{-1}^{(\alpha)} & s_{-2}^{(\alpha)} & s_{-3}^{(\alpha)} & \dots & s_{3-M}^{(\alpha)} & 0 \\ s_1^{(\alpha)} & s_0^{(\alpha)} & s_{-1}^{(\alpha)} & s_{-2}^{(\alpha)} & s_{-3}^{(\alpha)} & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ s_{M-5}^{(\alpha)} & \dots & s_1^{(\alpha)} & s_0^{(\alpha)} & s_{-1}^{(\alpha)} & s_{-2}^{(\alpha)} & 0 \\ s_{M-4}^{(\alpha)} & s_{M-5}^{(\alpha)} & \dots & s_1^{(\alpha)} & s_0^{(\alpha)} & s_{-1}^{(\alpha)} & 0 \\ s_{M-3}^{(\alpha)} & s_{M-4}^{(\alpha)} & s_{M-5}^{(\alpha)} & \dots & s_1^{(\alpha)} & s_0^{(\alpha)} & 0 \end{pmatrix}.$$

Scheme 2 : We discretize (1.1) as follows

$$\left[{}^{C}D_{0,t}^{\beta}u(x,t)\right]_{(x_i,t_n)} = \eta\left[\frac{\partial^{\alpha}u(x,t)}{\partial|x|^{\alpha}}\right]_{(x_i,t_n)} + g(x_i,t_n). \tag{3.9}$$

Substituting (3.2) and (3.3) in (3.9) yields

$$\begin{aligned}\frac{1}{\tau^{\beta}}\sum_{k=0}^{n}w_{n-k}^{(\beta)}(u_i^k-u_i^0) = \;& \eta\Bigg\{\frac{\alpha}{24h^{\alpha}}\sum_{k=-M+m+1}^{m-1}s_k^{(\alpha)}u_{m-(k+1)}^n \\ &+\frac{\alpha}{24h^{\alpha}}\sum_{k=-M+m+1}^{m-1}s_k^{(\alpha)}u_{m-(k-1)}^n \\ &-\left(1+\frac{\alpha}{12}\right)\frac{1}{h^{\alpha}}\sum_{k=-M+m+1}^{m-1}s_k^{(\alpha)}u_{m-k}^n\Bigg\} \\ &+g_i^n+O(\tau+h^4).\end{aligned} \tag{3.10}$$

After some simplifications and neglecting the truncation error, we get

$$\begin{aligned}&U_i^n+2\tau^{\beta-1}\left\{-\mu_1\sum_{k=-M+m+1}^{m-1}s_k^{(\alpha)}U_{m-(k+1)}^n-\mu_1\sum_{k=-M+m+1}^{m-1}s_k^{(\alpha)}U_{m-(k-1)}^n+\mu_2\sum_{k=-M+m+1}^{m-1}s_k^{(\alpha)}U_{m-k}^n\right\} \\ &\quad=-\sum_{k=0}^{n-1}w_{n-k}^{(\beta)}U_i^k+\sum_{k=0}^{n}w_{n-k}^{(\beta)}U_i^0+\tau^{\beta}g_i^n.\end{aligned} \tag{3.11}$$

The system (3.11) can be rewritten as follows

$$\begin{cases}\mathbf{C}\mathbf{U}^1=\mathbf{U}^0+\tau^{\beta}\mathbf{F}^1, \\ \mathbf{C}\mathbf{U}^n=-\displaystyle\sum_{k=1}^{n-1}w_{n-k}^{(\beta)}\mathbf{U}^k+\sum_{k=1}^{n}w_{n-k}^{(\beta)}\;\mathbf{U}^0+\tau^{\beta}\mathbf{F}^n,\;\; n>1,\end{cases} \tag{3.12}$$

where $\mathbf{C}=(\mathbf{I}+2\tau^{\beta-1}\mathbf{H})$ and $\mathbf{H}$ was defined in scheme 1.

4. Stability Analysis and Convergence Study

In this section, we perform the stability analysis and convergence study of the difference schemes (3.8) and (3.12). We need the following theorems and lemmas.

Lemma 4.1. *(Grünwall Lemma) Let $c \geq 0$ and $\{\phi_j\}_{j\geq 1}$ and $\{v_j\}_{j\geq 1}$ be nonnegative sequences. If*

$$\phi_j \leq c+\sum_{i=1}^{j-1}\phi_i v_i,\;\; j\geq 1,$$

then

$$\phi_j \leq c\prod_{i=1}^{j-1}(1+v_i)\leq c\;\exp\left(\sum_{i=1}^{j-1}v_i\right).$$

Proof. We refer the proof to [5]. □

Lemma 4.2. *Let $w_k^\beta = (-1)^k \binom{\beta}{k}$ and $0 < \beta < 1$ then*

$$w_0^\beta = 1,\ w_1^\beta = -\beta,\ w_k^\beta < 0,\ k \geq 1,\ \sum_{k=0}^{\infty} w_k^\beta = 0,\ -\sum_{k=1}^{n} w_k^\beta < 1,\ n \in \mathbb{N}.$$

Proof. We refer the proof to [2].

□

Theorem 4.3. *The matrix $\boldsymbol{H}$ (defined in schemes (3.8) and (3.12)) is positive definite and*

$$\begin{aligned} 0 < \lambda_{\boldsymbol{H}} &\leq max_{x\in[0,2\pi]} \left\{ \frac{\eta\tau}{12h^\alpha} \left[6 + \alpha \sin^2\left(\frac{x}{2}\right)\right] \left|2\sin^2\left(\frac{x}{2}\right)\right|^\alpha \right\} \\ &= \tfrac{\eta\tau}{12h^\alpha}(6+\alpha)2^\alpha, \end{aligned} \tag{4.1}$$

where $\lambda_{\boldsymbol{H}}$ denotes the eigenvalue of $\boldsymbol{H}$.

Proof. We refer the detailed proof to [4].

