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Existence and stability criterion for the results of fractional order Φ_p-Laplacian operator boundary value problem

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Abstract In this literature, we study the existence and stability of the solution of the boundary value problem of fractional differential equations with Φ_p-Laplacian operator. Our problem is based on Caputo fractional derivative of orders σ, ϵ, where $k-1 < \sigma, \epsilon \leq k$, and $k \geq 3$. By using the Schauder fixed point theory and properties of the Green function, some conditions are established which show the criterion of the existence and non-existence solution for the proposed problem. We also investigate some adequate conditions for the Hyers-Ulam stability of the solution. Illustrated examples are given as an application of our result.

Keywords. Fractional differential equations(FDEs), Caputo factional derivative, Boundary value problem(BVP), Schauder fixed point, Hyers-Ulams(UH) stability, Existence and uniqueness(EUS), Laplacian operator, Differential equations(DEs).

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1. Introduction

Recently, some physical phenomenons in different fields were described through fractional-order DEs and compared with an integer order differential equations which have better results and more accurate where non-integer models are more reliable than the classical models. Fractional-order DEs emerge in the scientific demonstration of numerous research due to the intense development and the wide applications of fractional order DEs in different fields of sciences such as fluid mechanics, electrochemistry, anomalous diffusion modeling, computer science and image processing. For a detailed description of the birth of fractional calculus, we refer to reading these

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studies [5, 9, 14, 15, 21, 22, 23, 24, 25, 26, 28, 33, 34]. In recent years, a large number of scientists have worked on generalizations of existing results, solutions, definitions, and models. For more details see [3, 13, 18, 20, 30, 31, 32] . We are interested to study existence and uniqueness for nonlinear BVP of fractional order differential equations because these areas of research acquired much interest in the field of applied mathematics and it generalizes to the BVP. For further information, see [1, 2, 4, 6, 8, 12, 16, 17, 27, 29, 36, 37]. In [10], M. El-Shahed studied sufficient conditions for existence of the solution as well as nonexistence of a positive solution to the two-point BVP

$$\begin{cases} \mathcal{D}^{\epsilon}\mu(\zeta) + \eta b(\zeta)\varrho(\mu(\zeta)) = 0 \qquad 2 < \epsilon < 3, \\ \mu(0) = \mu''(0) = 0, \quad \gamma\mu'(1) + \lambda\mu''(1) = 0, \end{cases}$$

where $0 < \zeta < 1$, η is a positive parameter, and $\mathcal{D}^{\epsilon}$ is Caputo fractional derivative. Guoqing Chai [7] studied existence and multiplicity of a positive solutions for BVP of FDEs with p-Laplacian

$$\begin{cases} \mathcal{D}^{\epsilon}(\phi_p(\mathcal{D}^{\delta}\mu(\zeta))) + \varrho(\zeta, \mu(\zeta)) = 0, \qquad 0 < \zeta < 1, \\ \mu(0) = 0, \mu(1) + \alpha\mathcal{D}^{\lambda}\mu(1) = 0 \qquad \mathcal{D}^{\delta}\mu(0) = 0, \end{cases}$$

where $\mathcal{D}^{\epsilon}, \mathcal{D}^{\delta}$ and $\mathcal{D}^{\lambda}$ are the fractional Riemann-Liouville of order $1 < \delta \leq 2$, $0 < \epsilon \leq 1$, $0 < \lambda \leq 1$, $0 \leq \delta - \lambda - 1$, α is a constant. In [35], Yuan and Yang investigated positive solution of four-points for BVP of fractional DEs with p-Laplacian

$$\begin{cases} \mathcal{D}^{\epsilon}(\phi_p(\mathcal{D}^{\delta}\mu(\zeta))) = \varrho(\zeta, \mu(\zeta)), \qquad 0 < \zeta < 1, 2 < \delta < 3, \\ \mu(0) = 0, \mu(1) = a\mu(\eta), \mathcal{D}^{\delta}\mu(0) = 0, \quad \mathcal{D}^{\delta}\mu(1) = a\mathcal{D}^{\delta}\mu(\gamma), \end{cases}$$

where $\mathcal{D}^{\epsilon}$ and $\mathcal{D}^{\delta}$ are the fractional Riemann-Liouville of order $1 < \epsilon, \delta < 2$. The main and important goal in our article is to study the EUS. In addition, we study Hyers-Ulam stability theorem for the following (FDEs):

$$\begin{cases} \mathcal{D}^{\epsilon}(\phi_p\mathcal{D}^{\sigma}\mu(\zeta)) + a(\zeta)Q(\zeta, \mu(\zeta)) = 0, \qquad k-1 \leq \epsilon, \sigma < k, k \geq 3, \\ (\phi_p(\mathcal{D}^{\sigma}\mu(\zeta)))^{(j)}|_{\zeta=0} = 0, \qquad for \quad j = 1, 3, 4, ..., k, \\ (\phi_p(\mathcal{D}^{\sigma}\mu(0))) = (\phi_p(\mathcal{D}^{\sigma}\mu(\gamma)))', \quad (\phi_p(\mathcal{D}^{\sigma}\mu(0)))' = (\phi_p(\mathcal{D}^{\sigma}\mu(1)))', \\ (\mu(0))^{(i)} = 0, \quad for \quad i = 2, 3, ..., k, \quad \gamma(\mu(0)) = \mu(1), \xi\mu'(1) = \mu'(0), \end{cases} \tag{1.1}$$

where $\mathcal{D}^{\epsilon}$ and $\mathcal{D}^{\sigma}$ are the Caputo derivative, $k-1 < \epsilon, \sigma \leq k, k \geq 3$. The ϕ_p denotes p-Laplacian, $\phi_p(\theta) = \theta|\theta|^{p-2}, \phi_p(0) = 0, \phi_q = \phi_p^{-1}$, such that $\frac{1}{p} + \frac{1}{q} = 1$. $0 \leq \xi, \gamma < 1$ are parameters, $a : (0, 1) \to [0, +\infty)$, and $Q[0, +\infty) \to [0, +\infty)$ are continuous. Our supposed problem will be more complicated and general than the problems studied before and mentioned above.

2. AUXILIARY RESULTS

Definition 2.1. [15] For $Q : (0, +\infty) \to R$, the Caputo fractional derivative of order $\sigma > 0$, is given by

$$^{c}\mathcal{D}^{\sigma}Q(\zeta) = \frac{1}{\Gamma(k-\sigma)} \int_0^{\zeta} (\zeta - \vartheta)^{k-\sigma-1} Q^{(k)}(\vartheta) d\vartheta,$$

where the integral point is defined on the right side on $(0, +\infty), Q(\zeta)$ is a continuous function, and $k - 1 < \sigma < k$.

Definition 2.2. [15] For $Q : (0, +\infty) \to R$, Fractional integral of order $\sigma > 0$ is given by

$$\mathcal{I}^{\sigma}Q(\zeta) = \frac{1}{\Gamma(\sigma)} \int_0^{\zeta} (\zeta - \vartheta)^{\sigma-1} Q(\vartheta) d\vartheta,$$

where the integral point is defined on the right side on $(0, +\infty)$.

$$\Gamma(\sigma) = \int_0^{+\infty} e^{-\gamma}\gamma^{\sigma-1} d\gamma,$$

Definition 2.3. [11] A cone $\mathcal{F}$ in a real Banach space $\mathcal{W}$ is called solid if interior of its interior $\mathcal{F}^0$ is non empty.

Definition 2.4. [11] Let $\mathcal{F}$ be a solid cone in a real Banach space $\mathcal{W}, \mathcal{B} : \mathcal{F}^0 \to \mathcal{F}^0$ be an operator and $0 < \lambda < 1$. Then $\mathcal{B}$ is called λ-concave if $\mathcal{B}(\kappa\mu) \geq \kappa^{\lambda}\mathcal{B}(\mu)$ for any $0 < \kappa < 1$ and $\mu \in \mathcal{F}^0$.

