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Exact solutions of the space time-fractional Klein-Gordon equation with cubic nonlinearities using some methods

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Abstract

Recently, finding exact solutions of nonlinear fractional differential equations has attracted great interest. In this work, the space time-fractional Klein-Gordon equation with cubic nonlinearities is examined. Firstly, suitable exact soliton solutions are formally extracted by using the solitary wave ansatz method. Some solutions are also illustrated by the computer simulations. Besides, the modified Kudryashov method is used to construct exact solutions of this equation.

Keywords. Space time fractional Klein-Gordon equation, Ansatz method, Modified Kudryashov method, Exact solutions.
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1. Introduction

Fractional differential equations are generalization of the differential equations. In recent years, non-linear fractional differential equations (FDEs) have gained importance in various disciplines and have become popular. Recently, the theory and applications of FDEs have been the focus of many studies since they appear frequently in various applications in mathematics, physics, biology, engineering, signal processing, systems identification, control theory, finance, fractional dynamics, and have increasingly fascinated the attention of many scientists. FDEs have been studied and many researchers published books and articles in this field [40, 43, 48]. Many methods have been introduced to obtain exact solutions of FDEs. For instance the first integral method [22, 25, 44, 49, 52], exp-function method [8, 9, 27], (G'/G) expansion method [6, 10, 14], sub-equation method [5, 11], functional variable method [28, 46], trial equation method [21, 47], local meshless method [4].

A dependable and powerful method called the ansatz method has been put forward to search for traveling wave solutions of nonlinear partial differential equations by Biswas [16, 38]. Although this method has been used by many authors, the applications of this method are very low in nonlinear FDEs. The installation of exact and analytical traveling wave solutions of nonlinear FDEs is one of the most significant and basic duties in nonlinear science because they will characterize miscellaneous natural cases such as vibrations, solitons, and finite speed distribution. The Ansatz method is one of the efficient methods used to obtain exact soliton solutions of FDEs.

The solitary wave study has made important progress recently. In mathematics and physics, a soliton or a solitary wave is a self-reinforcing single wave that moves at a constant velocity, while maintaining its shape. Solitons represent solutions of the class of largely weak nonlinear distributive partial differential equations associated with physical systems. This field of study has recently made a huge progress [1–3, 7, 12, 13, 15–19, 37, 39, 54, 57]. In the present study, FDEs will be converted into integer-order differential equations by fractional complex transformation, and then various exact solutions will be obtained to determine bright soliton solutions, dark soliton solutions, and the singular soliton solutions [29, 30, 45].

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One of the approaches that led to creating exact solutions of fractional differential equations is a modified version of the Kudryashov method [42]. The modified Kudryashov method is a powerful solution method for finding exact solutions of nonlinear partial differential equations (PDEs) in mathematical physics and biology. This method was first applied in fractional differential equations by Ege and Misirli [24]. Recently, this method has gained considerable attention due to the ability of PDEs to extract new complete solutions both in integer order and in fractional order [32, 41, 50].

Nonlinear Klein - Gordon equations have important application areas in science and engineering such as solid state physics, nonlinear optics, and quantum field theory [56]. This equation is a relativistic field equation for scalar particles and is a relativistic generalization of the well-known Schrödinger equation. Despite other relativistic wave equations, the Klein-Gordon equation (KGE) is the most frequently studied equation in quantum field theory, since it is used to describe particle dynamics [20]. They have been studied by many researchers and various methods have been used to solve them. Some of these studies can be listed as follows: Homotopy perturbation method [26], a semi-analytical method called fractional-reduced differential transformation method with the appropriate initial condition [53], modified Kudryashov method [33], fractional complex transformation, (G'/G) and (w/g) expansion methods [55], the well-organized ansatz method [34], a direct analytic method [23], the modified expanded Tanh method [51].

