Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 10, No. 3, 2022, pp. 608-616
DOI:10.22034/cmde.2021.45436.1911

Lie symmetries, exact solutions, and conservation laws of the nonlinear time-fractional Benjamin-Ono equation

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Abstract

In this work, we use the symmetry of the Lie group analysis as one of the powerful tools that deals with the wide class of fractional order differential equation in the Riemann-Liouville concept. We employ the classical Lie symmetries to obtain similarity reductions of nonlinear time-fractional Benjamin-Ono equation and then, we find the related exact solutions for the derived generators. Finally, according to the Lie symmetry generators obtained, we construct conservation laws for related classical vector fields of time-fractional Benjamin-Ono equation.

Keywords. Fractional equation, Lie symmetry analysis, Classical symmetry, Conservation laws.

2010 Mathematics Subject Classification. 76M60, 35R11.

1. Introduction

Lie group transformations were first put forward by Sophus Lie, who was an influential Norwegian mathematician in the early 19th century. These transformations play a fundamental role in the analysis of different kinds of differential equations [5, 6, 11, 17, 20, 22, 28]. The classical Lie symmetry method is utilized for fractional differential equations in [2, 9, 10, 24, 27, 29]. Various branches of the Lie symmetry method are introduced to find additional generators. Fractional calculus theory has received much attention from researchers and scientists in applied mathematics and physics because of its remarkable properties to the accurate description of different abnormal physical events and complex processes in engineering and applied science such that classical calculations were insufficient to explain them [14, 15, 19]. To investigate the fractional calculation theory, several abnormal events have been sampled through integral or fractional differential equations in the real–world.

The aim of this paper is assigned to Lie symmetry analysis of the (1+1)- dimensional Benjamin-Ono equation. We use the symmetry of the Lie group as an appropriate tool that deals with the wide class of fractional order differential equations in the Riemann-Liouville sense. In the current work, firstly, we employ the classical Lie symmetries to obtain similarity reductions of nonlinear generalized time fractional Benjamin-Ono equation. In the next step, we find the relevant exact solutions for the extracted generators, and finally, classical symmetries are used to find conservation laws. In summary, based on the advantages of the obtained exact solutions, one can find this method a promising one to solve different nonlinear time-fractional equations.

2. preliminaries

Now, we present some basic definitions for fractional operators and Lie symmetry analysis of fractional partial differential equations. We advise the readers to read more on these definitions and other properties of fractional differentiation to see [14, 15, 19].

Received: 14 April 2021 ; Accepted: 30 July 2021.
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2.1. **Fractional operators.** Here, we recall the following definition.

Definition 2.1. [14] Riemann–Liouville derivative operator of order α is defined as:

$$\frac{\partial^\alpha}{\partial t^\alpha} f(x,t) = \begin{cases} \dfrac{\partial^n}{\partial t^n} \displaystyle\int_0^t \frac{(t-\sigma)^{n-\alpha-1} f(x,\sigma)}{\Gamma(n-\alpha)} d\sigma & n-1 < \alpha < n, \\ \dfrac{\partial^n}{\partial t^n} f(x,t) & \alpha = n, \qquad n \in \mathbb{N}, \end{cases}$$

where $\Gamma(\cdot)$ denotes the well-known gamma function.

Definition 2.2. [9] The Erdélyi–Kober fractional integral operators for $\mathcal{F}(\zeta)$ is as:

$$\left(\mathcal{K}_\beta^{\tau,\alpha} \mathcal{F}\right) := \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^\infty (s-1)^{\alpha-1} s^{-(\tau+\alpha)} \mathcal{F}\big(\zeta s^{\frac{1}{\beta}}\big) ds, & \alpha > 0, \\ \mathcal{F}(\zeta), \quad \alpha = 0, \end{cases} \tag{2.1}$$

Definition 2.3. [9] The Erdélyi–Kober fractional derivative operators for $\mathcal{F}(\zeta)$ is as:

$$\left(\mathcal{P}_\beta^{\tau,\alpha} \mathcal{F}\right) := \prod_{j=0}^{n-1} \left(\tau + j - \frac{1}{\beta} \zeta \frac{d}{d\zeta}\right) \left(\mathcal{K}_\beta^{\tau+\alpha, n-\alpha} \mathcal{F}\right)(\zeta),$$

$$n = \begin{cases} [\alpha] + 1, & \alpha \notin \mathbb{N}, \\ \alpha, & \alpha \in \mathbb{N}. \end{cases}$$