Lemma 2.5. [26] *For $Q(\zeta) \in C(0, 1)$, the fractional order DE $D^{\sigma}Q(\zeta) = 0$ has a solution*

$$Q(\zeta) = a_1 + a_2\zeta + a_3\zeta^2 + ... + a_k\zeta^{k-1}, \quad a_i \in R, i = 1, 2, 3, ..., k.$$

Lemma 2.6. [26] *Let $\sigma > 0$. Assume that $Q \in C^k[0, 1]$. Then*

$$\mathcal{I}^{\sigma}\mathcal{D}^{\sigma}Q(y) = Q(y) + c_1 + c_2\zeta + c_3\zeta^2 + ... + c_k\zeta^{k-1},$$

$c_i \in R$ fori $= 0, 1, 2, 3, ..., k - 1$.

Lemma 2.7. [24] *For $\lambda \geq \sigma > 0$ and $f \in L_1[c, d]$, the following*

$$\mathcal{D}^{\sigma}\mathcal{I}^{\lambda}Q(\zeta) = I^{\lambda-\sigma}Q(\zeta),$$

Lemma 2.8. [19] *Suppose that $\mathcal{F}$ is solid cone in a real Banach space $\mathcal{W}$, $0 < \lambda < 1$ and $\mathcal{B} : \mathcal{F}^0 \to \mathcal{F}^0$ is a λ-concave non-decreasing operator. Then $\mathcal{B}$ has only one fixed point in $\mathcal{F}^0$.*

Lemma 2.9. [26]*(Schauder fixed point theory). Let $(\mathcal{O}, \mathcal{J})$ be a complete metric space, $\mathcal{V}$ be a closed convex subset of $\mathcal{O}$, and $\mathcal{Z} : \mathcal{V} \to \mathcal{V}$ be a mapping such that the set $\{\mathcal{Z}\mu : \mu \in \mathcal{V}\}$ is relatively compact in $\mathcal{O}$. Then $\mathcal{Z}$ has at least one fixed point.*

Lemma 2.10. [11] *Let $\phi_p : R \to R$ be p-Laplacian, $\phi_p(\vartheta) = |\vartheta|^{(p-2)}\vartheta, \vartheta \in R$. Then $(\frac{d\phi_p(\vartheta)}{d\vartheta}) = (p - 1)|\vartheta|^{(p-2)}$. And:*
(A_1) If $1 < p \leq 2, \vartheta_1, \vartheta_2 > 0$ and $|\vartheta_1|, |\vartheta_2| \geq \rho > 0$, then

$$|\phi_p(\vartheta_1) - \phi_p(\vartheta_2)| \leq (p - 1)\rho^{p-2}|\vartheta_1 - \vartheta_2|.$$

(A_2) *If* $p > 2, |\vartheta_1|, |\vartheta_2| \leq \rho^*$, *then*

$$|\phi_p(\vartheta_1) - \phi_p(\vartheta_2)| \leq (p-1)\rho^{*p-2}|\vartheta_1 - \vartheta_2|.$$

3. MAIN RESULTS

To prove our main results, we need the following lemmas:

Lemma 3.1. *For* $Q \in C[0,1]$, *the BVP for fractional differential equation*

$$\begin{cases} \mathcal{D}^\sigma \mu(\zeta) = Q(\zeta), & \sigma \geq k, k \geq 3, \\ (\mu(0))^{(i)} = 0, & i = 2,3,4,...,k, \\ \gamma(\mu(0)) = \mu(1), & \xi\mu'(1) = \mu'(0), \end{cases} \tag{3.1}$$

has a solution

$$\mu(\zeta) = \int_0^1 \mathcal{G}(\zeta,\vartheta)Q(\vartheta)d\vartheta, \tag{3.2}$$

where

$$\mathcal{G}(\zeta,\vartheta) = \begin{cases} \frac{(\zeta-\vartheta)^{\sigma-1}}{\Gamma(\sigma)} + \frac{(1-\vartheta)^{\sigma-1}}{(\gamma-1)\Gamma(\sigma)} - \frac{\xi(1-\vartheta)^{\sigma-2}}{(\gamma-1)(\xi-1)\Gamma(\sigma-1)} - \frac{\zeta\xi(1-\vartheta)^{\sigma-2}}{(\xi-1)\Gamma(\sigma-1)}, 0 \leq \vartheta \leq \zeta \leq 1, \\ \frac{(1-\vartheta)^{\sigma-1}}{(\gamma-1)\Gamma(\sigma)} - \frac{\xi(1-\vartheta)^{\sigma-2}}{(\gamma-1)(\xi-1)\Gamma(\sigma-1)} - \frac{\zeta\xi(1-\vartheta)^{\sigma-2}}{(\xi-1)\Gamma(\sigma-1)}, 0 \leq \zeta \leq \vartheta \leq 1. \end{cases} \tag{3.3}$$

Proof. By using Lemma 2.6 and applying integral operator $\mathcal{I}^\sigma$ on Eq.(3.1), we get

$$\mu(\zeta) = \mathcal{I}^\sigma Q(\zeta) + a_1 + a_2\zeta + a_3\zeta^2 + ... + a_k\zeta^{k-1}. \tag{3.4}$$

With the help of condition $(\mu(0))^{(i)} = 0, i = 2,3,4,5,...,k$, in (3.4) we proceed $a_3 = a_4 = a_5 = ... = c_k = 0$, we get

$$\mu(\zeta) = \mathcal{I}^\sigma Q(\zeta) + a_1 + a_2\zeta, \tag{3.5}$$

by using the condition $\gamma\mu(0) = \mu(1)$, in (3.5) , we get

$$\gamma a_1 = \mathcal{I}^\sigma Q(1) + a_1 + a_2 \Rightarrow a_1(\gamma-1) = \mathcal{I}^\sigma Q(1) + a_2,$$

$$a_1 = \frac{1}{(\gamma-1)}(\mathcal{I}^\sigma Q(1) + a_2). \tag{3.6}$$

And applying the boundary condition $\xi\mu'(1) = \mu'(0)$, in (3.5) , we have

$$\xi(\mathcal{I}^{\sigma-1}Q(1) + a_2) = a_2 \Rightarrow \xi\mathcal{I}^{\sigma-1}Q(1) + \xi a_2 = a_2$$
$$\Rightarrow a_2(\xi-1) = -\xi\mathcal{I}^{\sigma-1}Q(1),$$

$$a_2 = -\frac{\xi}{(\xi-1)}\mathcal{I}^{\sigma-1}Q(1). \tag{3.7}$$

Putting the value of a_2 in (3.6), we obtain

$$\begin{aligned} a_1 &= \frac{1}{(\gamma-1)}(\mathcal{I}^{\sigma}Q(1) - \frac{\xi}{(\xi-1)}\mathcal{I}^{\sigma-1}Q(1)) \\ &= \frac{1}{(\gamma-1)}\mathcal{I}^{\sigma}Q(1) - \frac{\xi}{(\xi-1)(\gamma-1)}\mathcal{I}^{\sigma-1}Q(1). \end{aligned} \tag{3.8}$$

By replacing the values a_1, a_2, in (3.5), we get

$$\begin{aligned} \mu(\zeta) &= \mathcal{I}^{\sigma}Q(\zeta) + \frac{1}{(\gamma-1)}\mathcal{I}^{\sigma}Q(1) - \frac{\xi}{(\xi-1)(\gamma-1)}\mathcal{I}^{\sigma-1}Q(1) \\ &-\zeta\frac{\xi}{(\xi-1)}\mathcal{I}^{\sigma-1}Q(1), \\ \mu(\zeta) &= \int_0^{\zeta}\frac{(\zeta-\vartheta)^{\sigma-1}}{\Gamma(\sigma)}Q(\vartheta)d\vartheta + \frac{1}{(\gamma-1)}\int_0^1\frac{(1-\vartheta)^{\sigma-1}}{\Gamma(\sigma)}Q(\vartheta)d\vartheta \\ &-\frac{\xi}{(\gamma-1)(\xi-1)}\int_0^1\frac{(1-\vartheta)^{\sigma-2}}{\Gamma(\sigma-1)}Q(\vartheta)d\vartheta \\ &-\frac{\zeta\xi}{(\xi-1)}\int_0^1\frac{(1-\vartheta)^{\sigma-2}}{\Gamma(\sigma-1)}Q(\vartheta)d\vartheta \\ &= \int_0^1\mathcal{G}(\zeta,\vartheta)Q(\vartheta)d\vartheta. \end{aligned}$$