□

Theorem 4.4. *Let $\boldsymbol{H}$ be a positive definite matrix. Then for any parameter $\delta \geq 0$, the following statements hold.*

$$\begin{aligned} &\|(\boldsymbol{I} + \delta\boldsymbol{H})^{-1}\|_\infty \leq 1, \\ &\|(\boldsymbol{I} + \delta\boldsymbol{H})^{-1}(\boldsymbol{I} - \delta\boldsymbol{H})\|_\infty \leq 1. \end{aligned} \tag{4.2}$$

Proof. We refer the detailed proof to [4].

□

Theorem 4.5. *Let $r = \eta\tau^\beta/h^\alpha$ and $\boldsymbol{B} = (\beta\boldsymbol{I} - 2\tau^{\beta-1}\boldsymbol{H})$. Then $\rho(\boldsymbol{B}) < 1$ if*

$$r < \frac{6(1+\beta)}{(6+\alpha)2^\alpha}. \tag{4.3}$$

Proof. We provide the conditions that $\rho(\mathbf{B}) \leq 1$, where $\mathbf{B}$ is the iteration matrix of the difference scheme (3.8). Note that $\rho(\mathbf{B})$ is the spectral radius of matrix $\mathbf{B}$. By the definition of $\mathbf{B}$ and (4.1) we have

$$\lambda_{\mathbf{B}} = \beta - 2\tau^{\beta-1}\lambda_{\mathbf{H}}.$$

Since $0 < \lambda_{\mathbf{H}} \leq \frac{\eta\tau}{12h^\alpha}(6+\alpha)2^\alpha$ then

$$\beta - \frac{1}{6}r(6+\alpha)2^\alpha \leq \lambda_{\mathbf{B}} < \beta.$$

Now $|\lambda_{\mathbf{B}}| < 1$ provides that $r < \frac{6(1+\beta)}{(6+\alpha)2^\alpha}$, and this completes the proof.

□

Theorem 4.6. *Let the step lengths h and τ are chosen such that $r = \eta\tau^\beta/h^\alpha < \frac{6(1+\beta)}{(6+\alpha)2^\alpha}$, then the difference scheme (3.8) is stable.*

Proof. Let the vector $\mathbf{U}^k$, $(k = 0, 1, \ldots, n)$ in (3.8) has perturbed by $\widetilde{\mathbf{U}}^k$, then the perturbed equation is as follows

$$\widetilde{\mathbf{U}}^n = \mathbf{B}\widetilde{\mathbf{U}}^{n-1} - \sum_{k=1}^{n-2} w_{n-k}^{(\beta)} \widetilde{\mathbf{U}}^k + \sum_{k=1}^{n} w_{n-k}^{(\beta)} \widetilde{\mathbf{U}}^0, \tag{4.4}$$

then

$$\|\widetilde{\mathbf{U}}^n\| \leq \|\mathbf{B}\| \|\widetilde{\mathbf{U}}^{n-1}\| + \sum_{k=1}^{n-2} |w_{n-k}^{(\beta)}| \, \|\widetilde{\mathbf{U}}^k\| + \left|\sum_{k=1}^{n} w_{n-k}^{(\beta)}\right| \|\widetilde{\mathbf{U}}^0\|. \tag{4.5}$$

By Grünwall's Lemma 4.1 we conclude that

$$\begin{aligned} \|\widetilde{\mathbf{U}}^n\| &\leq \left|\sum_{k=1}^{n} w_{n-k}^{(\beta)}\right| \|\widetilde{\mathbf{U}}^0\| \exp\left(\|\mathbf{B}\| + \sum_{k=1}^{n-2} |w_{n-k}^{(\beta)}|\right) \\ &\leq \|\widetilde{\mathbf{U}}^0\| \exp\left(\|\mathbf{B}\| + 1\right). \end{aligned} \tag{4.6}$$

We know that for any $\epsilon > 0$ there exists a matrix norm such that $\|\mathbf{B}\| < \rho(\mathbf{B}) + \epsilon$. Since $\rho(\mathbf{B}) < 1$ then by the equivalence of norms, for any norm we conclude that $\|\mathbf{B}\|$ is a finite value. Also by Lemmas 4.1, 4.2 and Theorem 4.5 the right hand side in (4.6) is finite value and does not tend infinity as $n \to \infty$. This proves the stability of the finite difference scheme (3.8).

□

Theorem 4.7. *The difference scheme (3.12) is unconditionally stable.*

Proof. The iteration matrix for scheme (3.12) is $\mathbf{C}^{-1}$ where $\mathbf{C} = (\mathbf{I} + 2\tau^{\beta-1}\mathbf{H})$ and it suffices to prove that $\rho(\mathbf{C}^{-1}) < 1$. By the definition of matrix $\mathbf{C}$ and (4.1) we have

$$\lambda_{\mathbf{C}} = 1 + 2\tau^{\beta-1}\lambda_{\mathbf{H}} > 1 + \frac{1}{6}r(6+\alpha)2^{\alpha}.$$

Since $\lambda_{\mathbf{C}} > 1$ then $\lambda_{\mathbf{C}^{-1}} < 1$. This provides $\rho(\mathbf{C}^{-1}) < 1$, and then $\|\mathbf{C}^{-1}\|$ is finite.
Now by (3.12)

$$\mathbf{U}^n = \mathbf{C}^{-1}\mathbf{U}^{n-1} - \sum_{k=1}^{n-2} w_{n-k}^{(\beta)} \mathbf{C}^{-1}\mathbf{U}^k + \sum_{k=1}^{n} w_{n-k}^{(\beta)} \mathbf{C}^{-1}\mathbf{U}^0 + \tau^{\beta}\mathbf{C}^{-1}F^n. \tag{4.7}$$

