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A novel local meshless scheme based on the radial basis function for pricing multi-asset options

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Abstract

A novel local meshless scheme based on the radial basis function (RBF) is introduced in this article for price multi-asset options of even European and American types based on the Black-Scholes model. The proposed approach is obtained by using operator splitting and repeating the schemes of Richardson extrapolation in the time direction and coupling the RBF technology with a finite-difference (FD) method that leads to extremely sparse matrices in the spatial direction. Therefore, it is free of the ill-conditioned difficulties that are typical of the standard RBF approximation. We have used a strong iterative idea named the stabilized Bi-conjugate gradient process (BiCGSTAB) to solve highly sparse systems raised by the new approach. Moreover, based on a review performed in the current study, the presented scheme is unconditionally stable in the case of independent assets when spatial discretization nodes are equispaced. As seen in numerical experiments, it has a low computational cost and generates higher accuracy. Finally, the proposed local RBF scheme is very versatile so that it can be used easily for solving numerous models and obstacles not just in the finance sector, as well as in other fields of engineering and science.

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1. Introduction

An option is a contract that provides the right, but not the obligation, to buy or sell one unit of a risky asset at a predetermined fixed price (strike price) within a determined period (maturity). Typically, two parties are involved in an option. One party is the writer who specifies the terms of the contract and sells the option. Another party is the holder who buys the option by paying the market price which is called option price. One of the hottest topics in financial markets is option pricing referred to compute a fair value of the option. There are two types of options: The call option which gives the holder the right to buy the underlying assets and the put option which gives the holder the right to sell them. Both kinds of options can be traded as a European style so that the option should be exercised only at the maturity or as an American style so that it also can be exercised at any time up to the maturity. Various mathematical models for option pricing have been developed based on partial differential equations. One of such models is the Black-Scholes (BS) model which is presented by Black and Scholes [6] in which the volatility (or standard deviation) of the option's underlying asset is assumed to be constant.

Some traditional mesh-based methods for solving the B-S equation are the finite difference (FD) and the finite element (FE) methods [32, 35]. They all suffer from mesh and mesh difficulties and also cannot develop to high dimensions easily. Recently, to deal with such problems, a new generation of meshless techniques on radial basis functions (RBFs) has also been developed. Authors in [13, 14, 19, 20, 27] have considered the one-dimensional B-S equation using such methods. However, most of them required solving a large full system matrix arisen from ill-conditioned systems. Authors in [1] and [2] have presented radial basis point interpolation (RBPI) and radial point interpolation (RPI) schemes respectively to overcome this difficulty. On the other hand, efforts for solving multi-dimensional B-S equation

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using meshless methods are extremely scarce, however, some works have been done [3, 18, 23]. For example, Ballestra and Pacelli have considered two-dimensional case by combining Gaussian radial basis functions with operator splitting scheme [3]. Some efforts have been done to use radial basis function generated finite difference (RBF-FD) approach for option pricing [16].

2. The B-S model for basket options

In this section, the B-S model is introduced for the basket options which means the underlying assets are two or more. Let the price of each underlying assets be S_i, $i = 1, 2, \ldots, d$, in the subsequent stochastic differential formula in the risk-neutral scale [15, 37]

$$dS_i = (r - D_i)S_i dt + \sigma_i S_i dW_i, \quad i = 1, 2, \ldots, d, \tag{2.1}$$

in which the constant interest rate is r. In addition, D_i and σ_i are respectively the dividend yield and the volatility of the ith asset which are constant, as well as, W_i shows a standard Wiener process for ith asset. If ρ_{ij} is the correlation coefficient between both the ith and jth processes, then the symmetric matrix with ρ_{ij} is named the correlation matrix in the following form as the entry in the ith row and jth column.

$$\mathbf{\Sigma} = \begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1d} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{d1} & \rho_{d2} & \cdots & \rho_{dd} \end{pmatrix}. \tag{2.2}$$

Notice that the correlation matrix, $\rho_{ii} = 1$ and $\rho_{ij} = \rho_{ji}$ are positive definite. Also, assume that $V(\mathbf{S}, t)$ be the option price at the vector of underlying assets $\underline{\mathbf{S}} = (S_1, \ldots, S_d) \in \Omega$ and time $t \in [0, T)$ where

$$\mathbf{\Omega} = (0, +\infty) \times \ldots \times (0, +\infty) \tag{2.3}$$

and T is the maturity. The following multi-dimensional linear parabolic partial differential problem named B-S equation is also satisfied by $V(\underline{\mathbf{S}}, t)$.