□

Lemma 3.2. *Let $Q \in C[0,1]$, then BVP for fractional differential equation*

$$\begin{cases} \mathcal{D}^{\epsilon}(\phi_p\mathcal{D}^{\sigma}\mu(\zeta)) + a(\zeta)Q(\zeta,\mu(\zeta)) = 0, & k-1 \le \epsilon, \sigma < k, k \ge 3, \\ (\phi_p(\mathcal{D}^{\sigma}\mu(\zeta)))^{(j)}|_{\zeta=0} = 0, \quad for \quad j = 1,3,4,...,k, \\ (\phi_p(\mathcal{D}^{\sigma}\mu(0))) = (\phi_p(\mathcal{D}^{\sigma}\mu(\gamma)))', \quad (\phi_p(\mathcal{D}^{\sigma}\mu(0)))' = (\phi_p(\mathcal{D}^{\sigma}\mu(1)))', \\ (\mu(0))^{(i)} = 0, \quad for \quad i = 2,3,...,k, \quad \gamma(\mu(0)) = \mu(1), \xi\mu'(1) = \mu'(0), \end{cases} \tag{3.9}$$

has a solution

$$\mu(\zeta) = \int_0^1 \mathcal{G}(\zeta,\vartheta)\phi_q\int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta, \tag{3.10}$$

where

$$\mathcal{H}(\zeta,\vartheta) = \begin{cases} \frac{-(\zeta-\vartheta)^{\epsilon-1}}{\Gamma(\epsilon)} - \frac{(\gamma-\vartheta)^{\epsilon-2}}{\Gamma(\epsilon-1)} + \frac{\gamma(1-\vartheta)^{\epsilon-2}}{\Gamma(\epsilon-1)} + \frac{\zeta^2(1-\vartheta)^{\epsilon-2}}{2\Gamma(\epsilon-1)}, 0 < \zeta \le \vartheta \le \gamma < 1, \\ -\frac{(\gamma-\vartheta)^{\epsilon-2}}{\Gamma(\epsilon-1)} + \frac{\gamma(1-\vartheta)^{\epsilon-2}}{\Gamma(\epsilon-1)} + \frac{\zeta^2(1-\vartheta)^{\epsilon-2}}{2\Gamma(\epsilon-1)}, \quad 0 < \vartheta \le \zeta \le \gamma < 1, \\ \frac{\gamma(1-\vartheta)^{\epsilon-2}}{\Gamma(\epsilon-1)} + \frac{\zeta^2(1-\vartheta)^{\epsilon-2}}{2\Gamma(\epsilon-1)}, \quad 0 < \zeta \le \gamma \le \vartheta < 1, \end{cases} \tag{3.11}$$

and $\mathcal{G}(\zeta,\vartheta)$ is given by (3.3).

Proof. Applying $\mathcal{I}^{\epsilon}$ on the FDE in (3.9) and with the help of lemma 2.6, we get

$$\phi_p(\mathcal{D}^{\sigma}\mu(\zeta)) = -\mathcal{I}^{\epsilon}a(\zeta)Q(\zeta,\mu(\zeta)) + b_1 + b_2\zeta + b_3\zeta^2 + ... + b_k\zeta^{k-1}. \tag{3.12}$$

By using condition $(\phi_p(\mathcal{D}^{\sigma}\mu(\zeta)))^{(j)}|_{\zeta=0} = 0$, for $j = 1, 3, 4, 5, ..., k$, in(3.12) we proceed $b_2 = b_4 = b_5 = ... = b_k = 0$, we get

$$\phi_p(\mathcal{D}^{\sigma}\mu(\zeta)) = -\mathcal{I}^{\epsilon}a(\zeta)Q(\zeta,\mu(\zeta)) + b_1 + b_3\zeta^2, \tag{3.13}$$

with the help of boundary condition $\phi_p(\mathcal{D}^{\sigma}\mu(0)) = \phi_p(\mathcal{D}^{\sigma}\mu(\gamma))'$ in(3.13), we get

$$\phi_p(\mathcal{D}^{\sigma}\mu(0)) = (\phi_p(\mathcal{D}^{\sigma}\mu(\gamma)))',$$

$$b_1 = -\mathcal{I}^{\epsilon-1}Q(\gamma) + 2\gamma b_3. \tag{3.14}$$

And applying the boundary condition $(\phi_p(\mathcal{D}^{\sigma}\mu(0)))' = (\phi_p(\mathcal{D}^{\sigma}\mu(1)))'$ in (3.13), we get

$$(\phi_p(\mathcal{D}^{\sigma}\mu(0)))' = (\phi_p(\mathcal{D}^{\sigma}\mu(1)))' \Rightarrow 0 = -\mathcal{I}^{\epsilon-1}Q(1) + 2b_3$$

$$\Rightarrow b_3 = \frac{1}{2}\mathcal{I}^{\epsilon-1}Q(1). \tag{3.15}$$

Putting the value of b_3 in (3.14), we get

$$b_1 = -\mathcal{I}^{\epsilon-1}Q(\gamma) + \gamma\mathcal{I}^{\epsilon-1}Q(1). \tag{3.16}$$

By replacing the values b_1, b_3, in (3.13), we get

$$\begin{aligned}\phi_p(\mathcal{D}^{\sigma}\mu(\zeta)) &= -\mathcal{I}^{\epsilon}Q(\zeta,\mu(\zeta)) - \mathcal{I}^{\epsilon-1}Q(\gamma) + \gamma\mathcal{I}^{\epsilon-1}Q(1) + \zeta^2\frac{1}{2}\mathcal{I}^{\epsilon-1}Q(1),\\ \phi_p(\mathcal{D}^{\sigma}\mu(\zeta)) &= \int_0^{\zeta}\frac{-(\zeta-\vartheta)^{\epsilon-1}}{\Gamma(\epsilon)}Q(\vartheta)d\vartheta - \int_0^{\gamma}\frac{(\gamma-\vartheta)^{\epsilon-2}}{\Gamma(\epsilon-1)}Q(\vartheta)d\vartheta\\ &\quad+\gamma\int_0^1\frac{(1-\vartheta)^{\epsilon-2}}{\Gamma(\epsilon-1)}Q(\vartheta)d\vartheta + \frac{\zeta^2}{2}\int_0^1\frac{(1-\vartheta)^{\epsilon-2}}{\Gamma(\epsilon-1)}Q(\vartheta)d\vartheta\\ &= \int_0^1\mathcal{H}(\zeta,\vartheta)Q(\vartheta)d\vartheta.\end{aligned} \tag{3.17}$$

Applying $\phi_p^{-1} = \phi_q$ for both sides of (3.17) , we get

$$(\mathcal{D}^{\sigma}\mu(\zeta)) = \phi_q\int_0^1\mathcal{H}(\zeta,\vartheta)Q(\vartheta)d\vartheta. \tag{3.18}$$

Applying the condition $(\mu(0))^{(i)} = 0$, for $i = 2, 3, ..., k$,
$\gamma(\mu(0)) = \mu(1), \xi\mu'(1) = \mu'(0)$, thus with the help of lemma 3.1, we have

$$\mu(\zeta) = \int_0^1\mathcal{G}(\zeta,\vartheta)\phi_q(\int_0^1\mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta. \tag{3.19}$$

□

Lemma 3.3. [37] *Let $k-1<\sigma,\epsilon\leq k, k>3$ the function $\mathcal{H}(\zeta,\vartheta)$ is continuous on $[0,1]\times[0,1]$, and satisfies*

- (∇_1) $\mathcal{H}(\zeta,\vartheta)\geq 0, \mathcal{H}(1,\vartheta)\geq\mathcal{H}(\zeta,\vartheta)$, *for* $\zeta,\vartheta\in[0,1]$,
- (∇_2) $\mathcal{H}(\zeta,\vartheta)\geq\zeta^{\epsilon-1}\mathcal{H}(1,\vartheta)$, *for* $\zeta,\vartheta\in(0,1)$.