The study consists of five sections. In the first section, brief information about fractional differential equations is given. In addition, an introduction about Nonlinear Klein - Gordon equations which form the basis of the publication, and the ansatz and modified Kudryashov method that will be used to solve this equation are explained. In the second section, the modified Riemann-Liouville derivative and methodology of the solution are explained. In section 3, modified Kudryashov method which is an important method for the solution is mentioned. The section 4 contains that the various new explicit exact solutions of the space time-fractional KGE with cubic nonlinearities are obtained by both the ansatz method and the modified Kudryashov method. Finally, the last section contains an explanation of the results.

2. The modified Riemann-Liouville derivative and methodology of solution

With recent studies, it is well known that the dynamics of many physical processes are accurately described using FDEs having different kinds of fractional derivatives. The most popular ones are the Caputo derivative, the Riemann-Liouville derivative and Grünwald-Letnikov derivative. A different definition of the fractional derivative is given by Jumarie with a little modification of the Riemann-Liouville derivative. In [35], $f : R \to R$, $\omega \to f(\omega)$ as a continuous function (not necessarily differentiable), the modified Riemann-Liouville derivative of order α is given as follows

$$D_\omega^\alpha f(\omega) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\omega} \int_0^\omega \frac{f(\tau)-f(0)}{(\omega-\tau)^\alpha} d\tau & ,0 < \alpha < 1, \\ (f^{(n)}(\omega))^{(\alpha-n)} & ,n \leq \alpha \leq n+1, n \geq 1 \end{cases} \tag{2.1}$$

where $\Gamma(.)$ is the Gamma function. In addition, some important properties of the fractional modified Riemann-Liouville derivative (mRLd) are listed as follows [36]:

$$D_\omega^\alpha \omega^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} \omega^{\gamma-\alpha} \qquad ,\gamma > 1, \tag{2.2}$$

$$D_\omega^\alpha (c) = 0 \text{ } (c \text{ constant}) \text{ }, \tag{2.3}$$

$$D_\omega^\alpha (af(\omega) + bg(\omega)) = aD_\omega^\alpha f(\omega) + bD_\omega^\alpha g(\omega), \tag{2.4}$$

where $a \neq 0$ and $b \neq 0$ are constants.

Now, we will take into account the following nonlinear space-time FDE of the type

$$H(u, D_t^\alpha u, D_x^\alpha u, D_{tt}^{2\alpha} u, D_{xx}^{2\alpha} u, D_t^\alpha D_x^\alpha u ...) = 0, \quad 0 < \alpha < 1 \tag{2.5}$$

where u is an unknown functions, H is a polynomial of u and its partial fractional derivatives, and α is order of the mRLd of the function $u = u(x,t)$.

The traveling wave transformation is

$$
\begin{aligned}
&u(x,t) = U(\varepsilon),\\
&\varepsilon = \frac{kx^{\alpha}}{\Gamma(1+\alpha)} - \frac{ct^{\alpha}}{\Gamma(1+\alpha)},
\end{aligned} \tag{2.6}
$$

with $k \neq 0$ and $c \neq 0$ are constants. We use the chain rule

$$
\begin{aligned}
&D_t^{\alpha} u = \sigma_t \frac{\partial U}{\partial \varepsilon} D_t^{\alpha} \varepsilon,\\
&D_x^{\alpha} u = \sigma_x \frac{\partial U}{\partial \varepsilon} D_x^{\alpha} \varepsilon,
\end{aligned} \tag{2.7}
$$

with σ_t, σ_x are sigma indexes [31] and they can be $\sigma_t = \sigma_x = L$, where L is a constant.
Substituting (2.6) and applying (2.2) and (2.7) to (2.5), we get following nonlinear ordinary differntial equation (ODE).

$$
N(U, \frac{dU}{d\varepsilon}, \frac{d^2U}{d\varepsilon^2}, \frac{d^3U}{d\varepsilon^3}, ...) = 0. \tag{2.8}
$$