$$\frac{\partial V(\underline{\mathbf{S}}, t)}{\partial t} + \frac{1}{2}\sum_{p=1}^{d}\sum_{j=1}^{d} \sigma_p \sigma_j \rho_{pj} S_p S_j \frac{\partial^2 V(\underline{\mathbf{S}}, t)}{\partial S_p \partial S_j} + \sum_{p=1}^{d}(r - D_p)S_p \frac{\partial V(\underline{\mathbf{S}}, t)}{\partial S_p} - rV(\underline{\mathbf{S}}, t) = 0. \tag{2.4}$$

One can obtain the European call problem by attaching the following final condition to (2.4)

$$V(\underline{\mathbf{S}}, T) = \mathcal{G}_c(\underline{\mathbf{S}}), \tag{2.5}$$

where $\mathcal{G}_c(\underline{\mathbf{S}})$ is called the payoff function and defined as follows

$$\mathcal{G}_c(\underline{\mathbf{S}}) = \max\Big(\sum_{j=1}^{d} \alpha_j S_j - K, 0\Big). \tag{2.6}$$

By taking its final condition as continues, then it changes to a European issue.

$$V(\underline{\mathbf{S}}, T) = \mathcal{G}_p(\underline{\mathbf{S}}), \tag{2.7}$$

where the payoff function in this case is

$$\mathcal{G}_p(\underline{\mathbf{S}}) = \max\Big(K - \sum_{j=1}^{d} \alpha_j S_j, 0\Big). \tag{2.8}$$

As well as, boundary conditions are derived from the exact solution given by [36]. Further details about the boundary conditions are provided in the subsequent sections.

3. RBF-FD scheme

As it is customary, using a global RBF approximation results in extremely dense and ill-conditioned RBF matrices, especially when the matrix dimension is enhanced. While Kansa et. al. [17] are presented some techniques to overcome this difficulty, others uses variable shape parameter strategies That corresponds to different shape parameters for each center due to decrease condition number [21, 22, 29].

To reduce the density and condition number of RBF matrices, we can use the remarkable idea of finite difference (FD) method. It uses local approximations to localize the collocation method which leads to sparse matrices. Let L be a linear partial differential operator, $\Xi = \{\underline{\mathbf{x}}_1, \ldots, \underline{\mathbf{x}}_N\} \subseteq \mathbb{R}^d$ be a set of test points and $\Xi^{(i)} = \{\underline{\mathbf{x}}_1^{(i)}, \ldots, \underline{\mathbf{x}}_{n_i}^{(i)}\} \subseteq \Xi$ be a subset of nearby nodes to $\underline{\mathbf{x}}_i$ named a stencil relevant to $\underline{\mathbf{x}}_i$ including n_i nodes so that $\underline{\mathbf{x}}_i \in \Xi^{(i)}$ and $n_i < N$. The FD method approximates $Lu(\underline{\mathbf{x}}_\mathbf{i})$ by a linear combination of function $u(\underline{\mathbf{x}})$ values, as follow

$$Lu(\underline{\mathbf{x}}_i) \simeq \sum_{j=1}^{n_i} \omega_j^{(i)} u(\underline{\mathbf{x}}_j^{(i)}), \tag{3.1}$$

then, the unknown weights $\{\omega_j^{(i)}\}_{j=1}^{n_i}$ are calculated by the polynomial interpolation [11]. This scheme for computing unknown weights is possible only for some types of nodal structures specially in high dimensions, for example, 3 uniform nodes or 5 uniform nodes in one or two dimension respectively which extremely limits the geometric flexibility of the FD method [9].