Proof. For $0<\vartheta\leq\zeta\leq\xi<1$, we have the following estimates

$$\begin{aligned}\mathcal{H}(\zeta,\vartheta)&=\frac{-(\zeta-\vartheta)^{\epsilon-1}}{\Gamma(\epsilon)}-\frac{(\gamma-\vartheta)^{\epsilon-2}}{\Gamma(\epsilon-1)}+\frac{\gamma(1-\vartheta)^{\epsilon-2}}{\Gamma(\epsilon-1)}+\frac{\zeta^2(1-\vartheta)^{\epsilon-2}}{2\Gamma(\epsilon-1)},\\&=\frac{-\zeta^{\epsilon-1}(1-\frac{\vartheta}{\zeta})^{\epsilon-1}}{\Gamma(\epsilon)}-\frac{\gamma^{\epsilon-2}(1-\frac{\vartheta}{\gamma})^{\epsilon-2}}{\Gamma(\epsilon-1)}+\frac{\gamma(1-\vartheta)^{\epsilon-2}}{\Gamma(\epsilon-1)}+\frac{\zeta^2(1-\vartheta)^{\epsilon-2}}{2\Gamma(\epsilon-1)},\\&\geq\frac{-\zeta^{\epsilon-1}(1-\vartheta)^{\epsilon-1}}{\Gamma(\epsilon)}-\frac{(1-\vartheta)^{\epsilon-2}}{\Gamma(\epsilon-1)}+\frac{(1-\vartheta)^{\epsilon-2}}{\Gamma(\epsilon-1)}+\frac{\zeta^{\epsilon-1}(1-\vartheta)^{\epsilon-2}}{2\Gamma(\epsilon-1)},\\&=\frac{\zeta^{\epsilon-1}}{\Gamma(\epsilon)}[-(1-\vartheta)^{\epsilon-1}+\frac{1}{2}(\epsilon-1)(1-\vartheta)^{\epsilon-2}],\\&=\frac{\zeta^{\epsilon-1}}{\Gamma(\epsilon)}[-(1-\vartheta)^{\epsilon-1}-\frac{1}{2}(\epsilon-1)(1-\vartheta)^{\epsilon-2}]\geq 0.\end{aligned}$$

In other cases, the proof is similar. Now

$$\frac{\partial\mathcal{H}}{\partial\zeta}(\zeta,\vartheta)=\begin{cases}\frac{-(\zeta-\vartheta)^{\epsilon-2}}{\Gamma(\epsilon-1)}+\frac{\zeta(1-\vartheta)^{\epsilon-2}}{\Gamma(\epsilon-1)}, & 0<\vartheta\leq\zeta\leq\gamma<1,\\ \frac{\zeta(1-\vartheta)^{\epsilon-2}}{\Gamma(\epsilon-1)}, & 0<\zeta\leq\vartheta\leq\gamma<1,\\ \frac{\zeta(1-\vartheta)^{\epsilon-2}}{\Gamma(\epsilon-1)}, & 0<\zeta\leq\gamma\leq\vartheta<1,\end{cases}\tag{3.20}$$

from (3.20), it is clear that $\mathcal{H}'(\zeta,\vartheta)>0$. That is, $\mathcal{H}(\zeta,\vartheta)$ is a non-decreasing function. Thus $\mathcal{H}(\zeta,\vartheta)\leq\mathcal{H}(1,\vartheta)$. For (∇_2) we have the following estimates

$$\begin{aligned}\frac{\mathcal{H}(\zeta,\vartheta)}{\mathcal{H}(1,\vartheta)}&=\frac{\frac{-(\zeta-\vartheta)^{\epsilon-1}}{\Gamma(\epsilon)}-\frac{(\gamma-\vartheta)^{\epsilon-2}}{\Gamma(\epsilon-1)}+\frac{\gamma(1-\vartheta)^{\epsilon-2}}{\Gamma(\epsilon-1)}+\frac{\zeta^2(1-\vartheta)^{\epsilon-2}}{2\Gamma(\epsilon-1)}}{\frac{-(1-\vartheta)^{\epsilon-1}}{\Gamma(\epsilon)}-\frac{(\gamma-\vartheta)^{\epsilon-2}}{\Gamma(\epsilon-1)}+\frac{\gamma(1-\vartheta)^{\epsilon-2}}{\Gamma(\epsilon-1)}+\frac{(1-\vartheta)^{\epsilon-2}}{2\Gamma(\epsilon-1)}},\\&\geq\frac{\frac{-\zeta^{\epsilon-1}(1-\vartheta)^{\epsilon-1}}{\Gamma(\epsilon)}-\frac{(\gamma-\vartheta)^{\epsilon-2}}{\Gamma(\epsilon-1)}+\frac{\gamma(1-\vartheta)^{\epsilon-2}}{\Gamma(\epsilon-1)}+\frac{\zeta^{\epsilon-1}(1-\vartheta)^{\epsilon-2}}{2\Gamma(\epsilon-1)}}{\frac{-(1-\vartheta)^{\epsilon-1}}{\Gamma(\epsilon)}-\frac{(\gamma-\vartheta)^{\epsilon-2}}{\Gamma(\epsilon-1)}+\frac{\gamma(1-\vartheta)^{\epsilon-2}}{\Gamma(\epsilon-1)}+\frac{(1-\vartheta)^{\epsilon-2}}{2\Gamma(\epsilon-1)}},\\&>\zeta^{\epsilon-1}.\end{aligned}$$

□

For complete the proof, we present the following suppositions.

- (N_1) $0<\int_0^1\mathcal{H}(1,\eta)a(\eta)d\eta<+\infty$;
- (N_2) There exist $0<\beta<1$ and $\mathcal{A}>0$ where

  $$Q(\mu)=\beta\mathcal{L}\phi_p(\mu),\quad for\quad 0\leq\mu\leq\mathcal{A},\tag{3.21}$$

  where $\mathcal{L}$ satisfies

  $$0<\mathcal{L}\leq(\phi_p(\Delta_1)\beta\int_0^1\mathcal{H}(1,\vartheta)a(\vartheta)d\vartheta)^{-1},\tag{3.22}$$

  for $\Delta_1=\frac{1}{\Gamma(\sigma+1)}+\frac{1}{(\gamma-1)\Gamma(\sigma+1)}-\frac{\xi}{(\gamma-1)(\xi-1)\Gamma(\sigma)}-\frac{\xi}{(\xi-1)\Gamma(\sigma)}$;

- (N_3) There exist $c > 0$, such that

$$Q(\mu) \leq \mathcal{M}\phi_p(\mu), \quad for \quad c < \mu < +\infty, \tag{3.23}$$

where $\mathcal{M}$ satisfies

$$0 < \mathcal{M} < (\phi_p(\Delta_1 2^{q-1}) \int_0^1 \mathcal{H}(1,\eta)a(\eta)d\eta)^{-1}; \tag{3.24}$$

