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A simulation study of the COVID-19 pandemic based on the Ornstein-Uhlenbeck processes

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Abstract

The rapid spread of coronavirus disease (COVID-19) has increased the attention to the mathematical modeling of spreading the disease in the world. The behavior of spreading is not deterministic in the last year. The purpose of this paper is to present a stochastic differential equation for modeling the data sets of the COVID-19 involving infected, recovered, and dead cases. At first, the time series of the covid-19 is modeled with the Ornstein-Uhlenbeck process and then using the Ito lemma and Euler approximation the analytical and numerical simulations for the stochastic differential equations are achieved. Parameters estimation is done using the maximum likelihood estimator. Finally, numerical simulations are performed using reported data by the world health organization for case studies of Italy and Iran. The numerical simulations and root mean square error criteria confirm the accuracy and efficiency of the findings of the present study.

Keywords. COVID-19, Numerical simulation, Ornstein Uhlenbeck process, Stochastic analysis.
2010 Mathematics Subject Classification. 60H10, 37H10.

1. Introduction

In December 2019, coronavirus was first reported in Wuhan, China, as an infectious disease caused by a newly discovered coronavirus. Currently, COVID 19 has a high rate of infection and mortality affecting governments, researchers, and all people worldwide. Since the conditions for an infectious disease to be wiped out or survive are complicated then presenting mathematical models is of great importance for predicting the subsequent behavior of the diseases. Following the prevalence of the new COVID-19 countries around the world have gradually taken measures to prevent the pandemic. These precautionary measures, which began with the closure of borders and the suspension of foreign flights, continued with the imposition of restrictions on domestic transportation and the announcement of traffic bans and traffic restrictions. Governments alleviate the burden of the corona by providing assistance and support to entrepreneurs and employees. Travel and transportation restrictions and measures to reduce social activities and business-related actions and transfer of citizens living abroad and international solidarity. Until now the behavior of the COVID-19 pandemic has been studied using several developed mathematical models. Fanelli and Piazza studied the spread of COVID-19 spreading in Italy, China, and France using the SIR and SIRD models [3]. Jiao et al. presented an SEIR epidemic model with infectivity in the incubation period and homestead isolation on the susceptible [5]. Ivorra et al. investigated mathematical modeling of the spread of the COVID-19 taking into account the undetected infections [4]. Melliani et al. use a fuzzy environment via a mathematical model to study the spread of COVID-19 in Brazil [11]. Dokuyucu and Çelik proposed a new model to analyze the infection due to the COVID-19. The model emphasizes the importance of environmental reservoirs in spreading the infection and infecting others [2]. Tilahun et al. purposed the stochastic model of measles transmission dynamics with double dose vaccination [14]. Khoshnaw et al. present sensitivity analysis and mathematical modeling for coronavirus disease in predicting future behaviors [7]. Sarkar et al. presented a model for forecasting the COVID-19 pandemic in India using the $SARII_qS_q$ model [13].

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Lina et al. extended the SEIR (susceptible-exposed- infectious-removed) compartment model to study the dynamics of COVID-19 incorporates public perception of risk and the number of cumulative cases [8]. An extended SEIR model was proposed by Khajanchi et al. They studied the transmission dynamics of COVID-19 using a short-term prediction based on the data from India [6]. Maleki et al. used the time series ARMA model to simulate and predict the spread and death rate of corona virus in the world [9]. Babaei et al. introduced a stochastic model to describe the spread of corona virus by considering several disease compartments related to different age groups [1]. The present paper aims at developing a stochastic differential equation based on the Ornstein-Uhlenbeck process to analyze the behavior of the infected, recovered, and lost people in the world. The remainder of this paper is as follows. Section 2, presents the preliminaries about the Ornstein-Uhlenbeck processes. In section 3 the parameter estimation of the model is discussed. Numerical simulations are carried out in section 4. Finally, section 5 introduces the discussion and conclusions.

2. Methodology

The number of infected and death cases has increased daily in some countries, especially in the United States of America, Italy, Spanish, Iran, Germany, and other countries. Particular policies and procedures are needed to manage the dangers of the spread of the COVID-19. Therefore, forecasting and predicting future confirmed cases are critical. The figures for reported data from some countries indicate that the behavior of infected, recovered , and death cases are not deterministic. It is appropriate to use a stochastic model to fit the data to capture this behavior. We want to investigate stochastic mathematical models for these time series. The Ornstein Uhlenbech (OU) processes are an effective and appropriate class of stochastic differential equations for simulating, modeling, and predicting real datasets. The OU process is a stochastic process with applications in the physical sciences and financial mathematics. This process describes the characteristics of the data that seemingly fluctuates randomly over time. The OU process is a stationary Gaussian process that is defined by the following stochastic differential equation [12].