3. Modified Kudryashov method

Let the exact solution of (2.8) can be showed as follows

$$
U(\varepsilon) = a_0 + a_1 Q(\varepsilon) + ... + a_N Q(\varepsilon)^N, \tag{3.1}
$$

where a_i values $(i = 0, 1, 2, ..., N)$ are arbitrary constants to be found later, but $a_N \neq 0$. $Q(\varepsilon)$ has the form

$$
Q(\varepsilon) = \frac{1}{1 + dA^{\varepsilon}} \tag{3.2}
$$

which is a solution to the Riccati equation

$$
Q'(\varepsilon) = \big(Q^2(\varepsilon) - Q(\varepsilon)\big) lnA \tag{3.3}
$$

where d and A are nonzero constants with $A > 0$ and $A \neq 1$. N is revealed by balancing the highest order derivative and nonlinear terms in (2.8). Substituting (3.1) into (2.8) and comparing the results of the terms with a series of nonlinear equations, new exact solutions will be taken for (2.5).

4. Applications

4.1. **Application of ansatz method to space time fractional KGE.**

We consider the space-time fractional KGE of the form

$$
D_{tt}^{2\alpha} u - a^2 D_{xx}^{2\alpha} u + b^2 u - \lambda u^3 = 0, \quad (t > 0, \quad 0 < \alpha \leq 1), \tag{4.1}
$$

where a, b, λ are constants [23]. The bright, dark and singular soliton solutions will be applied to the solitary wave ansatz method. In order to solve Eq. (4.1), using the traveling wave transformation (2.6), we obtain to an ODE

$$
L^2(a^2k^2 - c^2)U'' - b^2 U + \lambda U^3 = 0, \tag{4.2}
$$

with $U' = \frac{dU}{d\varepsilon}$.

4.1.1. *The bright soliton solution.*

For the bright soliton solution, we let A, k and, c be abritrary constants. Then suppose

$$
U(\varepsilon) = A\text{sech}^p(\varepsilon), \tag{4.3}
$$

where

$$
\varepsilon = \frac{kx^{\alpha}}{\Gamma(1+\alpha)} - \frac{ct^{\alpha}}{\Gamma(1+\alpha)}. \tag{4.4}
$$

It follows from ansatz (4.3) and (4.4) that

$$
\frac{d^2U}{d\varepsilon^2} = Ap^2\text{sech}^p(\varepsilon) - Ap(p+1)\text{sech}^{p+2}(\varepsilon), \tag{4.5}
$$

and

$$U^3 = A^3\text{sech}^{3p}(\varepsilon). \tag{4.6}$$

Substituting the ansatz (4.3)-(4.6) into (4.2), the following equation is obtained

$$\begin{aligned} &L^2(a^2k^2-c^2)Ap^2\text{sech}^p(\varepsilon) - L^2(a^2k^2-c^2)Ap(p+1)\text{sech}^{p+2}(\varepsilon) \\ &- b^2A\text{sech}^p(\varepsilon) + \lambda A^3\text{sech}^{3p}(\varepsilon) = 0. \end{aligned} \tag{4.7}$$

From (4.7), it is supposed that the exponents $p+2$ and $3p$ are equal and from that p is determined as 1. When this value is placed in (4.7), it is reduced to the following equation

$$\begin{aligned} &L^2(a^2k^2-c^2)A\text{sech}(\varepsilon) - 2L^2(a^2k^2-c^2)A\text{sech}^3(\varepsilon) - b^2A\text{sech}(\varepsilon) \\ &+ \lambda A^3\text{sech}^3(\varepsilon) = 0. \end{aligned} \tag{4.8}$$

From (4.8), we obtain the following system of algebraic equations

$$\begin{cases} \lambda A^2 - 2L^2(a^2k^2-c^2) = 0, \\ L^2(a^2k^2-c^2) - b^2 = 0. \end{cases}$$

Solving this system, we get

$$\begin{aligned} &A = \mp\sqrt{\frac{2L^2(a^2k^2-c^2)}{\lambda}}, \quad \Big(\frac{a^2k^2-c^2}{\lambda} > 0, \lambda \neq 0\Big) \\ &c = \mp\sqrt{\frac{L^2a^2k^2-b^2}{L^2}}, \quad (L^2a^2k^2-b^2>0). \end{aligned} \tag{4.9}$$