- (N_4) There exist $0 < \nu < 1$ and $b > 0$ such that

$$Q(\zeta, \mu(\zeta)) \geq \varrho\phi_p(\mu), \quad for \quad b < \mu < +\infty, \tag{3.25}$$

where ϱ satisfies

$$\varrho > (\phi_p(d_\alpha \int_0^1 \frac{(1-\vartheta)^{\sigma-1}}{\Gamma(\sigma)}\phi_q(\varpi^{\epsilon-1}d\vartheta)(\int_\alpha^1 \mathcal{H}(1,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta))^{-1}, \tag{3.26}$$

$$d_\alpha = \int_0^\zeta \sigma(\zeta-\vartheta)^{\sigma-1}\phi_q(\varpi^{\epsilon-1})d\vartheta \in (0,1); \tag{3.27}$$

- (N_5) $Q(\mu)$ is increasing and continuous in μ;
- (N_6) There exist $0 \leq \lambda < 1$ such that

$$Q(\kappa\mu) \geq (\phi_p(\kappa^\lambda))Q(\mu), \quad for \quad any \quad 0 < \kappa < 1 \quad and \quad 0 < \mu < +\infty; \tag{3.28}$$

- (N_7) There exist constants Υ_Q such that all $w, z \in \Omega_1$, we have

$$|Q(\zeta, w) - Q(\zeta, z)| = \Upsilon_Q|w(\zeta) - z(\zeta)|. \tag{3.29}$$

4. EXISTENCE OF SOLUTIONS

Theorem 4.1. *Suppose that (N_1) and (N_2) hold. Then the FDEs (3.9) with the boundary value condition has at least one positive solution.*

Proof. Let $\mathcal{A} > 0$ which is given in (N_2).
Define $\Omega_1 = \{\mu \in C[0,1] : 0 \leq \mu(\zeta) \leq \mathcal{A}$ on $[0,1]\}$ and the operator $\mathcal{V} : \Omega_1 \to C[0,1]$ by

$$\mathcal{V}\mu(\zeta) = \int_0^1 \mathcal{G}(\zeta,\vartheta)\phi_q(\int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta, \tag{4.1}$$

then, Ω_1 is a closed convex set. From lemma 3.2, $\mu(\zeta)$ is a solution of the BVP of fractional differential equations (3.9) if and only if $\mu(\zeta)$ is a fixed point of $\mathcal{V}$. Moreover, a standard argument can be used to show that $\mathcal{V}$ is compact. For any $\mu \in \Omega_1$, by (3.20) and (3.21) implies that $Q(\zeta,\mu(\zeta)) \leq \beta\mathcal{L}\phi_p\mu(\zeta) \leq \beta\mathcal{L}\phi_p(\mathcal{A})$ on $[0,1]$ and also , we have the following

$$\mathcal{V}\mu(\zeta) = \int_0^\zeta \frac{(\zeta-\vartheta)^{\sigma-1}}{\Gamma(\sigma)}(\phi_q \int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta$$
$$+\frac{1}{(\gamma-1)}\int_0^1 \frac{(1-\vartheta)^{\sigma-1}}{\Gamma(\sigma)}(\phi_q \int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta$$

$$
\begin{aligned}
&-\frac{\xi}{(\gamma-1)(\xi-1)}\int_0^1\frac{(1-\vartheta)^{\sigma-2}}{\Gamma(\sigma-1)}(\phi_q\int_0^1\mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta\\
&-\frac{\zeta\xi}{(\xi-1)}\int_0^1\frac{(1-\vartheta)^{\sigma-2}}{\Gamma(\sigma-1)}(\phi_q\int_0^1\mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta\\
&\le\int_0^\zeta\frac{(\zeta-\vartheta)^{\sigma-1}}{\Gamma(\sigma)}(\phi_q\int_0^1\mathcal{H}(1,\eta)a(\eta)\beta\mathcal{L}\phi_p(\mathcal{A})d\eta)d\vartheta\\
&+\frac{1}{(\eta-1)}\int_0^1\frac{(1-\vartheta)^{\sigma-1}}{\Gamma(\sigma)}(\phi_q\int_0^1\mathcal{H}(1,\eta)a(\eta)\beta\mathcal{L}\phi_p(\mathcal{A})d\eta)d\vartheta\\
&-\frac{\xi}{(\gamma-1)(\xi-1)}\int_0^1\frac{(1-\vartheta)^{\sigma-2}}{\Gamma(\sigma-1)}(\phi_q\int_0^1\mathcal{H}(1,\eta)a(\eta)\beta\mathcal{L}\phi_p(\mathcal{A})d\eta)d\vartheta\\
&-\frac{\zeta\xi}{(\xi-1)}\int_0^1\frac{(1-\vartheta)^{\sigma-2}}{\Gamma(\sigma-1)}(\phi_q\int_0^1\mathcal{H}(1,\eta)a(\eta)\beta\mathcal{L}\phi_p(\mathcal{A})d\eta)d\vartheta\\
&\le\frac{1}{\Gamma(\sigma+1)}+\frac{1}{(\gamma-1)\Gamma(\sigma+1)}-\frac{\xi}{(\gamma-1)(\xi-1)\Gamma(\sigma)}\\
&-\frac{\xi}{(\xi-1)\Gamma(\sigma)}(\phi_q\int_0^1\mathcal{H}(1,\eta)a(\eta)d\eta)\phi_q(\beta)\phi_q(\mathcal{L})\mathcal{A}\\
&=\Delta_1(\phi_q\int_0^1\mathcal{H}(\vartheta,\eta)a(\eta)d\eta)\phi_q(\beta)\phi_q(\mathcal{L})\mathcal{A}\le\mathcal{A},
\end{aligned}
$$

which implies that $\mathcal{V}(\Omega_1)\subseteq\Omega_1$. By Lemma 2.9, $\mathcal{V}$ has a fixed point in Ω_1, that is, the fractional differential equations BVP(3.9) has at least one positive solution. □

Theorem 4.2. *Suppose that (N_1) and (N_3) hold. Then the fractional differential equations BVP(3.9) has at least one positive solution.*

Proof. Let $c>0$ as given in (N_3). Define $U=\max_{(0\le\mu\le c)}Q(\mu)$. Then $Q(\mu)\le U$ for $0\le\mu\le c$. From (3.24), we get

$$\Delta_1 2^{q-1}\phi_q(\mathcal{M})\phi_q(\int_0^1\mathcal{H}(1,\eta)a(\eta)d\eta)<1.$$

Choose $c^*>c$ large enough where

$$\Delta_1 2^{q-1}(\phi_q(U)+\phi_q(\mathcal{M})c^*)\phi_q(\int_0^1\mathcal{H}(1,\eta)a(\eta)d\eta)<c^*. \tag{4.2}$$

Define $\Omega_2=\{\mu\in C[0,1]:0\le\mu(\zeta)\le c^* \text{ on } [0,1]\}$. For $\mu\in\Omega_2$.
Define $\varpi_1=\{\zeta\in[0,1]:0\le\mu(\zeta)\le c\}$, $\varpi_2=\{\zeta\in[0,1]:c<\mu(\zeta)\le c^*\}$.
Then we have $\varpi_1\cup\varpi_2=[0,1]$ and $\varpi_1\cap\varpi_2=\varphi$. From (3.23) we get that

$$Q(\zeta,\mu(\zeta))\le\mathcal{M}\phi_p(\mu(\zeta))\le\mathcal{M}\phi_p(c^*)\quad for\quad \zeta\in\varpi_2.$$