Let the vector $\mathbf{U}^k$, $(k = 0, 1, \ldots, n)$ in (4.7) has perturbed by $\widetilde{\mathbf{U}}^k$, then the perturbed equation is as follows

$$\widetilde{\mathbf{U}}^n = \mathbf{C}^{-1}\widetilde{\mathbf{U}}^{n-1} - \sum_{k=1}^{n-2} w_{n-k}^{(\beta)} \mathbf{C}^{-1}\widetilde{\mathbf{U}}^k + \sum_{k=1}^{n} w_{n-k}^{(\beta)} \mathbf{C}^{-1}\widetilde{\mathbf{U}}^0, \tag{4.8}$$

and then

$$\|\widetilde{\mathbf{U}}^n\| \leq \quad \|\mathbf{C}^{-1}\| \, \|\widetilde{\mathbf{U}}^{n-1}\| + \sum_{k=1}^{n-2} |w_{n-k}^{(\beta)}| \|\mathbf{C}^{-1}\| \, \|\widetilde{\mathbf{U}}^k\| \left|\sum_{k=1}^{n} w_{n-k}^{(\beta)}\right| \|\mathbf{C}^{-1}\| \, \|\widetilde{\mathbf{U}}^0\|. \tag{4.9}$$

Now by Grünwall's lemma we can write

$$\begin{aligned} \|\widetilde{\mathbf{U}}^n\| &\leq \left|\sum_{k=1}^{n} w_{n-k}^{(\beta)}\right| \|\mathbf{C}^{-1}\| \, \|\widetilde{\mathbf{U}}^0\| \, \exp\left[\|\mathbf{C}^{-1}\| \left(1 + \sum_{k=1}^{n-2} |w_{n-k}^{(\beta)}|\right)\right] \\ &\leq \|\mathbf{C}^{-1}\| \, \|\widetilde{\mathbf{U}}^0\| \, \exp\left(2\|\mathbf{C}^{-1}\|\right). \end{aligned} \tag{4.10}$$

Lemmas 4.1, 4.2 show that the right hand side in (4.10) is finite and does not tend infinity as $n \to \infty$. So the finite difference scheme (3.12) is unconditionally stable. □

It is obvious that the local truncation error for both schemes is $O(\tau + h^4)$. The following theorems give the convergence properties for the difference schemes (3.8) and (3.12).

Theorem 4.8. *The explicit finite difference scheme (3.8) for solving the fractional Riesz diffusion equation (1.1) is convergent if*

$$r < \frac{6(1+\beta)}{(6+\alpha)2^{\alpha}}.$$

Proof. Let U_m^k be approximate value for $u_m^k = u(x_m, t_k)$ and $e_{m,k}^n = u_m^k - U_m^k$ be the error term at the n-th time level, then with scheme (3.7) and (3.8), it is easy to get the matrix form of error equation as follows

$$\mathbf{E}^n = \mathbf{B}\mathbf{E}^{n-1} - \sum_{k=1}^{n-2} w_{n-k}^{(\beta)} \mathbf{E}^k + \sum_{k=1}^{n} w_{n-k}^{(\beta)} \mathbf{E}^0 + c(\tau^{1+\beta} + \tau^\beta h^4), \tag{4.11}$$

where $\mathbf{E}^k = [e_1^k, e_2^k, \ldots, e_{M-1}^k]^T$ and c is a constant. Then (4.11) can be rewritten as following form

$$\|\mathbf{E}^n\| \leq \quad \|\mathbf{B}\| \ \|\mathbf{E}^{n-1}\| + \sum_{k=1}^{n-2} |w_{n-k}^{(\beta)}| \|\mathbf{E}^k\| + \left|\sum_{k=1}^{n} w_{n-k}^{(\beta)}\right| \|\mathbf{E}^0\| + |c(\tau^{1+\beta} + \tau^\beta h^4)|. \tag{4.12}$$

Now we use the Grünwall's lemma and get

$$\begin{aligned} \|\mathbf{E}^n\| \quad &\leq |c(\tau^{\beta+1} + \tau^\beta h^4)| \exp\left(\|\mathbf{B}\| + \sum_{k=1}^{n-2} |w_{n-k}^{(\beta)}| + \left|\sum_{k=1}^{n} w_{n-k}^{(\beta)}\right| \right) \\ &\leq |c(\tau^{1+\beta} + \tau^\beta h^4)| \exp\left(\|\mathbf{B}\| + 2\right). \end{aligned} \tag{4.13}$$

It is obvious that the right hand side in (4.13) tends zero as h and τ tend zero. □

Next theorem proves the convergence of scheme (3.12).

Theorem 4.9. *The implicit difference scheme (3.12) for solving the fractional Riesz diffusion equation (1.1) is convergent.*

Proof. According to the notations of previous theorem and using(3.11) and (3.12), we write the matrix form of error equation as follows

$$\mathbf{E}^n = \mathbf{C}^{-1}\mathbf{E}^{n-1} - \sum_{k=1}^{n-2} w_{n-k}^{(\beta)} \mathbf{C}^{-1}\mathbf{E}^k + \sum_{k=1}^{n} w_{n-k}^{(\beta)} \mathbf{C}^{-1}\mathbf{E}^0 + c(\tau^{1+\beta} + \tau^\beta h^4). \tag{4.14}$$

Now (4.14) can be rewritten as following format

$$\|\mathbf{E}^n\| \leq \quad \|\mathbf{C}^{-1}\| \ \|\mathbf{E}^{n-1}\| + \sum_{k=1}^{n-2} |w_{n-k}^{(\beta)}| \|\mathbf{C}^{-1}\| \ \|\mathbf{E}^k\| + \left|\sum_{k=1}^{n} w_{n-k}^{(\beta)}\right| \|\mathbf{C}^{-1}\| \ \|\mathbf{E}^0\| + |c(\tau^{1+\beta} + \tau^\beta h^4)|. \tag{4.15}$$