Finally, we obtain the bright soliton solution for the Fractional Klein-Gordon as follows

$$u(x,t) = \mp\sqrt{\frac{2L^2(a^2k^2-c^2)}{\lambda}}\text{sech}\Big(\frac{kx^\alpha}{\Gamma(1+\alpha)} \mp \sqrt{\frac{L^2a^2k^2-b^2}{L^2}}\frac{t^\alpha}{\Gamma(1+\alpha)}\Big). \tag{4.10}$$

The physical behavior of (4.10) is displayed in Figure 1, in the interval $0 < x < 10$ and $0 < t < 1$.

4.1.2. *The dark soliton solution.*
To obtain dark soliton solution , suppose that

$$U(\varepsilon) = A\tanh^p(\varepsilon), \tag{4.11}$$

where

$$\varepsilon = \frac{kx^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}, \tag{4.12}$$

which k, c and A are nonzero constant coefficients. From ansatz (4.11) and (4.12), we get

$$\frac{d^2U}{d\varepsilon^2} = Ap(p-1)\tanh^{p-2}(\varepsilon) - 2Ap^2\tanh^p(\varepsilon) + Ap(p+1)\tanh^{p+2}(\varepsilon), \tag{4.13}$$

and

$$U^3 = A^3\tanh^{3p}(\varepsilon). \tag{4.14}$$

Thus, substituting the ansatz (4.11)-(4.14) into (4.2), it is achieved

$$L^2(c^2-a^2k^2)[Ap(p-1)\tanh^{p-2}(\varepsilon) - 2Ap^2\tanh^p(\varepsilon) + Ap(p+1)\tanh^{p+2}(\varepsilon)] + b^2A\tanh^p(\varepsilon) - \lambda A^3\tanh^{3p}(\varepsilon) = 0. \tag{4.15}$$

From (4.15), equating exponents p+2 and $3p$, that gives rise to p=1. By using this value, Eq. (4.15) reduces to

$$L^2(c^2-a^2k^2)[-2A\tanh(\varepsilon) + 2A\tanh^3(\varepsilon)] + b^2A\tanh(\varepsilon) - \lambda A^3\tanh^3(\varepsilon) = 0. \tag{4.16}$$

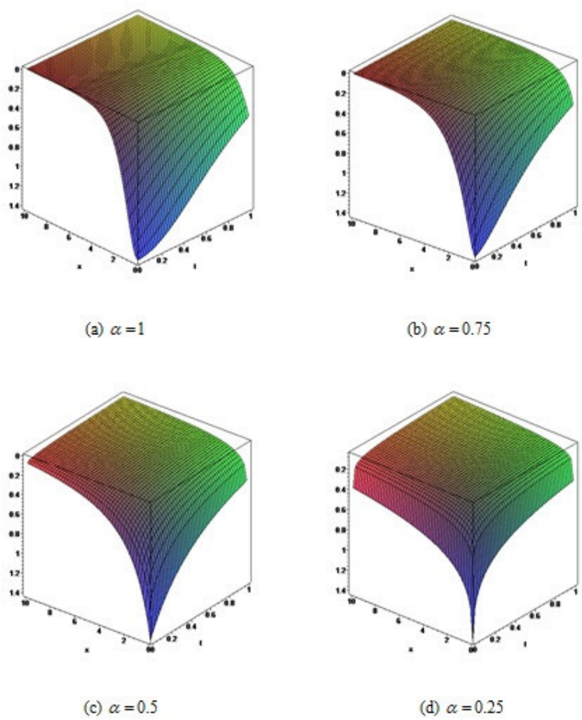


FIGURE 1. The physical behavior of $u(x,t)$ when $a = 2, k = 1, b = 1, L = 1, \lambda = 1$.