To overcome this difficulty related to calculating the unknown weights $\{\omega_j^{(i)}\}_{j=1}^{n_i}$, one can use this idea that the relation (3.1) must be held for RBFs centered at nodes inside the stencil $\Xi^{(i)}$ [24–26, 31], *i.e.* $\{\phi_j(\underline{\mathbf{x}}, c)\}_{j=1}^{n_i}$, so that

$$L\phi_k(\underline{\mathbf{x}}_i, c) = \sum_{j=1}^{n_i} \omega_j^{(i)} \phi_j(\underline{\mathbf{x}}_k, c), \quad k = 1, \ldots, n_i. \tag{3.2}$$

By collocating (3.2) at nodes of $\Xi^{(i)}$, one can obtain the following linear system

$$\mathbf{\Phi}\omega^{(i)} = [\mathbf{L\Phi}]^{(i)}, \tag{3.3}$$

where the entries of interpolant matrix $\mathbf{\Phi}$ of dimension $n_i \times n_i$ are as follows

$$\phi_{kj} = \phi_j(\underline{\mathbf{x}}_k, c), \quad k, j = 1, \ldots, n_i. \tag{3.4}$$

The vector $\omega^{(i)}$ of dimension $n_i \times 1$ contains the unknown weights $\{\omega_j^{(i)}\}_{j=1}^{n_i}$ called RBF-FD coefficients, and $[\mathbf{L\Phi}]^{(i)}$ is a $n_i \times 1$ vector containing $L\phi_k(\underline{\mathbf{x}}_i, c)$ for $k = 1, \ldots, n_i$. For determining the weights vector $\omega^{(i)}$, one can use the fact of nonsingularity of the matrix $\mathbf{\Phi}$ [7, 28, 34] so that

$$\omega^{(i)} = \mathbf{\Phi}^{-1}[\mathbf{L\Phi}]^{(i)}. \tag{3.5}$$

The above approach which is formed by coupling the FD and RBF methods is called RBF-FD approach. In fact, these can be perceived as an improved FD procedure because they are the same except in the scheme of computing the unknown weights $\{\omega_j^{(i)}\}_{j=1}^{n_i}$.

The global method's major advantage is its simplicity of programming and possible spectral precision, but its main disadvantage is the resulting linear system's ill-conditioning. One of the really recent and innovative solutions to this problem is to localize the collocation process. It uses local estimations to generate sparse system matrices. One of the local RBF methods proposes applying RBF in Finite Difference Mode [4, 5, 8]. A local distinction has been extended to RBFs, and it is widely used in RBF research, particularly when dealing with time-dependent PDEs. Recently, so many effects of the local RBF-FD approach are considered in comparison with the RBF and FD approaches [10, 12, 30]. Bayona et al. [5] employed the Maple software and have presented some formulae for RBF-FD coefficients when L is just the first or second order derivative of the processor and the MQ function as basis function is used. Some of their results for stencils with $n_i = 3, 5$ uniform nodes are reported in Table 2 in the limit $c \gg h$. We apply results presented in this table when Ξ is equispaced.

Table 1. RBF-FD for the first and second derivatives.

Node	First derivative		Second derivative	
	$N=3$	$N=5$	$N=3$	$N=5$
S_{i-2h}		$\frac{1}{12h}\left(1+\frac{8h^2}{c^2}\right)$		$-\frac{1}{12h^2}\left(1+\frac{74h^2}{7c^2}\right)$
S_{i-h}	$-\frac{1}{2h}\left(1+\frac{h^2}{2c^2}\right)$	$-\frac{2}{3h}\left(1+\frac{2h^2}{c^2}\right)$	$\frac{1}{h^2}\left(1+\frac{h^2}{c^2}\right)$	$\frac{4}{3h^2}\left(1+\frac{37h^2}{14c^2}\right)$
S_i	0	0	$-\frac{2}{h^2}\left(1+\frac{h^2}{c^2}\right)$	$-\frac{5}{2h^2}\left(1+\frac{74h^2}{35c^2}\right)$
S_{i+h}	$\frac{1}{2h}\left(1+\frac{h^2}{2c^2}\right)$	$\frac{2}{3h}\left(1+\frac{2h^2}{c^2}\right)$	$\frac{1}{h^2}\left(1+\frac{h^2}{c^2}\right)$	$\frac{4}{3h^2}\left(1+\frac{37h^2}{14c^2}\right)$
S_{i+2h}		$-\frac{1}{12h}\left(1+\frac{8h^2}{c^2}\right)$		$-\frac{1}{12h^2}\left(1+\frac{74h^2}{7c^2}\right)$