So we use the Grünwall's lemma and get

$$\begin{aligned} \|\mathbf{E}^n\| \quad &\leq |c(\tau^{1+\beta} + \tau^\beta h^4)| \exp\left(\|\mathbf{C}^{-1}\| + \sum_{k=1}^{n-2} |w_{n-k}^{(\beta)}| \|\mathbf{C}^{-1}\| + \left|\sum_{k=1}^{n} w_{n-k}^{(\beta)}\right| \|\mathbf{C}^{-1}\| \right) \\ &\leq |c(\tau^{1+\beta} + \tau^\beta h^4)| \exp\left(3\| \ \mathbf{C}^{-1}\|\right). \end{aligned} \tag{4.16}$$

It is obvious that the right hand side in (4.16) tends zero as h and τ tend zero. □

5. Numerical results

In this section, We present some numerical examples to demonstrate the theoretical analysis. To compare the results numerically, we use the following error norm

$$L_\infty = \max_{0 \leq i \leq M} \left| u_i^N - U_i^N \right|, \tag{5.1}$$

where the exact value u_i^n is approximated by U_i^n [3]. Also, the computational orders of the method presented in this paper (C-order) are calculated with the following formula [3]

$$\text{C-order} = \frac{\log(E_2/E_1)}{\log(h_2/h_1)}, \tag{5.2}$$

in which E_1 and E_2 are errors correspond to grids with mesh size h_1 and h_2, respectively.

Example 1: Consider the time fractional Riesz space fractional diffusion equation (1.1) with the initial and boundary conditions

$$u(0,t) = u(1,t) = 0, \; u(x,0) = 0, \; 0 \leq x \leq 1,$$

and the inhomogeneous term

$$\begin{aligned} g(x,t) = \;& \frac{\Gamma(2\alpha+2)}{\Gamma(2\alpha+2-\beta)} x^4(1-x)^4 - \eta c_\alpha t^{2\alpha+1} \Bigg\{ \frac{12}{\Gamma(5-\alpha)} \left(x^{4-\alpha} + (1-x)^{4-\alpha}\right) \\ & - \frac{240}{\Gamma(6-\alpha)} (x^{5-\alpha} + (1-x)^{5-\alpha}) + \frac{2160}{\Gamma(7-\alpha)} (x^{6-\alpha} + (1-x)^{6-\alpha}) \\ & - \tfrac{10080}{\Gamma(8-\alpha)} (x^{7-\alpha} + (1-x)^{7-\alpha} + \frac{20160}{\Gamma(9-\alpha)} (x^{8-\alpha} + (1-x)^{8-\alpha}) \Bigg\}. \end{aligned}$$

Under these assumptions, the exact solution is $u(x,t) = t^{2\alpha+1} x^4 (1-x)^4$.

This example has been solved for $\beta = 1$ in [4]. We solved this example with $\eta = 1$ for different values of β and α. Figure 1 represents the changes in absolute error concerning time (left) and N (right), which is obtained by implicit method ($M = 20$). Table 1 represents the maximum error at time $t = 1$ for different values of α and β which was obtained using the implicit scheme 2. The results confirm the ones obtained in [4] for $\beta = 1$.
Table 2 shows the errors for the explicit scheme 1. In this table, the dashed line indicates where scheme 1 was unstable for the selected τ and h. Table 3 represents the maximum absolute errors and computational convergence orders for some values of α and β. The computational convergence orders confirm the theoretical convergence order.

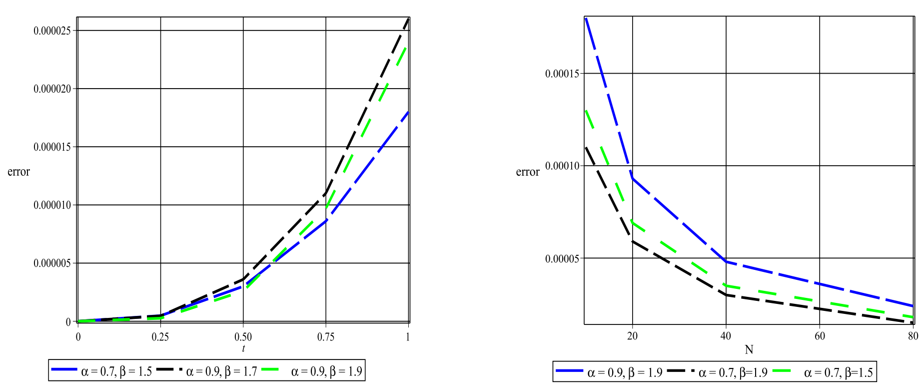

FIGURE 1. The graph of maximum absolute error for Example 1.

Example 2: Consider the time fractional Riesz space fractional diffusion equation (1.1) with the initial and boundary conditions as follows

$$u(0,t) = u(1,t) = 0, \; u(x,0) = 0, \; 0 \leq x \leq 1,$$

and the inhomogeneous term

$$\begin{aligned} g(x,t) = \;& x^2(1-x)^2 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(2k+2-\beta)} t^{2k+1-\beta} - \eta c_\gamma \sin t \left\{ \frac{2}{\Gamma(3-\beta)} \left(x^{2-\beta} + (1-x)^{2-\beta}\right) \right. \\ & \left. - \frac{12}{\Gamma(4-\beta)} \left(x^{3-\beta} + (1-x)^{3-\beta}\right) + \frac{24}{\Gamma(5-\beta)} \left(x^{4-\beta} + (1-x)^{4-\beta}\right) \right\} \end{aligned}$$

TABLE 1. The maximum absolute errors at $t = 1$ for Example 1 (Scheme II).