To proceed, the operator $\mathcal{V}$ be defined by (4.1). Then with the help of Lemma 3.3 and condition (N_3), we get

$$\mathcal{V}\mu(\zeta)=\int_0^\zeta\frac{(\zeta-\vartheta)^{\sigma-1}}{\Gamma(\sigma)}(\phi_q\int_0^1\mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta$$

$$
\begin{aligned}
&+\frac{1}{(\gamma-1)}\int_0^1 \frac{(1-\vartheta)^{\sigma-1}}{\Gamma(\sigma)}(\phi_q\int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta \\
&-\frac{\xi}{(\gamma-1)(\xi-1)}\int_0^1 \frac{(1-\vartheta)^{\sigma-2}}{\Gamma(\sigma-1)}(\phi_q\int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta \\
&-\frac{\zeta\xi}{(\xi-1)}\int_0^1 \frac{(1-\vartheta)^{\sigma-2}}{\Gamma(\sigma-1)}(\phi_q\int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta \\
&\leq \Delta_1\phi_q(\int_0^1 \mathcal{H}(1,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta) \\
&= \Delta_1\phi_q(\int_{\varpi_1} \mathcal{H}(1,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta + \int_{\varpi_2} \mathcal{H}(1,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta) \\
&\leq \Delta_1\phi_q(U\int_{\varpi_1} \mathcal{H}(1,\eta)a(\eta)d\eta + \mathcal{M}\phi_p(c^*)\int_{\varpi_2} \mathcal{H}(1,\eta)a(\eta)d\eta) \\
&\leq \Delta_1\phi_q(U+\mathcal{M}\phi_p(c^*)\phi_q(\int_0^1 \mathcal{H}(1,\eta)a(\eta)d\eta).
\end{aligned}
$$

From (4.2) and the inequality $(\mathfrak{e}+\mathfrak{y})^{\mathfrak{l}} \leq 2^{\mathfrak{l}}(\mathfrak{e}^{\mathfrak{l}}+\mathfrak{y}^{\mathfrak{l}})$ for any $\mathfrak{e},\mathfrak{y},\mathfrak{l}>0$, we have

$$0 \leq \mathcal{V}\mu(\zeta) \leq \Delta_1 2^{q-1}(\phi_q(U)+\phi_q(\mathcal{M})c^*)\phi_q(\int_0^1 (\mathcal{H}(1,\eta)a(\eta))d\eta) \leq c^*,$$

thus $\mathcal{V}(\Omega_2) \subseteq \Omega_2$. Consequently, by lemma 2.9, $\mathcal{V}$ has a fixed point $\mu \in \Omega_2$, thus the FDE (3.9) has at least one positive solution. □

5. UNIQUENESS OF SOLUTIONS

Theorem 5.1. *Suppose that (N_1), (N_5) and (N_6) hold. Then the fractional differential equations BVP(3.9) has a unique positive solution.*

Proof. Define $\mathcal{F} = \{\mu \in C[0,1] : \mu(\zeta) \geq 0 \text{ on } [0,1]\}$. Then $\mathcal{F}$ is solid cone in $C[0,1]$ with $\mathcal{F}^0 = \{\mu \in C[0,1] : \mu(\zeta) > 0 \text{ on } [0,1]\}$ let $\mathcal{V} : \mathcal{F} \to C[0,1]$ be defined by (4.1)

$$
\begin{aligned}
\mathcal{V}\mu(\zeta) = &\int_0^\zeta \frac{(\zeta-\vartheta)^{\sigma-1}}{\Gamma(\sigma)}(\phi_q\int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta \\
&+\frac{1}{(\gamma-1)}\int_0^1 \frac{(1-\vartheta)^{\sigma-1}}{\Gamma(\sigma)}(\phi_q\int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta \\
&-\frac{\xi}{(\gamma-1)(\xi-1)}\int_0^1 \frac{(1-\vartheta)^{\sigma-2}}{\Gamma(\sigma-1)}(\phi_q\int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta \\
&-\frac{\zeta\xi}{(\xi-1)}\int_0^1 \frac{(1-\vartheta)^{\sigma-2}}{\Gamma(\sigma-1)}(\phi_q\int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta.
\end{aligned}
$$

Clearly $\mathcal{V} : \mathcal{F} \to C[0,1]$. Now we proof that $\mathcal{V}$ is λ-concave non-decreasing operator. For $\mu_1, \mu_2 \in \mathcal{F}$ with $\mu_1(\zeta) \geq \mu_2(\zeta)$ on [0,1] we obtain $\mathcal{V}(\mu_1(\zeta)) \geq \mathcal{V}(\mu_2(\zeta))$ and for $Q(\mathcal{K}\mu) \geq \phi_p(\kappa^\lambda)Q(\mu)$, we have

$$\mathcal{V}\kappa\mu(\zeta) \geq \int_0^1 \mathcal{G}(\zeta,\vartheta)\phi_q(\int_0^0 \mathcal{H}(\vartheta,\eta)\phi_q(\kappa^\lambda)Q(\mu(\eta))d\eta)d\vartheta$$

$$= \kappa^{\lambda} \int_0^1 \mathcal{G}(\zeta,\vartheta)\phi_q(\int_0^1 \mathcal{H}(\vartheta,\eta)Q(\mu(\eta))d\eta)d\vartheta$$
$$= \kappa^{\lambda}\mathcal{V}(\mu(\zeta)).$$

This mean that $\mathcal{V}$ is λ-concave operator. Thus, $\mathcal{V}$ has a unique fixed point. □

6. NONEXISTENCE OF SOLUTIONS

Here we consider Banach space $C[0,1]$ be endowed with the norm $\|\mu\| = \max_{0\leq\zeta\leq 1}|\mu(\zeta)|$.

Theorem 6.1. *Suppose (N_1) holds and let $0 < \alpha < 1$ be defined in (N_4). Then the unique solution $\mu(\zeta)$ for fractional differential equations BVP (3.9) satisfies*

$$\mu(\zeta) \geq d_{\alpha}\|\mu\| \quad for \quad \alpha \leq \zeta \leq 1,$$

where d_{α} is given by (3.27).

Proof. In view of Lemma 3.3 and equation (3.10), we have

$$\mu(\zeta) \leq \int_0^{\zeta} \frac{(\zeta-\vartheta)^{\sigma-1}}{\Gamma(\sigma)}(\phi_q \int_0^1 \mathcal{H}(1,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta$$
$$+\frac{1}{(\gamma-1)}\int_0^1 \frac{(1-\vartheta)^{\sigma-1}}{\Gamma(\sigma)}(\phi_q \int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta$$
$$-\frac{\xi}{(\gamma-1)(\xi-1)}\int_0^1 \frac{(1-\vartheta)^{\sigma-2}}{\Gamma(\sigma-1)}(\phi_q \int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta$$
$$-\frac{\zeta\xi}{(\xi-1)}\int_0^1 \frac{(1-\vartheta)^{\sigma-2}}{\Gamma(\sigma-1)}(\phi_q \int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta$$
$$\leq \frac{1}{\Gamma(\sigma+1)}(\phi_q \int_0^1 \mathcal{H}(1,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)$$
$$+\frac{1}{(\gamma-1)}\int_0^1 \frac{(1-\vartheta)^{\sigma-1}}{\Gamma(\sigma)}(\phi_q \int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta$$
$$-\frac{\xi}{(\gamma-1)(\xi-1)}\int_0^1 \frac{(1-\vartheta)^{\sigma-2}}{\Gamma(\sigma-1)}(\phi_q \int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta$$
$$-\frac{\zeta\xi}{(\xi-1)}\int_0^1 \frac{(1-\vartheta)^{\sigma-2}}{\Gamma(\sigma-1)}(\phi_q \int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta,$$