From (4.16), we find the algebraic system

$$\begin{aligned} &2L^2(c^2 - a^2k^2) - \lambda A^2 = 0, \\ &-2L^2(c^2 - a^2k^2) + b^2 = 0. \end{aligned} \tag{4.17}$$

Solving the system (4.17)

$$\begin{aligned} &A = \mp\sqrt{\frac{2L^2(c^2 - a^2k^2)}{\lambda}}, \quad \Big(\frac{c^2 - a^2k^2}{\lambda} > 0, \lambda \neq 0\Big) \\ &c = \mp\sqrt{\frac{b^2 + 2L^2a^2k^2}{2L^2}}. \end{aligned} \tag{4.18}$$

Finally, we get the dark soliton solution for the Fractional Klein-Gordon as follows:

$$u(x,t) = \mp\sqrt{\frac{2L^2(c^2 - a^2k^2)}{\lambda}} \ \tanh\Big(\frac{kx^\alpha}{\Gamma(1+\alpha)} \mp \sqrt{\frac{b^2 + 2L^2a^2k^2}{2L^2}}\frac{t^\alpha}{\Gamma(1+\alpha)}\Big). \tag{4.19}$$

The physical characteristic of (4.19) is shown in Figure 2, in the interval $0 < x < 10$ and $0 < t < 1$.

4.1.3. *The singular soliton solution.*

In finding singular soliton solution we assume

$$U(\varepsilon) = A\mathrm{csch}^p(\varepsilon), \tag{4.20}$$

with

$$\varepsilon = \frac{kx^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}, \tag{4.21}$$

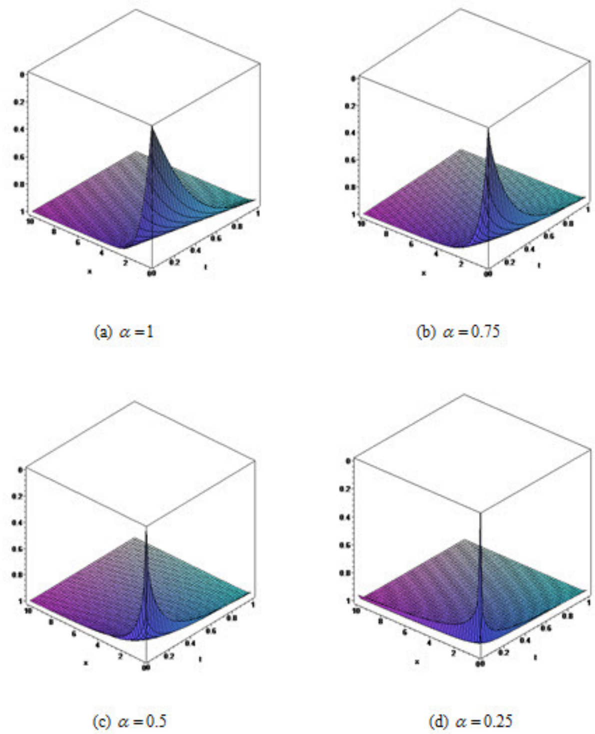


FIGURE 2. The physical characteristic of $u(x,t)$ when $a = 2, k = 1, b = 1, L = 1, \lambda = 1$.

where k, c and A are nonzero constant coefficients. From ansatz (4.20) and (4.21), we find

$$\frac{d^2U}{d\varepsilon^2} = Ap^2\text{csch}^p(\varepsilon) + Ap(p+1)\text{csch}^{p+2}(\varepsilon), \tag{4.22}$$

and

$$U^3 = A^3\text{csch}^{3p}(\varepsilon). \tag{4.23}$$

Substituting ansatz (4.20)-(4.23) into (4.2), yields

$$L^2(c^2 - a^2k^2)Ap^2\text{csch}^p(\varepsilon) + L^2(c^2 - a^2k^2)Ap(p+1)\text{csch}^{p+2}(\varepsilon) + b^2A\text{csch}^p(\varepsilon) - \lambda A^3\text{csch}^{3p}(\varepsilon) = 0. \tag{4.24}$$