4. Spatial and time discretization

In this section, we are continuing to develop the RBF-FD method to solve the multi-dimensional B-S equation (2.4). For simplicity in formulation, we focus on two-dimensional put option *i.e.* we put $d = 2$ in (2.4), however, one can easily employ it for the call option or higher dimensions with a little modifications.
In the first place, let us impose the change of variables

$$\tau = T - t, \quad \tilde{V}(\underline{\mathbf{S}}, \tau) = e^{r\tau} V(\underline{\mathbf{S}}, T - \tau), \tag{4.1}$$

therefore, Problem (2.4) and (2.7) turns to

$$\frac{\partial \tilde{V}(\underline{\mathbf{S}}, \tau)}{\partial \tau} = \tilde{\alpha}_1(\underline{\mathbf{S}}) \frac{\partial^2 \tilde{V}(\underline{\mathbf{S}}, \tau)}{\partial S_1^2} + \tilde{\zeta}(\underline{\mathbf{S}}) \frac{\partial^2 \tilde{V}(\underline{\mathbf{S}}, \tau)}{\partial S_1 \partial S_2} + \tilde{\alpha}_2(\underline{\mathbf{S}}) \frac{\partial^2 \tilde{V}(\underline{\mathbf{S}}, \tau)}{\partial S_2^2} \quad \tilde{\beta}_1(\underline{\mathbf{S}}) \frac{\partial \tilde{V}(\underline{\mathbf{S}}, \tau)}{\partial S_1} + \tilde{\beta}_2(\underline{\mathbf{S}}) \frac{\partial \tilde{V}(\underline{\mathbf{S}}, \tau)}{\partial S_2}, \tag{4.2}$$

and

$$\tilde{V}(\underline{\mathbf{S}}, \tau) = \mathcal{G}_p(\underline{\mathbf{S}}), \tag{4.3}$$

where $\underline{\mathbf{S}} = (S_1, S_2)$ and $\mathcal{G}_p(\underline{\mathbf{S}})$ can be calculated as before using (2.8) when $d = 2$, also

$$\tilde{\alpha}_i(\underline{\mathbf{S}}) = \frac{1}{2}\sigma_i^2 S_i^2, \quad \tilde{\beta}_i(\underline{\mathbf{S}}) = (r - D_i)S_i, \quad i = 1, 2, \quad \tilde{\zeta}(\underline{\mathbf{S}}) = \sigma_1 \sigma_2 \rho S_1 S_2. \tag{4.4}$$

Also, we consider the following boundary conditions

$$\tilde{V}(0, S_2, \tau) = e^{r\tau} \alpha_2 f(S_2, \frac{K}{\alpha_2}, \tau), \quad \lim_{S_1 \to +\infty} \tilde{V}(S_1, S_2, \tau) = 0, \tag{4.5}$$

$$\tilde{V}(S_1, 0, \tau) = e^{r\tau} \alpha_1 f(S_1, \frac{K}{\alpha_1}, \tau), \quad \lim_{S_2 \to +\infty} \tilde{V}(S_1, S_2, \tau) = 0, \tag{4.6}$$

where $f(S_2, \frac{K}{\alpha_2}, \tau)$ and $f(S_1, \frac{K}{\alpha_1}, \tau)$ are the solutions of the basic BS equation of a normal put with strike prices $\frac{K}{\alpha_2}$ and $\frac{K}{\alpha_1}$ respectively which are available by the analytical solution of one-dimensional B-S equation given in [36].

4.1. **Temporal discretization.** In order to discretize Problem (4.2)-(4.6) in time, we use the splitting scheme in which the equation (4.2) is solved along separate direction. We can rewrite (4.2) as follows

$$\frac{\partial \tilde{V}(\underline{\mathbf{S}}, \tau)}{\partial \tau} = (\tilde{\mathcal{L}}_1 + \tilde{\mathcal{L}}_2 + \tilde{\mathcal{L}}_{12}) \tilde{V}(\underline{\mathbf{S}}, \tau), \tag{4.7}$$

where

$$\tilde{\mathcal{L}}_i = \tilde{\alpha}_i(\underline{\mathbf{S}})\frac{\partial^2}{\partial S_i^2} + \tilde{\beta}_i(\underline{\mathbf{S}})\frac{\partial}{\partial S_i} \quad i=1,2, \quad \tilde{\mathcal{L}}_{12} = \tilde{\zeta}(\underline{\mathbf{S}})\frac{\partial^2}{\partial S_1 \partial S_2}. \tag{4.8}$$