for $\zeta \in [0,1]$, and

$$\mu(\zeta) \geq \int_0^{\zeta} \frac{(\zeta-\vartheta)^{\sigma-1}}{\Gamma(\sigma)}(\phi_q \int_0^1 \varpi^{\epsilon-1}\mathcal{H}(1,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta$$
$$+\frac{1}{(\gamma-1)}\int_0^1 \frac{(1-\vartheta)^{\sigma-1}}{\Gamma(\sigma)}(\phi_q \int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta$$
$$-\frac{\xi}{(\gamma-1)(\xi-1)}\int_0^1 \frac{(1-\vartheta)^{\sigma-2}}{\Gamma(\sigma-1)}(\phi_q \int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta$$
$$-\frac{\zeta\xi}{(\xi-1)}\int_0^1 \frac{(1-\vartheta)^{\sigma-2}}{\Gamma(\sigma-1)}(\phi_q \int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta$$
$$= \int_0^{\zeta} \sigma(\zeta-\vartheta)^{\sigma-1}\phi_q(\varpi^{\epsilon-1})ds\frac{1}{\Gamma(\sigma+1)}\phi_q(\int_0^1 \mathcal{H}(1,\eta)a(\eta)Q(\eta,\mu(\eta))d\mu)$$
$$+\frac{1}{(\gamma-1)}\int_0^1 \frac{(1-\vartheta)^{\sigma-1}}{\Gamma(\sigma)}(\phi_q \int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta$$

$$
\begin{aligned}
&-\frac{\xi}{(\gamma-1)(\xi-1)}\int_0^1\frac{(1-\vartheta)^{\sigma-2}}{\Gamma(\sigma-1)}(\phi_q\int_0^1\mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta\\
&-\frac{\zeta\xi}{(\xi-1)}\int_0^1\frac{(1-\vartheta)^{\sigma-2}}{\Gamma(\sigma-1)}(\phi_q\int_0^1\mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta\\
&\geq d_\alpha\frac{1}{\Gamma(\sigma+1)}\phi_q(\int_0^1\mathcal{H}(1,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)\\
&+\frac{1}{(\gamma-1)}\int_0^1\frac{(1-\vartheta)^{\sigma-1}}{\Gamma(\sigma)}(\phi_q\int_0^1\mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta\\
&-\frac{\xi}{(\gamma-1)(\xi-1)}\int_0^1\frac{(1-\vartheta)^{\sigma-2}}{\Gamma(\sigma-1)}(\phi_q\int_0^1\mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta\\
&-\frac{\zeta\xi}{(\xi-1)}\int_0^1\frac{(1-\vartheta)^{\sigma-2}}{\Gamma(\sigma-1)}(\phi_q\int_0^1\mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\mu)d\vartheta\\
&\geq d_\alpha[\frac{1}{\Gamma(\sigma+1)}\phi_q(\int_0^1\mathcal{H}(1,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)\\
&+\frac{1}{(\gamma-1)}\int_0^1\frac{(1-\vartheta)^{\sigma-1}}{\Gamma(\sigma)}(\phi_q\int_0^1\mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta\\
&-\frac{\xi}{(\gamma-1)(\xi-1)}\int_0^1\frac{(1-\vartheta)^{\sigma-2}}{\Gamma(\sigma-1)}(\phi_q\int_0^1\mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta\\
&-\frac{\zeta\xi}{(\xi-1)}\int_0^1\frac{(1-\vartheta)^{\sigma-2}}{\Gamma(\sigma-1)}(\phi_q\int_0^1\mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta],
\end{aligned}
$$

for $\zeta\in[\alpha,1]$. Therefore, $\mu(\zeta)>d_\alpha\|\mu\|$ for $\alpha\leq\zeta\leq 1$.
The proof is complete. □

Theorem 6.2. *Suppose that* $(N_1),(N_4)$ *hold. Then the fractional differential equation BVP* (3.9) *has no positive solution.*

Proof. Suppose, to the contrary, then fractional differential equation BVP (3.9) has a positive solution $\mu(\zeta)$. Then by Lemma 3.2, we have

$$
\begin{aligned}
\mu(\zeta)=&\int_0^\zeta\frac{(\zeta-\vartheta)^{\sigma-1}}{\Gamma(\sigma)}(\phi_q\int_0^1\mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta\\
&+\frac{1}{(\gamma-1)}\int_0^1\frac{(1-\vartheta)^{\sigma-1}}{\Gamma(\sigma)}(\phi_q\int_0^1\mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta\\
&-\frac{\xi}{(\gamma-1)(\xi-1)}\int_0^1\frac{(1-\vartheta)^{\sigma-2}}{\Gamma(\sigma-1)}(\phi_q\int_0^1\mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta\\
&-\frac{\zeta\xi}{(\xi-1)}\int_0^1\frac{(1-\vartheta)^{\sigma-2}}{\Gamma(\sigma-1)}(\phi_q\int_0^1\mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta.
\end{aligned}
$$

Therefore, $\mu(\zeta)>b$ on $[0,1]$. In view of (3.25) and (3.26), we obtain

$$
\begin{aligned}
&Q(\zeta,\mu(\zeta))\geq\varrho(\mu(\zeta))\quad on\quad[0,1],\\
&d_\alpha\phi_q(\varrho)\phi_q(\int_\alpha^1\mathcal{H}(1,\eta)a(\eta)Q(\eta,\mu(\eta)d\eta)\int_0^1\frac{(1-\vartheta)^{\sigma-1}}{\Gamma(\sigma)}\phi_q(\varpi^{\epsilon-1})d\vartheta>1.
\end{aligned}
$$

By lemma 3.3 and theorem 4.1, we obtain

$$\begin{aligned}
\|\mu\| = \mu(1) &> \int_0^1 \frac{(1-\vartheta)^{\sigma-1}}{\Gamma(\sigma)}\phi_q\Big(\int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta\Big)d\vartheta \\
&\geq \int_0^1 \frac{(1-\vartheta)^{\sigma-1}}{\Gamma(\sigma)}\phi_q(\varpi^{\epsilon-1})d\vartheta\phi_q\Big(\int_0^1 \mathcal{H}(1,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta\Big) \\
&\geq \int_0^1 \frac{(1-\vartheta)^{\sigma-1}}{\Gamma(\sigma)}\phi_q(\varpi^{\epsilon-1})d\vartheta\phi_q(\varrho)\phi_q\Big(\int_\delta^1 \mathcal{H}(1,\eta)a(\eta)\phi_p(\mu(\eta))d\eta\Big) \\
&\geq \|\mu\|d_\alpha \int_0^1 \frac{(1-\vartheta)^{\sigma-1}}{\Gamma(\sigma)}\phi_q(\varpi^{\epsilon-1})d\vartheta\phi_q(\varrho)\phi_q\Big(\int_\delta^1 \mathcal{H}(1,\eta)a(\eta))d\eta\Big) \\
&\geq \|\mu\|.
\end{aligned}$$

This contradiction completes the proof. □

7. HYERS-ULAM STABILITY

Here, we present Hyers-Ulam stability of the solution for the fractional differential equations BVP with nonlinear ϕ_p-Laplacian operator in suggested problem (3.9).

Definition 7.1. [18] We say that integral Equation (3.10) is HU- stability if there exists positive constant value Δ_2, satisfying: For every $\psi > 0$,
if

$$|\mu(\zeta) - \int_0^1 \mathcal{G}(\zeta,\vartheta)\phi_q \int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta)d\vartheta| \leq \psi, \tag{7.1}$$

there exist $y(\zeta)$ satisfying

$$y(\zeta) = \int_0^1 \mathcal{G}(\zeta,\vartheta)\phi_q \int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,y(\eta))d\eta)d\vartheta, \tag{7.2}$$

such that the pair $(\mu(\zeta), y(\zeta))$ satisfies

$$|\mu(\zeta) - y(\zeta)| = \psi\Delta_2. \tag{7.3}$$

Theorem 7.2. *With the presumptions (N_5) and (N_7), the fractional differential equations BVP with nonlinear ϕ_p-Laplacian operator(3.9) is Hyers-Ulam stable.*