2.2. **Lie symmetry analysis method for fractional partial differential equation (FPDE).** The following FPDE of order $\alpha \in (0,1)$ is considered as:

$$\Delta \equiv \frac{\partial^\alpha u}{\partial t^\alpha} - \delta(x,t,u,u_x,u_{xx}) = 0, \tag{2.2}$$

where x, t are independent variables, and u is depend to x, t.
Suppose that equation (2.2) is invariant with respect to the following 1-parameter Lie symmetry transformations

$$\begin{aligned} x^\star &= x + \epsilon \xi(x,t,u) + O(\epsilon^2), \\ t^\star &= t + \epsilon \tau(x,t,u) + O(\epsilon^2), \\ u^\star &= u + \epsilon \eta(x,t,u) + O(\epsilon^2), \\ \frac{\partial^\alpha u^\star}{\partial t^{\star\alpha}} &= \frac{\partial^\alpha u}{\partial t^\alpha} + \epsilon \eta^{\alpha,t}(x,t,u) + O(\epsilon^2), \\ \frac{\partial u^\star}{\partial x^\star} &= \frac{\partial u}{\partial x} + \epsilon \eta^x(x,t,u) + O(\epsilon^2), \\ \frac{\partial^2 u^\star}{\partial x^{\star 2}} &= \frac{\partial^2 u}{\partial x^2} + \epsilon \eta^{xx}(x,t,u) + O(\epsilon^2), \end{aligned} \tag{2.3}$$

where the group parameter is shown by ϵ, and its corresponding infinitesimal generator is

$$V = \xi(x,t,u) \frac{\partial}{\partial x} + \tau(x,t,u) \frac{\partial}{\partial t} + \eta(x,t,u) \frac{\partial}{\partial u}. \tag{2.4}$$

This vector field must be creates a symmetry of (2.2) iff the following is true in invariance conditions:

$$Pr^{(\alpha,2)} V(\Delta)\Big|_{\Delta=0} = 0, \tag{2.5}$$

where 2 show the highest integer-orders of extended infinitesimals in the equation of (2.2), and it is worth noting that the extended operator of fractional prolongation $Pr^{(\alpha,2)}V$ is introduced by [9]:

$$Pr^{(\alpha,2)}V = V + \eta^{\alpha,t}\frac{\partial}{\partial(\partial_t^\alpha u)} + \eta^x\frac{\partial}{\partial u_x} + \eta^{xx}\frac{\partial}{\partial u_{xx}}, \tag{2.6}$$

where η^x, η^{xx} and η^{xxx} represent integer-order extended infinitesimals which is defined as:

$$\begin{aligned}
\eta^x &= D_x\eta - D_x(\xi)u_x - D_x(\tau)u_t,\\
\eta^{xx} &= D_x\eta^x - D_x(\xi)u_{xx} - D_x(\tau)u_{xt},
\end{aligned} \tag{2.7}$$

and the operator of total derivative along x, i.e, D_x is:

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + \cdots. \tag{2.8}$$

Also, the α-order extended infinitesimal operator $\eta^{\alpha,t}$ is:

$$\eta^{\alpha,t} = D_t^\alpha\eta + \xi D_t^\alpha u_x - D_t^\alpha(\xi u_x) + D_t^\alpha\left(D_t(\tau)u\right) + \tau D_t^{\alpha+1}u - D_t^{\alpha+1}(\tau u),$$

where D_t^α represent the total α-order fractional derivative in a course of time. Applying the Leibnitz formula and the chain rule [14, 15], the α-order extension of infinitesimal $\eta^{\alpha,t}$ can be obtained explicitly as:

$$\begin{aligned}
\eta^{\alpha,t} =& \frac{\partial^\alpha\eta}{\partial t^\alpha} + \Big(\eta_u - \alpha D_t(\tau)\Big)\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha\eta_u}{\partial t^\alpha}\\
&+ \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^n\eta_u}{\partial t^n} - \binom{\alpha}{n+1}D_t^{n+1}(\tau)\right]D_t^{\alpha-n}(u)\\
&- \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)D_t^{\alpha-n}(u_x) + \mu_1,
\end{aligned} \tag{2.9}$$

where

$$\mu_1 = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{r=2}^{m}\sum_{s=0}^{r-1}\binom{\alpha}{n}\binom{n}{m}\binom{r}{s}\frac{t^{n-\alpha}}{r!\Gamma(n-\alpha+1)}(-1)^s \times \left(u^s\frac{\partial^m(u^{r-s})}{\partial t^m}\frac{\partial^{n-m+r}\eta}{\partial t^{n-m}\partial u^r}\right).$$