In (4.24), when equating exponents p+2 and $3p$, leads p=1. Similarly using $p = 1$, Eq. (4.24) reduces to

$$L^2(c^2 - a^2k^2)A\text{csch}(\varepsilon) + 2L^2(c^2 - a^2k^2)A\text{csch}^3(\varepsilon) + b^2A\text{csch}(\varepsilon) - \lambda A^3\text{csch}^3(\varepsilon) = 0. \tag{4.25}$$

From (4.25), we find the algebraic equation system

$$\begin{cases} 2L^2(c^2 - a^2k^2) - \lambda A^2 = 0, \\ L^2(c^2 - a^2k^2) + b^2 = 0. \end{cases}$$

Solving this system, we get

$$\begin{aligned} A &= \mp\sqrt{\frac{2L^2(c^2 - a^2k^2)}{\lambda}} \quad \left(\frac{c^2 - a^2k^2}{\lambda} > 0, \lambda \neq 0\right), \\ c &= \mp\sqrt{\frac{L^2a^2k^2 - b^2}{L^2}} \ (L^2a^2k^2 - b^2 > 0). \end{aligned} \tag{4.26}$$

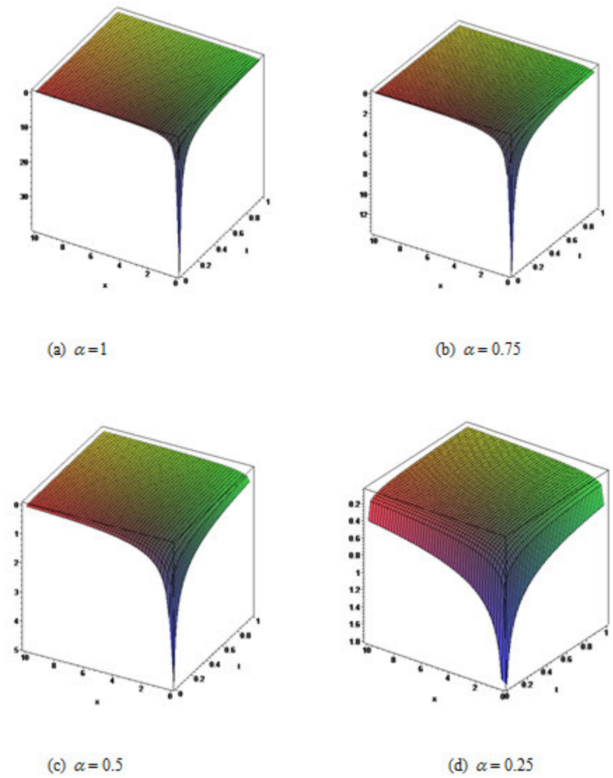


FIGURE 3. The physical attitude of $u(x,t)$ when $a = 2, k = 1, b = 1, L = 1, \lambda = -1$.

Finally, we find the singular soliton solution for the Fractional Klein-Gordon as follows

$$u(x,t) = \mp\sqrt{\frac{2L^2(c^2 - a^2k^2)}{\lambda}}\operatorname{csch}\Big(\frac{kx^\alpha}{\Gamma(1+\alpha)} \mp \sqrt{\frac{L^2a^2k^2 - b^2}{L^2}}\frac{t^\alpha}{\Gamma(1+\alpha)}\Big). \tag{4.27}$$

The physical attitude of (4.27) is indicated in Figure 3, in the interval $0 < x < 10$ and $0 < t < 1$.

4.2. **Application of modified Kudryashov method to space time fractional KGE.**
We consider the space-time fractional KGE of the form (4.1). In order to solve Eq. (4.1), using the traveling wave transformation (2.6), we obtain to an ODE