Let $\tau_0, \tau_1, \dots, \tau_M$ be M equally spaced time levels for $[0,T]$ with $\Delta\tau = \frac{T}{M}$ so that $\tau_k = k\Delta\tau$, $k = 0,1,\dots,M$. Then for each time step $k = 1,2,\dots,M$, we define the following multi-steps operator splitting so that, in first step, $\tilde{D}^k(\underline{\mathbf{S}})$ is calculated using implicit Euler problems

$$\frac{\tilde{D}^k(\underline{\mathbf{S}}) - \tilde{V}^{k-1}(\underline{\mathbf{S}})}{\Delta\tau} = \tilde{\mathcal{L}}_1 \tilde{D}^k(\underline{\mathbf{S}}), \tag{4.9}$$

$$\tilde{D}(0,S_2,\tau) = e^{r\tau}\alpha_2 f(S_2, \frac{K}{\alpha_2}, \tau), \quad \lim_{S_1\to+\infty} \tilde{D}(S_1,S_2,\tau) = 0, \tag{4.10}$$

in second step, we compute $\tilde{Z}^k(\underline{\mathbf{S}})$ as

$$\frac{\tilde{Z}^k(\underline{\mathbf{S}}) - \tilde{D}^k(\underline{\mathbf{S}})}{\Delta\tau} = \tilde{\mathcal{L}}_2 \tilde{Z}^k(\underline{\mathbf{S}}), \tag{4.11}$$

$$\tilde{Z}(S_1,0,\tau) = e^{r\tau}\alpha_1 f(S_1, \frac{K}{\alpha_1}, \tau), \quad \lim_{S_2\to+\infty} \tilde{Z}(S_1,S_2,\tau) = 0, \tag{4.12}$$

which are also implicit Euler problems. Finally, the option price $\tilde{V}^k(\underline{\mathbf{S}})$ at time step k can be obtained using the explicit Euler scheme

$$\frac{\tilde{V}^k(\underline{\mathbf{S}}) - \tilde{Z}^k(\underline{\mathbf{S}})}{\Delta\tau} = \tilde{\mathcal{L}}_{12} \tilde{Z}^k(\underline{\mathbf{S}}). \tag{4.13}$$

Remark 1. The Euler scheme is (theoretically) first order accurate. To recover the rate of covergence in time direction and enrich the option price $\tilde{V}^M(\underline{\mathbf{S}})$ in numerical experiments, we use the repeated Richardson extrapolation as follows

$$\tilde{V}(\underline{\mathbf{S}}) \simeq \frac{1}{3}\Big(8\tilde{V}^{4M}(\underline{\mathbf{S}}) - 6\tilde{V}^{2M}(\underline{\mathbf{S}}) + \tilde{V}^M(\underline{\mathbf{S}})\Big). \tag{4.14}$$

4.2. **Spatial discretization using RBF-FD approach.** In this section, we employ RBF-FD approach to discretize Problems (4.9)-(4.13). First of all, the infinite spatial domain of problem is replaced with a bounded one

$$\tilde{\mathbf{\Omega}} = [0, S_{1\infty}] \times [0, S_{2\infty}]$$

where $S_{1\infty}$ and $S_{2\infty}$ will be chosen suitably large. Assume $\underline{\mathbf{S}} \in \tilde{\mathbf{\Omega}}$ and $\Xi = \{\underline{\mathbf{S}}_1, \dots, \underline{\mathbf{S}}_N\}$ be a partition of $\tilde{\mathbf{\Omega}}$ so that it contains N_I inner nodes and N_B boundary nodes in which $N = N_I + N_B$. Furthermore $\Xi^{(i)} = \{\underline{\mathbf{S}}_1^{(i)}, \dots, \underline{\mathbf{S}}_{n_i}^{(i)}\} \subseteq \Xi$ be a stencil relevant to $\underline{\mathbf{S}}_i = (S_{1i}, S_{2i})$ so that $i = 1,\dots,N_I$ and $n_i \le N$. By starting from relation (4.9) and collocating it at inner nodes $\underline{\mathbf{S}}_i$, $i = 1,2,\dots,N_I$ and also using (4.8) one can obtain

$$\tilde{D}^k(\underline{\mathbf{S}}_i) = \tilde{\alpha}_1(\underline{\mathbf{S}}_i)\Delta\tau \frac{\partial^2 \tilde{D}^k(\underline{\mathbf{S}}_i)}{\partial S_1^2} + \tilde{\beta}_1(\underline{\mathbf{S}}_i)\frac{\partial \tilde{D}^k(\underline{\mathbf{S}}_i)}{\partial S_1} + \tilde{V}^{k-1}(\underline{\mathbf{S}}_i), \tag{4.15}$$