Furthermore, since η is linear w.r.t variable u, then derivatives $\dfrac{\partial^n\eta}{\partial u^n}$ for $n \geq 2$ are vanished and immediately we conclude that $\mu_1 = 0$.

3. Implementation of Lie symmetry analysis method for time–fractional Benjamin-Ono equation

In this section, we are interested to discuss about the fractional version of a well-known Benjamin-Ono equation. This equation with integer order is analyzed and investigated in literature from the many points of view [7, 8].

Time-fractional Benjamin-Ono equation is as the following form[1, 18, 25]

$$\mathcal{F}: \ \frac{\partial^\alpha}{\partial t^\alpha}u + Hu_{xx} + uu_x = 0, \tag{3.1}$$

where H is the Hilbert transform. We suppose that it is not dependent on x and t, then it is an arbitrary constant. The Benjamin-Ono (BO) equation is a fully integrable equation which used to describe internal waves. This equation has N-soliton solutions.

3.1. **Classical symmetries.** If we apply prolongation of the fractional vector field to the this equation, we obtain

$$\eta^\alpha + u_x\eta + u\eta^x + H\eta^{xx} = 0, \tag{3.2}$$

by substituting (2.7) and (2.9) in (3.2) and solving the determining equations we obtain the following symmetry generators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \alpha x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - \alpha u\frac{\partial}{\partial u}. \tag{3.3}$$

Invariant solution of vector field X_1 is $u(x,t) = f(t)$, then equation (3.1) reduced to the following time-fractional ordinary differential equation (FODE)

$$\frac{\partial^\alpha}{\partial t^\alpha} f(t) = 0, \tag{3.4}$$

hence, the exact solution of equation (3.1) is $u(x,t) = c_1 t^{\alpha-1}$.
According to the vector field X_2 we have the following invariant solution for equation (3.1)

$$u(x,t) = t^{\frac{-\alpha}{2}} f(\zeta), \qquad \zeta = xt^{\frac{-\alpha}{2}}. \tag{3.5}$$

Theorem 3.1. *Using the above transformation and substituting in equation (3.1) reduced the governing equation and obtain the following FODE*

$$\left(\mathcal{P}^{1-\frac{3\alpha}{2},\alpha}_{\frac{2}{\alpha}} f\right)(\zeta) + Hf''(\zeta) + f(\zeta)f'(\zeta) = 0. \tag{3.6}$$

Proof. Let $\alpha \in (n-1,n), n \in \mathbb{N}$, the Rieman-Lioville fractional derivative of relation (3.5) can be written as follows

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1} s^{-\frac{\alpha}{2}} F(xs^{-\frac{\alpha}{2}})ds\right], \tag{3.7}$$

let $w = \dfrac{t}{s}$, then $ds = \dfrac{-t}{w^2}dw$. Equation (3.7) becomes

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left[\frac{t^{n-\frac{3\alpha}{2}}}{\Gamma(n-\alpha)}\int_1^\infty (w-1)^{n-\alpha-1} w^{\frac{3\alpha}{2}-n-1} F(xt^{-\frac{\alpha}{2}} w^{\frac{\alpha}{2}})dw\right]$$

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left[t^{n-\frac{3\alpha}{2}}\left(\mathcal{K}^{1-\frac{\alpha}{2},n-\alpha}_{\frac{2}{\alpha}} F\right)(\zeta)\right].$$

Furthermore, considering $\zeta = xt^{-\frac{\alpha}{2}}$, $\rho \in (0,\infty)$ gives

$$t\frac{\partial}{\partial t}\rho(\zeta) = t\frac{\partial\zeta}{\partial t}\frac{d\rho(\zeta)}{d\zeta} = -\frac{\alpha}{2}\zeta\frac{d\rho(\zeta)}{d\zeta}.$$