β		$\alpha = 1.3$	$\alpha = 1.5$	$\alpha = 1.7$	$\alpha = 1.9$
	$\tau = \frac{1}{2},\ h = \frac{1}{4}$	3.1675×10^{-3}	4.2379×10^{-3}	5.5724×10^{-3}	7.2185×10^{-3}
0.5	$\tau = \frac{1}{8},\ h = \frac{1}{8}$	1.1592×10^{-4}	1.0996×10^{-4}	1.0031×10^{-4}	8.7192×10^{-5}
	$\tau = \frac{1}{32},\ h = \frac{1}{16}$	2.8258×10^{-5}	2.6813×10^{-5}	2.4602×10^{-5}	2.1768×10^{-5}
	$\tau = \frac{1}{2},\ h = \frac{1}{4}$	3.1445×10^{-3}	4.1751×10^{-3}	5.4676×10^{-3}	7.0723×10^{-3}
0.7	$\tau = \frac{1}{8},\ h = \frac{1}{8}$	1.8156×10^{-4}	1.7564×10^{-4}	1.6400×10^{-4}	1.4708×10^{-4}
	$\tau = \frac{1}{32},\ h = \frac{1}{16}$	4.5396×10^{-5}	4.4072×10^{-5}	4.1433×10^{-5}	3.7666×10^{-5}
	$\tau = \frac{1}{2},\ h = \frac{1}{4}$	3.1447×10^{-3}	4.1344×10^{-3}	5.3804×10^{-3}	6.9365×10^{-3}
0.9	$\tau = \frac{1}{8},\ h = \frac{1}{8}$	2.6325×10^{-4}	2.5982×10^{-4}	2.4804×10^{-4}	2.2845×10^{-4}
	$\tau = \frac{1}{32},\ h = \frac{1}{16}$	6.6810×10^{-5}	6.6276×10^{-5}	6.3743×10^{-5}	5.9392×10^{-5}
	$\tau = \frac{1}{2},\ h = \frac{1}{4}$	1.1678×10^{-3}	1.1403×10^{-3}	1.0722×10^{-3}	9.6728×10^{-4}
1	$\tau = \frac{1}{8},\ h = \frac{1}{8}$	3.1083×10^{-4}	3.0979×10^{-4}	2.9900×10^{-4}	2.7866×10^{-4}
	$\tau = \frac{1}{32},\ h = \frac{1}{16}$	7.9323×10^{-5}	7.9492×10^{-5}	7.7284×10^{-5}	7.2846×10^{-5}

Under these assumptions, the exact solution is $u(x,t) = x^2(1-x)^2 \sin t$.
We solve this example with $\eta = 1$. Figure 2 represents the changes in absolute error concerning time (left) and N (right), which is obtained by implicit method ($M = 40$). Tables 4 and 5 show the maximum absolute errors for scheme 2 and scheme 1 respectively.

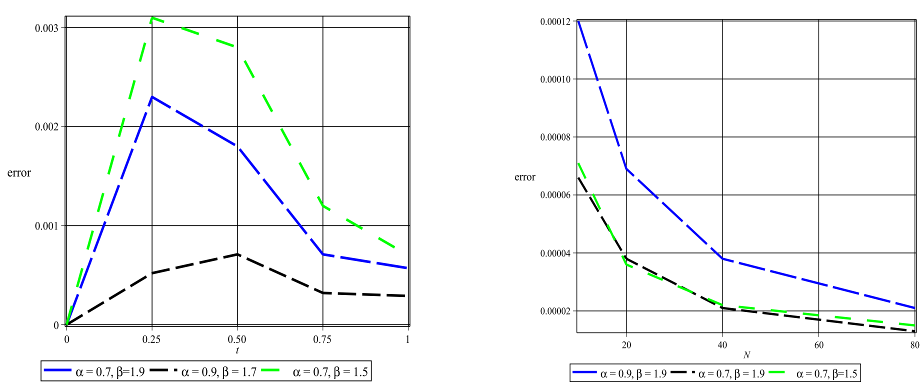

FIGURE 2. The graph of maximum absolute error for Example 2.

Table 2. The maximum absolute errors at $t = 1$ for Example 1 (Scheme I).

β		$\alpha = 1.3$	$\alpha = 1.5$	$\alpha = 1.7$	$\alpha = 1.9$
	$\tau = \frac{1}{200},\ h = \frac{1}{10}$	1.2778×10^{-5}	–	–	–
0.7	$\tau = \frac{1}{400},\ h = \frac{1}{10}$	9.5737×10^{-6}	1.1786×10^{-5}	–	–
	$\tau = \frac{1}{800},\ h = \frac{1}{10}$	7.9739×10^{-6}	1.0174×10^{-5}	–	–
	$\tau = \frac{1}{200},\ h = \frac{1}{10}$	1.2547×10^{-5}	1.4828×10^{-5}	–	–
0.8	$\tau = \frac{1}{400},\ h = \frac{1}{10}$	9.3853×10^{-6}	1.1603×10^{-5}	1.4261×10^{-5}	–
	$\tau = \frac{1}{800},\ h = \frac{1}{10}$	7.8067×10^{-6}	9.9934×10^{-6}	1.2690×10^{-5}	1.4536×10^{-4}
	$\tau = \frac{1}{200},\ h = \frac{1}{10}$	1.2257×10^{-5}	1.4569×10^{-5}	1.7182×10^{-5}	–
0.9	$\tau = \frac{1}{400},\ h = \frac{1}{10}$	9.1623×10^{-6}	1.1378×10^{-5}	1.4530×10^{-5}	1.7321×10^{-5}
	$\tau = \frac{1}{800},\ h = \frac{1}{10}$	7.6171×10^{-6}	9.7851×10^{-6}	1.2457×10^{-5}	1.5828×10^{-5}
	$\tau = \frac{1}{200},\ h = \frac{1}{10}$	1.1905×10^{-5}	1.4235×10^{-5}	1.6870×10^{-5}	2.0015×10^{-5}
1	$\tau = \frac{1}{400},\ h = \frac{1}{10}$	8.9028×10^{-6}	1.1108×10^{-5}	1.3747×10^{-5}	1.7014×10^{-5}
	$\tau = \frac{1}{800},\ h = \frac{1}{10}$	7.4040×10^{-6}	9.5475×10^{-6}	1.2188×10^{-5}	1.5518×10^{-5}

Table 3. The maximum absolute errors and convergence orders for Example 1 (Scheme II).