where

$$\frac{\partial \tilde{D}^k(\underline{\mathbf{S}}_i)}{\partial S_1} = \sum_{m=1}^{n_i} \omega_m^{(1,i)} D^k(\underline{\mathbf{S}}_m^{(i)}), \quad \frac{\partial^2 \tilde{D}^k(\underline{\mathbf{S}}_i)}{\partial S_1^2} = \sum_{m=1}^{n_i} \omega_m^{(11,i)} D^k(\underline{\mathbf{S}}_m^{(i)}), \tag{4.16}$$

and $\{\omega_m^{(1,i)}\}_{m=1}^{n_i}$ and $\{\omega_m^{(11,i)}\}_{m=1}^{n_i}$ are the RBF-FD coefficients. Substituting relations (4.16) in (4.15) leads to the following equation

$$\sum_{\substack{m=1 \\ \underline{\mathbf{S}}_m^{(i)} \neq \underline{\mathbf{S}}_i}}^{n_i} \big[\tilde{\alpha}_1(\underline{\mathbf{S}}_i)\Delta\tau\omega_m^{(11,i)} + \tilde{\beta}_1(\underline{\mathbf{S}}_i)\omega_m^{(1,i)}\big]\tilde{D}^k(\underline{\mathbf{S}}_m)$$

$$+\big[\tilde{\alpha}_1(\underline{\mathbf{S}}_i)\Delta\tau\omega_i^{(11,i)} + \tilde{\beta}_1(\underline{\mathbf{S}}_i)\omega_i^{(1,i)} - 1\big]\tilde{D}^k(\underline{\mathbf{S}}_i) = \tilde{V}^{k-1}(\underline{\mathbf{S}}_i). \tag{4.17}$$

It leads to the following linear system in the matrix form

$$\tilde{\mathcal{A}}_{\mathbf{1}}\tilde{\mathbf{D}}^{\mathbf{k}} = -\tilde{\mathbf{V}}^{\mathbf{k-1}}, \tag{4.18}$$

The matrix $\tilde{\mathcal{A}}_{\mathbf{1}}$ of dimention $N_I \times N_I$ turns to a n_i-diagonal matrix in the case of uniform nodes. It also turns to a highly sparse one when the nodes are scattered. Note that in the both cases the condition number is very low and near one.

Similarly, for the relation (4.11)

$$\tilde{Z}^k(\underline{\mathbf{S}}_i) = \tilde{\alpha}_2(\underline{\mathbf{S}}_i)\Delta\tau\frac{\partial^2\tilde{Z}^k(\underline{\mathbf{S}}_i)}{\partial S_2^2} + \tilde{\beta}_2(\underline{\mathbf{S}}_i)\frac{\partial\tilde{Z}^k(\underline{\mathbf{S}}_i)}{\partial S_2} + \tilde{D}^k(\underline{\mathbf{S}}_i), \tag{4.19}$$

where

$$\frac{\partial\tilde{Z}^k(\underline{\mathbf{S}}_i)}{\partial S_2} = \sum_{m=1}^{n_i}\omega_m^{(2,i)}Z^k(\underline{\mathbf{S}}_m^{(i)}), \quad \frac{\partial^2\tilde{Z}^k(\underline{\mathbf{S}}_i)}{\partial S_2^2} = \sum_{m=1}^{n_i}\omega_m^{(22,i)}Z^k(\underline{\mathbf{S}}_m^{(i)}). \tag{4.20}$$

And then by substituting (4.20) in (4.19)

$$\sum_{\substack{m=1\\ \underline{\mathbf{S}}_m^{(i)}\neq\underline{\mathbf{S}}_i}}^{n_i}\big[\tilde{\alpha}_2(\underline{\mathbf{S}}_i)\Delta\tau\omega_m^{(22,i)} + \tilde{\beta}_2(\underline{\mathbf{S}}_i)\omega_m^{(2,i)}\big]\tilde{Z}^k(\underline{\mathbf{S}}_m) + \big[\tilde{\alpha}_2(\underline{\mathbf{S}}_i)\Delta\tau\omega_i^{(22,i)} + \tilde{\beta}_2(\underline{\mathbf{S}}_i)\omega_i^{(2,i)} - 1\big]\tilde{Z}^k(\underline{\mathbf{S}}_i) = \tilde{D}^k(\underline{\mathbf{S}}_i), \tag{4.21}$$