$$L^2(c^2 - a^2k^2)U'' + b^2U - \lambda U^3 = 0, \tag{4.28}$$

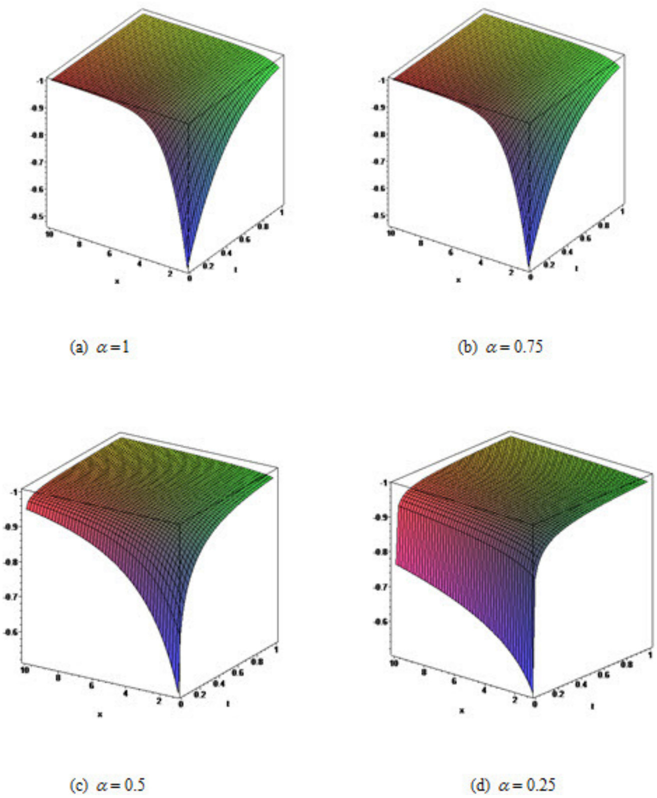
with $U' = \frac{dU}{d\varepsilon}$. The balance of U^3 and U'' gives $N = 1$. Therefore, we have

$$U(\varepsilon) = a_0 + a_1Q(\varepsilon), \quad a_1 \neq 0. \tag{4.29}$$

Substituting the solution (4.29) and its derivative into (4.28) gets

$$\begin{aligned}\Big(2a_1L^2(c^2 - a^2k^2)(lnA)^2 - \lambda a_1^3\Big)Q^3(\varepsilon) - 3\Big(a_1L^2(c^2 - a^2k^2)(lnA)^2 + \lambda a_0a_1^2\Big)Q^2(\varepsilon)\\ + \Big(a_1L^2(c^2 - a^2k^2)(lnA)^2 + b^2a_1 - 3\lambda a_0^2a_1\Big)Q(\varepsilon) + b^2a_0 - \lambda a_0^3 = 0.\end{aligned} \tag{4.30}$$

Equating the coefficients of each power of $Q(\varepsilon)$ and the constant term to zero, solving the resulting system of algebraic equations, we get the following solutions.



(a) $\alpha=1$ (b) $\alpha=0.75$ (c) $\alpha=0.5$ (d) $\alpha=0.25$

FIGURE 4. The physical characteristic of $u(x,t)$ when $A=e, a=1, k=1, b=1, d=1, L=1, \lambda=1$.

Case 1:

$$a_0=-\frac{b}{\lambda\sqrt{\frac{1}{\lambda}}},\quad a_1=2b\sqrt{\frac{1}{\lambda}},\quad c=\mp\sqrt{(lnA)^2a^2k^2L^2+2b^2} \tag{4.31}$$