	$\alpha = 0.9,\ \beta = 1.9$		$\alpha = 0.7,\ \beta = 1.7$	
N	error	C-order	error	C-order
10	1.7973×10^{-4}		1.2606×10^{-4}	
20	9.3135×10^{-5}	0.9484	6.4887×10^{-5}	0.9581
40	4.7505×10^{-5}	0.9713	3.3027×10^{-5}	0.9743
80	2.4087×10^{-5}	0.9798	1.6771×10^{-5}	0.9777
160	1.2225×10^{-5}	0.9784	8.5612×10^{-6}	0.9701

Example 3: Consider the time fractional Riesz-space fractional diffusion equation (1.1) as follows

$$ {}^{C}D_{0,t}^{\beta} u(x,t) = \eta \frac{\partial^{\alpha} u(x,t)}{\partial |x|^{\alpha}}, $$

with the initial and boundary conditions

$$ u(0,t) = u(\pi,t) = 0,\ u(x,0) = x^2(\pi - x),\ 0 \le x \le \pi. $$

TABLE 4. The maximum absolute errors at $t = 1$ for Example 2 (Scheme II).

β		$\alpha = 1.3$	$\alpha = 1.5$	$\alpha = 1.7$	$\alpha = 1.9$
	$\tau = \frac{1}{2},\ h = \frac{1}{4}$	2.8188×10^{-3}	3.9311×10^{-3}	5.3064×10^{-3}	7.1229×10^{-3}
0.5	$\tau = \frac{1}{8},\ h = \frac{1}{8}$	6.7994×10^{-4}	9.5197×10^{-4}	1.2931×10^{-3}	1.7156×10^{-3}
	$\tau = \frac{1}{32},\ h = \frac{1}{16}$	1.7144×10^{-4}	2.4299×10^{-4}	3.2868×10^{-4}	4.3969×10^{-4}
	$\tau = \frac{1}{2},\ h = \frac{1}{4}$	2.4693×10^{-3}	3.6431×10^{-3}	5.0758×10^{-3}	6.8578×10^{-3}
0.7	$\tau = \frac{1}{8},\ h = \frac{1}{8}$	6.3018×10^{-4}	9.1022×10^{-4}	1.2602×10^{-3}	1.6880×10^{-3}
	$\tau = \frac{1}{32},\ h = \frac{1}{16}$	1.6750×10^{-4}	2.3721×10^{-4}	3.2429×10^{-4}	4.3356×10^{-4}
	$\tau = \frac{1}{2},\ h = \frac{1}{4}$	1.8192×10^{-3}	3.1206×10^{-3}	4.6676×10^{-3}	6.4758×10^{-3}
0.9	$\tau = \frac{1}{8},\ h = \frac{1}{8}$	5.2829×10^{-4}	8.3045×10^{-4}	1.2012×10^{-3}	1.6468×10^{-3}
	$\tau = \frac{1}{32},\ h = \frac{1}{16}$	1.5225×10^{-4}	2.2571×10^{-4}	3.1624×10^{-4}	4.2591×10^{-4}
	$\tau = \frac{1}{2},\ h = \frac{1}{4}$	1.3516×10^{-3}	2.7508×10^{-3}	4.3837×10^{-3}	6.2641×10^{-4}
1	$\tau = \frac{1}{8},\ h = \frac{1}{8}$	4.4699×10^{-4}	7.7078×10^{-4}	1.1598×10^{-3}	1.6197×10^{-3}
	$\tau = \frac{1}{32},\ h = \frac{1}{16}$	1.3933×10^{-4}	2.1692×10^{-4}	3.1062×10^{-4}	4.2259×10^{-4}

When $\beta = 1$, the analytic solution is obtained in [19] as follows

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-(n^2)^{\alpha/2}\eta t}, \tag{5.3}$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} u(x,0)\sin(nx)dx = (-1)^{n+1}\frac{8}{n^3} - \frac{4}{n^3}.$$

We plot the exact and approximated solution surfaces in Figure 3 ($\eta = 0.1$). Figure 4 represents the exact and approximate solutions for $\beta = 1$ at $t = 1$. The figure shows that the Numerical solution is in excellent agreement with the exact one. for $\beta < 1$, the exact solution is not known, then we plote the numerical solutions are plotted at $t = 0.5$ for different values of β and α in Figure 5.

Example 4: Consider the time fractional Riesz-space fractional diffusion equation (1.1)

$${}^{C}D_{0,t}^{\beta} u\,(x,t) = \eta \frac{\partial^{\alpha} u\,(x,t)}{\partial\,|x|^{\alpha}},$$

with the initial and boundary conditions

$$u(0,t) = u(\pi,t) = 0,\ u(x,0) = \sin(4x),\ 0 \leq x \leq \pi,$$

When $\beta = 1$, the analytic solution is $u(x,t) = \sin(4x)e^{-4^{\alpha}\eta\ t}$ (see [19]).
We plot the results in Figures 6 and 7.

TABLE 5. The maximum absolute errors at $t = 1$ for Example 2 (Scheme I).