Hence,

$$\begin{aligned}
\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{3\alpha}{2}}\left(\mathcal{K}^{1-\frac{\alpha}{2},n-\alpha}_{\frac{2}{\alpha}} F\right)(\zeta)\right] &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\frac{3\alpha}{2}}\left(\mathcal{K}^{1-\frac{\alpha}{2},n-\alpha}_{\frac{2}{\alpha}} F\right)(\zeta)\right)\right] \\
&= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\frac{3\alpha}{2}-1}\left(n - \frac{3\alpha}{2} - \frac{\alpha}{2}\zeta\frac{d}{d\zeta}\right)\left(\mathcal{K}^{1-\frac{\alpha}{2},n-\alpha}_{\frac{2}{\alpha}} F\right)(\zeta)\right] \\
&\vdots \\
&= t^{-\frac{3\alpha}{2}}\Pi_{j=0}^{n-1}\left(1 - \frac{3\alpha}{2} + j - \frac{\alpha}{2}\zeta\frac{d}{d\zeta}\right)\left(\mathcal{K}^{1-\frac{\alpha}{2},n-\alpha}_{\frac{2}{\alpha}} F\right)(\zeta) \\
&= t^{-\frac{3\alpha}{2}}\left(\mathcal{P}^{1-\frac{3\alpha}{2},\alpha}_{\frac{2}{\alpha}} F\right)(\zeta).
\end{aligned}$$

Thus,

$$\frac{\partial^\alpha u}{\partial t^\alpha} = t^{-\frac{3\alpha}{2}}\left(\mathcal{P}^{1-\frac{3\alpha}{2},\alpha}_{\frac{2}{\alpha}} F\right)(\zeta).$$

$\square$

We can find the exact solution of (3.6) by using power series method. Let

$$f(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n, \tag{3.8}$$

then,

$$\begin{aligned} f'(\zeta) &= \sum_{n=0}^{\infty} a_{n+1}(n+1)\zeta^n, \\ f''(\zeta) &= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)\zeta^n, \\ \left(\mathcal{P}_{\beta}^{\tau,\alpha} f\right)(\zeta) &= \sum_{n=0}^{\infty} a_n \frac{\Gamma(\tau - \frac{n}{\beta} + 1)}{\Gamma(\tau - \frac{n}{\beta} + 1 - \alpha)} \zeta^n, \end{aligned} \tag{3.9}$$

substituting above relation in (3.6) we have

$$\sum_{n=0}^{\infty} a_n \frac{\Gamma(1 - \frac{3}{2} - \frac{n\alpha}{2} + 1)}{\Gamma(1 - \frac{5\alpha}{2} - \frac{n\alpha}{2} + 1)} \zeta^n + \sum_{n=0}^{\infty} \sum_{k=0}^{n} (n-k+1) a_k a_{n-k+1} \zeta^n + H \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} \zeta^n = 0. \tag{3.10}$$

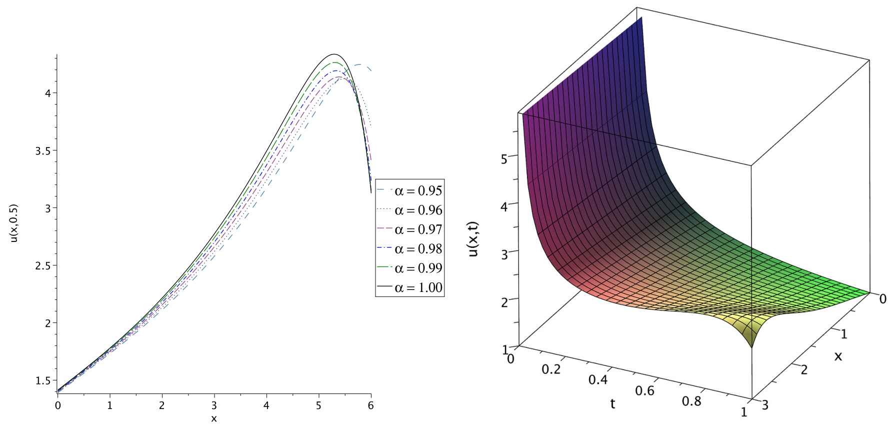
According to the equation (3.10), for $n \geq 0$ we have the following recursive relation