Proof. With the help of definition 7.1, theorem 4.1 and 4.2, let $\mu(\zeta)$ be the real solution of the fractional differential equations BVP of equation (4.1) and $y(\zeta)$ be an approximate solution satisfying (7.2). Then, we have

$$\begin{aligned}
|\mu(\zeta) - y(\zeta)| = |&\int_0^1 \mathcal{G}(\zeta,\vartheta)\phi_q\Big(\int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta\Big)d\vartheta \\
&- \int_0^1 \mathcal{G}(\zeta,\vartheta)\phi_q\Big(\int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,y(\eta))d\eta\Big)d\vartheta| \\
&\leq \int_0^1 |\mathcal{G}(\zeta,\vartheta)||\phi_q\Big(\int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,\mu(\eta))d\eta\Big)d\vartheta
\end{aligned}$$

$$-\phi_q(\int_0^1 \mathcal{H}(\vartheta,\eta)a(\eta)Q(\eta,y(\eta))d\eta)d\vartheta|$$
$$\leq \rho^{q-2}(q-1)\int_0^1 |\mathcal{G}(1,\vartheta)|[\int_0^1 \mathcal{H}(1,\eta)\|a(\eta)\|$$
$$\times|Q(\eta,\mu(\eta))-Q(\eta,y(\eta))|d\eta d\vartheta$$
$$\leq \rho^{q-2}(q-1)\Upsilon_Q[\frac{\gamma}{(\gamma-1)\Gamma(\sigma+1)}-\frac{(\gamma\xi+2\xi)}{(\gamma-1)(\xi-1)\Gamma(\sigma)}]$$
$$\times(\frac{1}{\Gamma(\epsilon+1)}+\frac{1+\gamma+\gamma^{\epsilon-1}}{\Gamma(\epsilon)})\|\mu-y\|\|a\|$$
$$\leq \|\mu-y\|\rho^{q-2}(q-1)\Upsilon_Q(\frac{\gamma(\xi-1)-\sigma(\gamma\xi+2\xi)}{(\gamma-1)(\xi-1)\Gamma(\sigma+1)})$$
$$\times[\frac{1+\epsilon(1+\gamma-\gamma^{\epsilon-1})}{\Gamma(\epsilon+1)}]\|a\|$$
$$\leq \|\mu-y\|\Delta_2, \tag{7.4}$$

where $\Delta_2=\rho^{q-2}(q-1)\Upsilon_Q(\frac{\gamma(\xi-1)-\sigma(\gamma\xi+2\xi)}{(\gamma-1)(\xi-1)\Gamma(\sigma+1)})[\frac{1+\epsilon(1+\gamma-\gamma^{\epsilon-1})}{\Gamma(\epsilon+1)}]\|a\|$. Hence, by (7.3), the integral equation (3.10) is HU-stable. Consequently, the solution to fractional differential equations BVP with nonlinear ϕ_p-Laplacian operator (3.9) is Hyers-Ulam stable. □

8. ILLUSTRATIVE EXAMPLES

In this section, we give application for the characterization of the results proved in sections 4, 5, and 6 to our suggested poblem (3.1).

Example 8.1. Consider

$$\begin{cases}\mathcal{D}^{\frac{5}{2}}(\phi_p\mathcal{D}^{\frac{3}{2}}\mu(\zeta))+a(\zeta)Q(\zeta,\mu(\zeta))=0,\\ (\phi_p(D^{\frac{3}{2}}\mu(\zeta)))^{(j)}|_{\zeta=0}=0, \qquad for \quad j=1,3,4,...,k,\\ (\phi_p(\mathcal{D}^{\frac{3}{2}}\mu(0)))=(\phi_p(\mathcal{D}^{\frac{3}{2}}\mu(\gamma)))', \quad (\phi_p(\mathcal{D}^{\frac{3}{2}}\mu(0)))'=(\phi_p(D^{\frac{3}{2}}\mu(1)))',\\ (\mu(0))^{(i)}=0, \quad for \quad i=2,3,...,k, \quad \gamma(\mu(0))=\mu(1),\xi\mu'(1)=\mu'(0),\end{cases} \tag{8.1}$$

for the existence solution to the BVP (8.1), let $\rho=1, p=2, \beta=\frac{1}{2}$, $\epsilon=5/2, \sigma=3/2, \xi=\gamma=1/4, a(\zeta)=2\zeta$, $Q(\mu(\zeta))=\sqrt{\mu(\zeta)}$, by a simple calculation, we obtain $0<\mathcal{L}<10.025$, and thus considering $\mathcal{L}=10$, we get those conditions N_1, N_2 and the problem (8.1) are satisfied. So, by theorem 4.2, we have BVP (8.1) has at least one positive solution.

Example 8.2. Consider

$$\begin{cases} \mathcal{D}^{\frac{7}{2}}(\phi_p \mathcal{D}^{\frac{5}{2}}\mu(\zeta)) + a(\zeta)Q(\zeta,\mu(\zeta)) = 0, \quad , \\ (\phi_p(D^{\frac{5}{2}}\mu(\zeta)))^{(j)}|_{\zeta=0} = 0, \quad for \quad j = 1,3,4,...,k, \\ (\phi_p(\mathcal{D}^{\frac{5}{2}}\mu(0))) = (\phi_p(\mathcal{D}^{\frac{5}{2}}\mu(\gamma)))', \quad (\phi_p(\mathcal{D}^{\frac{5}{2}}\mu(0)))' = (\phi_p(\mathcal{D}^{\frac{5}{2}}\mu(1)))', \\ (\mu(0))^{(i)} = 0, \quad for \quad i = 2,3,...,k, \quad \gamma(\mu(0)) = \mu(1), \xi\mu'(1) = \mu'(0), \end{cases} \tag{8.2}$$

for the uniqueness of solution to the BVP (8.2), by using theorem 5.1. Let $\epsilon = 7/2, \sigma = 5/2, \xi = \gamma = 1/3, a(\zeta) = t^3, Q(\mu(\zeta)) = (\mu(\zeta))^{\frac{1}{5}}$. Clearly, (N_1) and (N_2) hold. Choosing $\lambda = 1/3$, then (N_6) is satisfied. Thus, by Theorem 5.1, we get that BVP has a unique solution.

Example 8.3. Consider

$$\begin{cases} \mathcal{D}^{\frac{3}{2}}(\phi_p \mathcal{D}^{\frac{5}{2}}\mu(\zeta)) + a(\zeta)Q(\zeta,\mu(\zeta)) = 0, \quad , \\ (\phi_p(\mathcal{D}^{\frac{5}{2}}\mu(\zeta)))^{(j)}|_{\zeta=0} = 0, \quad for \quad j = 1,3,4,...,k, \\ (\phi_p(\mathcal{D}^{\frac{5}{2}}\mu(0))) = (\phi_p(\mathcal{D}^{\frac{5}{2}}\mu(\gamma)))', \quad (\phi_p(D^{\frac{5}{2}}\mu(0)))' = (\phi_p(\mathcal{D}^{\frac{5}{2}}\mu(1)))', \\ (\mu(0))^{(i))} = 0, \quad for \quad i = 2,3,...,k, \gamma(\mu(0)) = \mu(1), \xi\mu'(1) = \mu'(0), \end{cases} \tag{8.3}$$

for non-existence solution to the BVP (8.3), let $p = 2$, we have $\epsilon = 3/2, \sigma = 5/2, \gamma = \xi = 3/2, a(\zeta) = \zeta^3, Q(\mu(\zeta)) = (\mu(\zeta))^{\frac{1}{3}}$. Clearly, (N_1) holds . Choosing $\alpha = 1/2$, by a simple calculation, we obtain $d_\alpha = 0.3495, \varrho > 11.912$. Let $\varrho = c = 12$. Then (N_4) is satisfied. By using theorem 6.2, the BVP (8.3) has no positive solution.

9. CONCLUSION

With the help of Schauder theory and functional analysis, we achieved both of existence, uniqueness, and non-existence criterion of the solution for FDEs with the ϕ_p-Laplacian operator. For these goals, we converted boundary value problem of FDEs into integral form by using the concept of Green functions. As well as we used the Hyers-Ulam technique to prove the stability of the solution to our proposed problem. For an application of our results, we included expressive examples using Mathematical program.

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