Substituting (4.31) into (4.29), we have

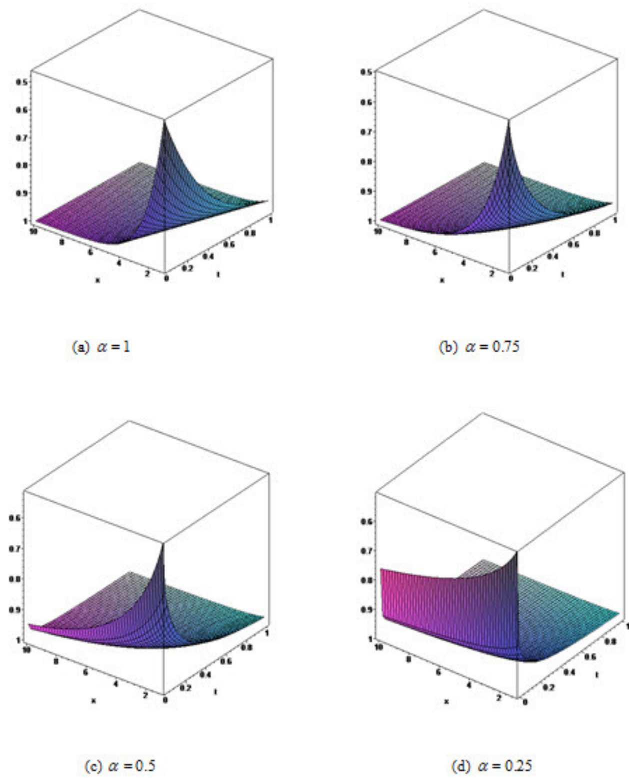
$$U(\varepsilon)=-\frac{b}{\lambda\sqrt{\frac{1}{\lambda}}}+2b\sqrt{\frac{1}{\lambda}}\Big(\frac{1}{1+dA^{\varepsilon}}\Big),\quad (\lambda>0). \tag{4.32}$$

Finally, we obtain the exact solution of (4.1)

$$u_1(x,t)=-\frac{b}{\lambda\sqrt{\frac{1}{\lambda}}}+2b\sqrt{\frac{1}{\lambda}}\Big(\frac{1}{1+dA^{\frac{kx^{\alpha}}{\Gamma(1+\alpha)}\mp\frac{\sqrt{(lnA)^2a^2k^2L^2+2b^2}t^{\alpha}}{\Gamma(1+\alpha)}}}\Big),\quad (\lambda>0). \tag{4.33}$$

The physical characteristic of (4.33) is displayed in Figure 4, in the interval $0<x<10$ and $0<t<1$.
Case 2:

$$a_0=\frac{b}{\lambda\sqrt{\frac{1}{\lambda}}},\quad a_1=-2b\sqrt{\frac{1}{\lambda}},\quad c=\mp\sqrt{(lnA)^2a^2k^2L^2+2b^2} \tag{4.34}$$



(a) $\alpha = 1$ (b) $\alpha = 0.75$

(c) $\alpha = 0.5$ (d) $\alpha = 0.25$

FIGURE 5. The physical behavior of $u(x,t)$ when $A = e, a = 1, k = 1, b = 1, d = 1, L = 1, \lambda = 1$.

Substituting (4.34) into (4.29), we get

$$U(\varepsilon) = \frac{b}{\lambda\sqrt{\frac{1}{\lambda}}} - 2b\sqrt{\frac{1}{\lambda}}\Big(\frac{1}{1 + dA^{\varepsilon}}\Big), \quad (\lambda > 0). \tag{4.35}$$

Finally, we obtain the exact solution of (4.1)

$$u_2(x,t) = \frac{b}{\lambda\sqrt{\frac{1}{\lambda}}} - 2b\sqrt{\frac{1}{\lambda}}\Big(\frac{1}{1 + dA^{\frac{kx^{\alpha}}{\Gamma(1+\alpha)} \mp \frac{\sqrt{(lnA)^2a^2k^2L^2+2b^2}\,t^{\alpha}}{\Gamma(1+\alpha)}}}\Big), \quad (\lambda > 0). \tag{4.36}$$

The physical behavior of (4.36) is indicated in Figure 5, in the interval $0 < x < 10$ and $0 < t < 1$.

5. CONCLUSION

In this article, the space time-fractional KGE with cubic nonlinearities has been investigated for soliton and exact solutions. Complex fractional transformation is utilized to attain the nonlinear ODE from this equation. Bright, dark and singular soliton solutions have been obtained with solitary wave ansatz method and some exact solutions have been found with modified Kudryashov method which may be useful for describing some physical events. The results are proof that these methods are accurate and effective. Therefore, it can be applied to solve other linear and nonlinear fractional partial differential equations in engineering and mathematical physics. In addition, graphs of all soliton solutions and exact solutions have been drawn for the appropriate coefficients.

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