$$a_{n+2} = \frac{-1}{(n+1)(n+2)H} \left(a_n \frac{\Gamma(2 - \frac{3\alpha}{2} - \frac{n\alpha}{2})}{\Gamma(2 - \frac{5\alpha}{2} - \frac{n\alpha}{2})} + \sum_{k=0}^{n} (n-k+1) a_k a_{n-k+1} \right), \tag{3.11}$$

also a_0 and a_1 are arbitrary constants. According to (3.5), the exact solution of equation (3.1) is

$$u(x,t) = t^{\frac{-\alpha}{2}} \sum_{n=0}^{\infty} a_n (x t^{\frac{-\alpha}{2}})^n. \tag{3.12}$$

In order to illustrate the related plots of the obtained exact solution (3.12), we truncate its series with $N = 24$. Figure 1 shows the plots of $u(x,t)$ for $a_0 = 1, a_1 = 0.5$ and $H = 1$. Figure 1 (A) shows the convergence properties of the obtained exact solutions to the exact solution of the integer-order Benjamin-Ono equation when α tends to 1 and Figure 1 (B) shows the 3D plots of (3.12) for the value of $\alpha = 0.9$. In [21, 23], the fractional Benjamin-Ono equation with conformable fractional derivative is discussed. However, to the best of authors knowledge, there is no paper about the exact solutions of fractional Benjamin-Ono equation with Riemann-Liouville derivative.



(a) The plot of $u(x,0.5)$ for different values of α.

(b) The plot of $u(x,t)$ in 3D for the value of $\alpha = 0.9$.

Figure 1. The related plots of the exact solutions of (3.12) which obtained by classical generator of X_2 for different values of α in 2D and 3D.

4. Conservation laws of the time-fractional Benjamin-Ono equation

In this part, we attention to the nonlinear self–adjointness [12] and new conservation laws theorem that firstly used to construct the conservation laws by Ibragimov in [13]. This theorem uses the Lie symmetries to find conserved vectors [3, 4, 16, 26]. Here, we intend to construct conservation laws for (3.1) according to this method. Conservation laws of equation (3.1) is given by

$$D_t\mathcal{T}_i^t + D_x\mathcal{T}_i^x = 0, \tag{4.1}$$

where $\mathcal{T}_i^t = \mathcal{T}_i^t(t,x,u,...)$, $\mathcal{T}_i^x = \mathcal{T}_i^x(t,x,u,...)$ are the conserved vectors

$$\begin{aligned} \mathcal{T}_i^x &= \left({}^u\mathcal{W}_i\frac{\delta\mathcal{H}}{\delta u_x} + \sum_{k\geq 1} D_x...D_x({}^u\mathcal{W}_i)\frac{\partial\mathcal{H}}{\partial u_{(k+1)x}}\right), \\ \mathcal{T}_i^t &= \sum_{k=0}^{n-1}(-1)^k\partial_t^{\alpha-1-k}({}^u\mathcal{W}_i)D_t^k\left(\frac{\partial\mathcal{H}}{\partial(\partial_t^\alpha u)}\right) - (-1)^n\mathcal{J}\left({}^u\mathcal{W}_i, D_t^n(\frac{\partial\mathcal{H}}{\partial(\partial_t^\alpha u)})\right), \end{aligned} \tag{4.2}$$

where $n = [\alpha]+1$, ${}^u\mathcal{W}_i = \eta_i - \xi_i u_x - \tau_i u_t$ and the integral $\mathcal{J}$ is given by

$$\mathcal{J}(h,g) = \int_0^t\int_t^T \frac{h(\lambda,x)g(\mu,x)(\mu-\lambda)^{n-(\alpha+1)}}{\Gamma(n-\alpha)}d\lambda d\mu. \tag{4.3}$$