which leads to the following matrix form of a linear system

$$\tilde{\mathcal{A}}_{\mathbf{2}}\tilde{\mathbf{Z}}^{\mathbf{k}} = -\tilde{\mathbf{D}}^{\mathbf{k}}, \tag{4.22}$$

where the matrix $\tilde{\mathcal{A}}_{\mathbf{2}}$ has the same conditions as matrix $\tilde{\mathcal{A}}_{\mathbf{1}}$.

Finally, by collocating (4.13) in the same inner nodes and some simplifications we have

$$\tilde{V}^k(\underline{\mathbf{S}}_i) = \tilde{\zeta}(\underline{\mathbf{S}}_i)\Delta\tau\frac{\partial^2\tilde{Z}^k(\underline{\mathbf{S}}_i)}{\partial S_1\partial S_2} + \tilde{Z}^k(\underline{\mathbf{S}}_i), \tag{4.23}$$

where

$$\frac{\partial^2\tilde{Z}^k(\underline{\mathbf{S}}_i)}{\partial S_1\partial S_2} = \sum_{m=1}^{n_i}\omega_m^{(12,i)}Z^n(\underline{\mathbf{S}}_m^{(i)}), \tag{4.24}$$

in which the $\{\omega_m^{(12,i)}\}_{m=1}^{n_i}$ are the RBF-FD coefficients so that by substituting (4.24) in (4.23) we can calculate the option price vector $\tilde{V}^k(\underline{\mathbf{S}}_i)$ at desired internal nodes and time step k.

Remark 2. With solving for M times, i.e. $k = 1, 2, \ldots, M$, the option price vector $\tilde{V}^k(\underline{\mathbf{S}}_i)$ is calculated by Systems (4.18) and (4.22). Utilizing effective ways for sparse matrices focused on either direct or iterative procedures, these extremely sparse linear system are solved. Advanced techniques, such as the process of LU factorization, can be extended to every non-singular matrix and are very well suited to linear system solutions. When the coefficient matrix has been high and sparse, such procedures can indeed be costly, since the triangular variables of a sparse matrix typically have even more non-zero components than themselves. Thus a significant shared memory is needed and so many graphics processing calculations price indeed the answer of the floating method. It includes any use of iterative algorithms to conserve the coefficient matrix sparsity. The biconjugate gradient stabilized algorithm (BiCGSTAB), established by Van de Vorst [33] to solve sparse linear systems [1], is now the most efficient iterative algorithm of these forms. We employ the BiCGSTAB method for solving such highly sparse systems in numerical experiments due to decrease computational cost.

TABLE 2. Two-asset problem parameters

K	σ_1	σ_2	r	T	ρ	α_1	α_2	D_1	D_2
1	0.2	0.3	0.2	1	0	0.4	0.6	0	0

5. Numerical Experiments

The evaluation of the proposed splitting RBF-FD method is considered in this section using some option pricing problems. The accuracy of conventional RBF method depends on the value of shape parameter significantly due to illconditioned matrices. The RBF-FD approach eliminates this issue, but notice that now the value of the shape parameter is reasonably large in any and all simulation results, enabling us to do this formulas of the RBF-FD coefficients in Table 2 when Ξ is equivalent. All of comparisons are based on following norms.

$$L_2 = \frac{1}{N-2}\sqrt{\sum_{i=2}^{N-1}\left(V_{exact}(S_i) - V_{app.}^{M}(S_i)\right)^2}, \tag{5.1}$$

$$L_\infty = \max_{i=2,\ldots,N-1}\left|V_{exact}(S_i) - V_{app.}^{M}(S_i)\right|, \tag{5.2}$$

$$Err_{Fin2} = \max_{S\in\Omega_{Fin}}\left|V_{exact}(S_i) - V_{app.}^{M}(S_i)\right|, \tag{5.3}$$

where the $V_{exact}(\underline{\mathbf{S}}_i)$ are the exact solutions and the $V_{app}(\underline{\mathbf{S}}_i)$ M (Si) are the approximation of option values. As well, Ω_{Fin} in (5.3) denotes the set of all internal nodes $\underline{\mathbf{S}} = (S_1, S_2)$ such that $\frac{K}{3} \leq S_1 \leq \frac{5K}{3}$ and$\frac{K}{3} \leq S_2 \leq \frac{5K}{3}$. Note that all numerical simulations are carried out by the MQ function using Matlab software by PC Laptop with Intel(R) Core(TM)2 Duo CPU T6400 2 GHz 2 GB RAM.