β		$\alpha = 1.3$	$\alpha = 1.5$	$\alpha = 1.7$	$\alpha = 1.9$
	$\tau = \frac{1}{200},\ h = \frac{1}{10}$	4.3494×10^{-4}	–	–	–
0.7	$\tau = \frac{1}{400},\ h = \frac{1}{10}$	4.3406×10^{-4}	1.9781×10^{-3}	–	–
	$\tau = \frac{1}{800},\ h = \frac{1}{10}$	4.3362×10^{-4}	6.1212×10^{-4}	–	–
	$\tau = \frac{1}{200},\ h = \frac{1}{10}$	4.3714×10^{-4}	6.1631×10^{-4}	–	–
0.8	$\tau = \frac{1}{400},\ h = \frac{1}{10}$	4.3557×10^{-4}	6.1462×10^{-4}	8.3632×10^{-4}	–
	$\tau = \frac{1}{800},\ h = \frac{1}{10}$	4.3465×10^{-4}	6.1345×10^{-4}	8.3555×10^{-4}	–
	$\tau = \frac{1}{200},\ h = \frac{1}{10}$	4.3942×10^{-4}	6.2385×10^{-4}	8.4137×10^{-4}	–
0.9	$\tau = \frac{1}{400},\ h = \frac{1}{10}$	4.3725×10^{-4}	6.1101×10^{-4}	8.3936×10^{-4}	1.1233×10^{-3}
	$\tau = \frac{1}{800},\ h = \frac{1}{10}$	4.3529×10^{-4}	6.0321×10^{-4}	8.3628×10^{-4}	1.1218×10^{-3}
	$\tau = \frac{1}{200},\ h = \frac{1}{10}$	4.4445×10^{-4}	6.2360×10^{-4}	8.4585×10^{-4}	1.1348×10^{-3}
1	$\tau = \frac{1}{400},\ h = \frac{1}{10}$	4.4042×10^{-4}	6.2031×10^{-4}	8.4327×10^{-4}	1.1311×10^{-3}
	$\tau = \frac{1}{800},\ h = \frac{1}{10}$	4.3440×10^{-4}	6.1867×10^{-4}	8.4199×10^{-4}	1.1292×10^{-3}

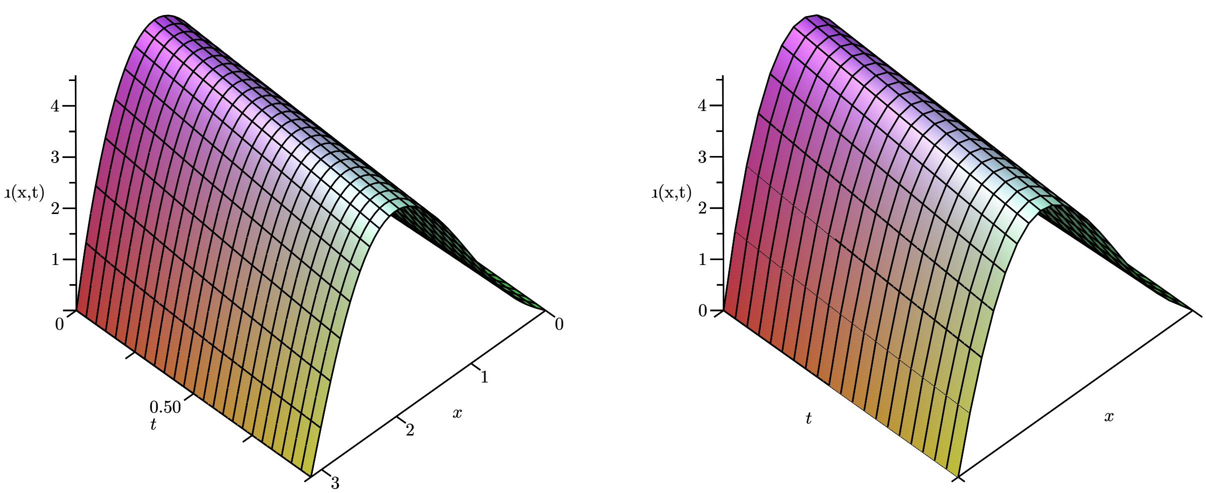

FIGURE 3. The graph for exact solution (left) and approximate solution ($M = 20,\ N = 40$) (right) for Example 3 with $\alpha = 1.9,\ \beta = 1$.

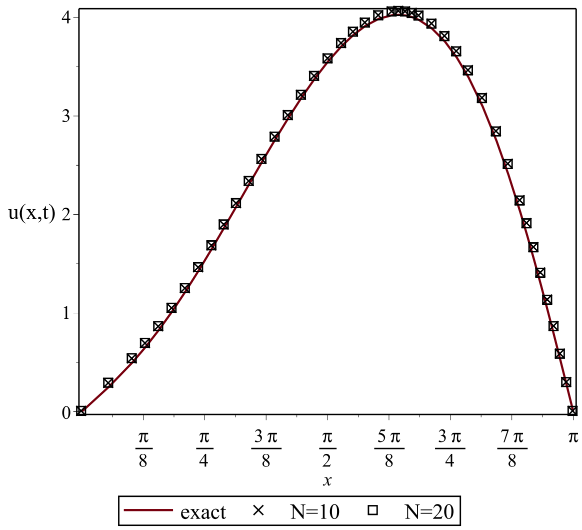

FIGURE 4. The graph for exact and approximate solutions ($M = 20$) for Example 3 with $\alpha = 1.7,\ \beta = 1$.

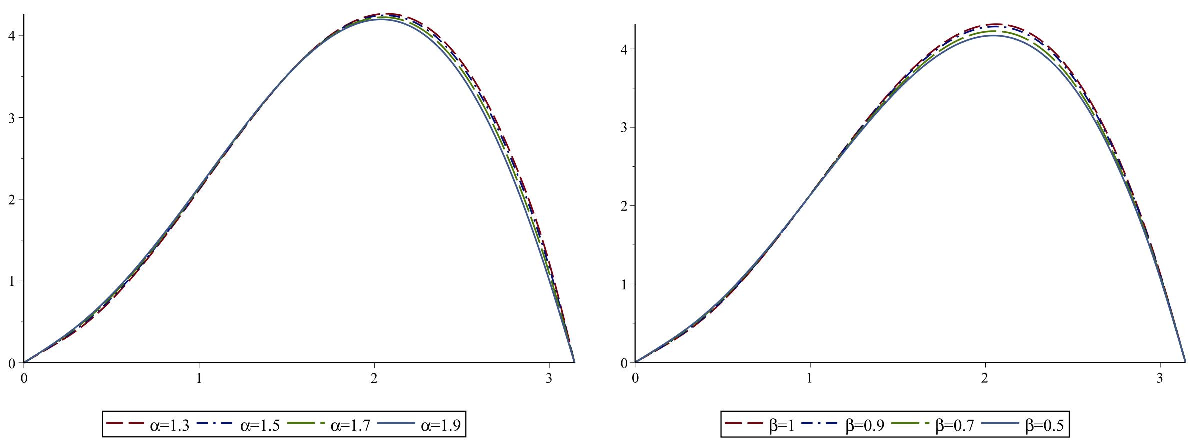

FIGURE 5. The graph for approximate solution with $\beta = 0.9$ (left) and $\alpha = 1.7$ (right) at $t = 0.5$ for Example 3.