The formal Lagrangian for (3.1) is given by

$$\mathcal{H} = \Phi(x,t)\left(\frac{\partial^\alpha u}{\partial t^\alpha} + uu_x + Hu_{xx}\right), \tag{4.4}$$

where $\Phi(x,t) = \Psi(x,t,u)$ is new dependent variable and the adjoint equations of the fractional Benjamin-Ono equation can be specified as follows

$$\mathcal{F}^* \equiv \frac{\delta\mathcal{H}}{\delta u} = 0, \tag{4.5}$$

where the operators of Euler-Lagrange are given by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (\partial_t^\alpha)^*\frac{\partial}{\partial(\partial_t^\alpha u)} + \sum_{k\geq 1}(-1)^k D_x...D_x\frac{\partial}{\partial u_{kx}},$$

where $(\partial_t^\alpha)^*$ denotes the adjoint operator for the ∂_t^α. According to the Riemann–Liouville fractional differential operator

$$(\partial_t^\alpha)^* = (-1)^n \mathcal{I}_T^{n-\alpha}(\partial_t^n) = (\partial_T^\alpha)_t^{\mathcal{C}},$$
$$\mathcal{I}_T^{n-\alpha} h(t,x) = \int_t^T \frac{h(\tau,x)(\tau-t)^{n-(1+\alpha)}}{\Gamma(n-\alpha)} d\tau, \qquad n = [\alpha]+1,$$

where $(\partial_T^\alpha)_t^{\mathcal{C}}$ is the operator of right-sided Caputo derivative.
If we have the following relation for the time-fractional nonlinear equation (4.4), then we can say that the equation (3.1) is self adjoint

$$\mathcal{F}^* \equiv \frac{\delta \mathcal{H}}{\delta u} = \mu F, \tag{4.6}$$

where μ unknowns to be determined. Thus, we can write the nonlinear self adjoint condition (4.6) as follows

$$\mu = 0, \qquad \Psi(x,t,u) = A, \qquad A \in \mathbb{R}.$$

Hence, if we suppose $A = 1$, then

$$\mathcal{H} = \frac{\partial^\alpha u}{\partial t^\alpha} + u u_x + H u_{xx}. \tag{4.7}$$

According to the above analysis and Lie symmetry generators, we consider the conserved vectors for classical generators of time-fractional Benjamin-Ono equation. We have the following cases for classical generators:
Case 1: Here for the first classical generator X_1, the respective Lie characteristic function is

$${}^u\mathcal{W}_1 = -u_x. \tag{4.8}$$

Substituting (4.8) into (4.2) yields the conserved vector as follows:

$$\mathcal{T}_1^x = {}^u\mathcal{W}_1\left(\frac{\partial \mathcal{H}}{\partial u_x} - D_x \frac{\partial \mathcal{H}}{\partial u_{xx}}\right) + D_x({}^u\mathcal{W}_1)\frac{\partial \mathcal{H}}{\partial u_{xx}},$$
$$\mathcal{T}_1^t = \mathcal{I}^{1-\alpha}({}^u\mathcal{W}_1)\Psi + \mathcal{J}\Big({}^u\mathcal{W}_1, \Psi_t\Big),$$

then

$$\mathcal{T}_1^x = -u_x(u-H) - H u_{xx}$$
$$\mathcal{T}_1^t = -\mathcal{I}^{1-\alpha} u_x \Psi + \mathcal{J}\big(-u_x, \Psi_t\big).$$

Case 2: For the generator $X_2 = \alpha x \frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - \alpha u \frac{\partial}{\partial u}$. the respective Lie characteristic function is obtained by

$${}^u\mathcal{W}_2 = -\alpha u - \alpha x u_x - 2t u_t. \tag{4.9}$$

Substituting (4.9) into (4.2) yields

$$\mathcal{T}_2^x = -\left(u-H\right)\left(\alpha u + \alpha x u_x + 2t u_t\right) - H\left(2\alpha u_x + \alpha x u_{xx} + 2t u_{xt}\right),$$
$$\mathcal{T}_2^t = -\mathcal{I}^{1-\alpha}(\alpha u + \alpha x u_x + 2t u_t).$$

5. Conclusion

In this work, we used the symmetry of the Lie group analysis for nonlinear time-fractional Benjamin-Ono equation and employed the classical Lie symmetries to obtain similarity reductions of this equation and find the related exact solutions for the derived generators. Finally, according to the Lie symmetry generators obtained, we construct conservation laws for related classical vector fields of the mentioned equation.

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