As the first example, we consider two-asset European put option with parameters given in Table 2 and $S_{1\infty} = S_{2\infty} = 4$. Results for uniform nodes with $n_i = 5$ are presented in Table 3 where they shows the high accuracy of the proposed approach. Also, we see that the convergence order (CO) is about 3.2 in Table 4. The convergence order is calculated using the below relation

$$CO = \log_2^{\frac{Max.\ Error\ for\ previousrow}{Max.\ Error\ for\ currentrow}}.$$

As well, the approach demonstrates an excellent time performance because not only does it lead to the highly sparse RBF-FD systems, but also they are solved by BiCGSTAB iterative algorithm. For instance, the banded structure of matrices $\tilde{\mathcal{A}}_{\mathbf{1}}$ and $\tilde{\mathcal{A}}_{\mathbf{2}}$ in the case of $N = 1681$ is shown in Figure 1. In Figure 2, we demonstrate the absolute error for RBF-FD method with the different values.

TABLE 3. Efficiency of the splitting RBF-FD method with $n_i = 5$ using repeated Richardson extrapolation and smoothing scheme

N	M	L_2	L_∞	Err_{Fin2}	CPU Time
441	21	$9.6899e-03$	$8.3964e-05$	$4.5997e-03$	0.72
1681	41	$4.5446e-04$	$4.2624e-06$	$2.8919e-04$	4.65
6561	81	$2.1185e-05$	$2.1452e-07$	$1.9617e-06$	15.21

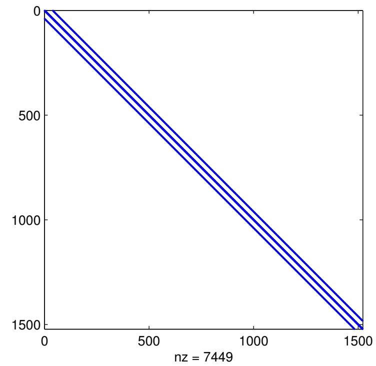

FIGURE 1. Banded structure of matrices $\tilde{\mathcal{A}}_1$ and $\tilde{\mathcal{A}}_1$ with $N_I = 1521 (N = 1681)$. The non-zero elements are about 0.3 percent.

TABLE 4. The convergence order (CO) of the splitting RBF-FD method for pricing multi-asset options

N	M	L_2	CO	L_∞	CO
64	64	$5.3067e-03$		$1.1349e-03$	
128	128	$6.5421e-04$	3.02	$1.3145e-04$	3.11
256	256	$7.5250e-05$	3.12	$1.4205e-05$	3.21
512	512	$8.3029e-06$	3.18	$1.5140e-06$	3.23

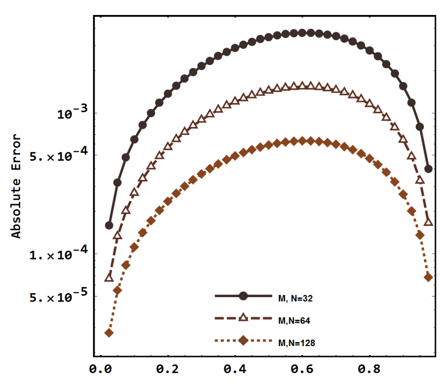

FIGURE 2. Comparison of the absolute error for RBF-FD method with the different values.

6. Conclusion

This paper presented the RBF-FD method to price multi-asset options of both European and American types under B-S model. The proposed approach is obtained by using operator splitting and repeated Richardson extrapolation schemes in time direction and coupling the RBF technology with FD scheme in spatial direction which leads to highly sparse matrices. Therefore, it is free of the ill-conditioned difficulties that are typical of the standard RBF approximation. We also used a powerful iterative algorithm named BiCGSTAB to solve highly sparse systems raised by the new approach. It has been shown that the presented scheme is unconditionally stable in the case of independent assets when spatial discretization nodes are equispaced. As well as, it has low computational cost and as seen in experimental measurements, it provides precise results.

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