6. Conclusion

In this study, one explicit and one implicit finite difference scheme was constructed for the fractional-order diffusion equation with the Riesz space derivative. The stability and convergence study of the numerical methods were provided. It was proven that the explicit finite difference scheme is conditionally stable while the implicit one is stable unconditionally, and the order of convergence is equal to one. The numerical illustrations confirmed the theoretical results.

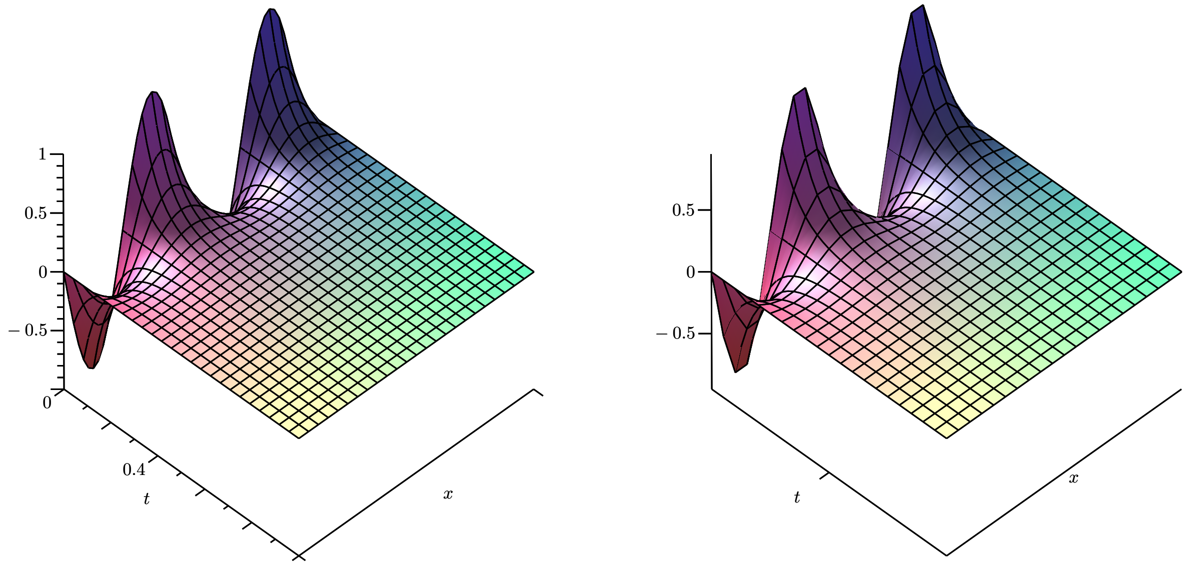

FIGURE 6. The graph for exact solution (left) and approximate solution for ($M = 20,\ N = 40$) (right) for Example 4 with $\alpha = 1.7,\ \beta = 1,\ \eta = 0.1$.

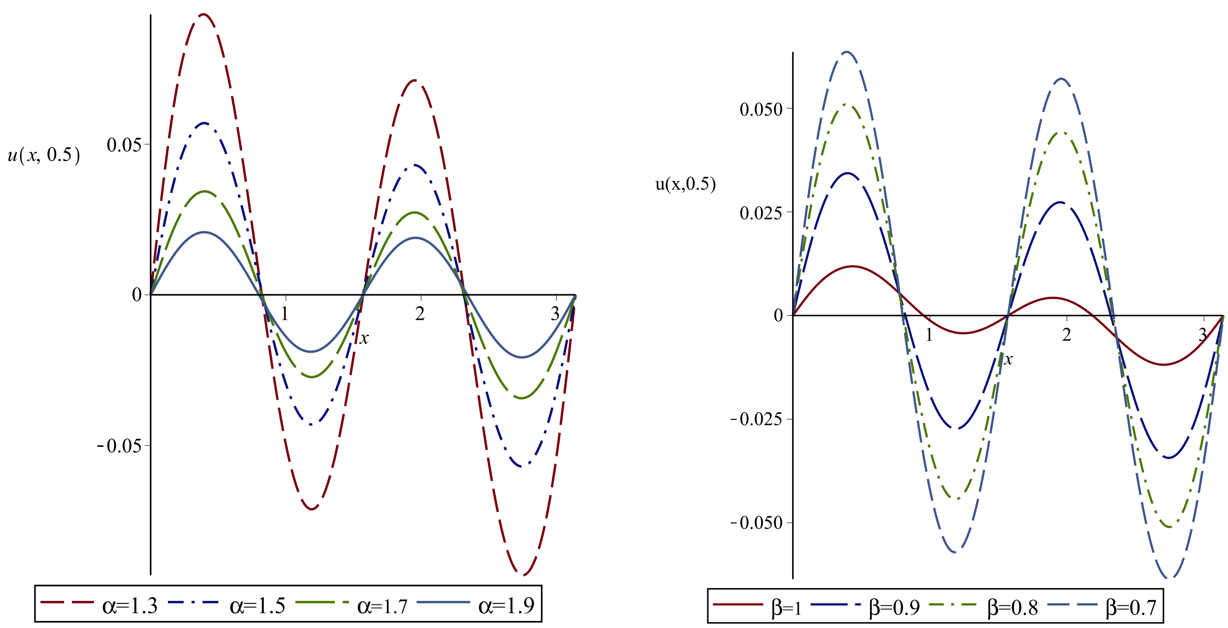

FIGURE 7. The graph for approximate solution with $\beta = 0.7$ (left) and $\alpha = 1.7$ (right) at $t = 0.5$ for Example 4 ($\eta = 0.1$).

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