Definition 1: The process $\{X_t\}$ is an Ornstein-Uhlenbeck process if it is cadlag (It is right continued and has a left limit at every point) and satisfies the stochastic differential equation,

$$dX_t = \lambda(\mu - X_t)dt + \sigma dB_t,\ X_0 > 0 \tag{2.1}$$

where λ is the mean reversion rate, μ is the mean, σ is the volatility and B_t is a Brownian motion. Moreover, it is a process that describes the characteristics of the process that drifts toward the mean, a mean-reverting process. The analytical solution of equation (2.1) is,

$$X(t) = X_0 \exp(-\lambda t) + \mu(1 - exp(-\lambda t)) + \sigma \int_0^t \exp(-\lambda s) dB(s) \tag{2.2}$$

The mean and variance of $X(t)$ with some algebraic manipulation can be obtained,

$$\begin{aligned} &E(X_t) = exp(-\lambda t)E(X_0) + \mu(1 - exp(-\lambda t)) \\ &var(X_t) = \tfrac{\sigma^2}{2\lambda} + \left(var(X_0) - \tfrac{\sigma^2}{2\lambda}\right)\exp(-2\lambda t) \end{aligned} \tag{2.3}$$

respectively. Let $\Pi_n = \{0, \delta t, 2\delta t, \dots, n\delta t\}$ is a partition for time interval $[0, T]$, where $\delta t = \frac{T}{n}$. The Euler Maruyama scheme for model (2.1) is as follows,

$$X(t + \delta t) = X(t) + \lambda\big(\mu - X(t)\big)\delta t + \sigma \delta B_t \tag{2.4}$$

where $\delta B_t \sim N(0, \delta t)$.

3. Parameter estimation

The parameters estimation is a serious challenge in any mathematical model-based study. In statistics, the maximum likelihood estimation (MLE) is a method of estimating the unknown parameters by maximizing a likelihood function when the observed data is most probable. From a statistical standpoint, a given set of observations is a random sample from an unknown population. The distribution of $X_{t+\delta t}$ in the Euler scheme is given by,

$$f(X_{t+\delta t}|X_t) = \frac{1}{\sqrt{2\pi\sigma^2\delta t}} \exp \frac{-[X_{t+\delta t} - (X_t + \lambda(\mu - X_t)\delta t)]^2}{2\sigma^2 \delta t} \tag{3.1}$$

so the log-likelihood is as follows,

$$\log(L(\theta)) = \log f_0(X_0|\theta) + \sum_{i=1}^{n-1} \log f(X_i|X_{i-1}, \theta) \tag{3.2}$$

The goal of maximum likelihood estimation is to find the values of the model parameters that maximize the likelihood function over the parameters space, that is $\theta = argmax(L(\theta, x))$. The logharitm is a monotonic function, the maximum of $L(\theta; X)$ occurs at the same value of θ as does the maximum of L_n. It is easy to show that the parameters are defined as follows,

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} \frac{X_i - X_{i-1}(1 - \lambda\delta t)}{\lambda\delta t} \tag{3.3}$$

$$\hat{\lambda} = \frac{1}{\delta t} \frac{\sum_{i=1}^{n}(X_i - X_{i-1})(\mu - X_{i-1})}{\sum_{i=1}^{n}(\mu - X_{i-1})^2} \tag{3.4}$$

$$\hat{\sigma^2} = \frac{1}{n} \sum_{i=1}^{n} \frac{\big(X_i - (X_{i-1} + \lambda(\mu - X_{i-1})\delta t\big)^2}{\delta t} \tag{3.5}$$

The model parameters have been estimated assuming the initial population size and fitting the model simulation with the observed COVID-19 cases.

4. Numerical simulation

In this section, we briefly present the simulation techniques for the reported data on the daily infected, recovered, and death cases in Iran and Italy. Consider the random vector $X_t = (I_t, R_t, D_t)$, where I_t denotes the Infected people, R_t denotes the recovered people, and D_t denotes the death people at time t respectively. This process expresses the characteristics of the data that seemingly fluctuate randomly over time. Now let us consider $X(t)$ satisfies in equation (2.1). So we have

$$\begin{aligned} dI_t &= \lambda_I(\mu_I - I_t)dt + \sigma_I dB_I(t) \\ dR_t &= \lambda_R(\mu_R - R_t)dt + \sigma_R dB_R(t) \\ dD_t &= \lambda_D(\mu_D - D_t)dt + \sigma_D dB_D(t) \end{aligned} \tag{4.1}$$

Where B_I, B_R and B_D are i.i.d brownian motions. We used the real data provided by the world health organization from 24 Feb to 9 Aug 2020 [15]. The simulation of process is carried out using MATLAB programming with 1000 iteration. Table 1. summarized the parameter estimation for case studies of Italy and Iran. In the time series data points, the data sets are normalized because the local variance of the series was larger when the level of the series was higher [10]. By performing this change of scale, a stationary or integrated model can likely be fitted after the transformation. The normalized data sets for Iran are shown in Figure 1. To check this model, the mean square error criteria are achieved for each data set. The root means square error measures the performance of our model to the given data sets. Table 2 presents these criteria for simulated cases in Italy and Iran. The simulated and real reported data for infected recovered and death cases of Italy are presented in Figures 2-4 respectively. Figures 5-7 show the simulated data for Iran.

5. Conclusion

This paper presents a stochastic differential equation for modeling the data sets of COVID-19 involving infected, recovered, and death cases. Numerical simulations and parameter estimations using MATLAB programming display the accuracy and efficiency of the present work. The analysis presented in this report can be replicated for other countries and updated regularly during the global outbreak. It is hoped that the results of the present study will be of benefit to public health authorities contributing to outbreak prevention, response, and recovery during these crucial moments of the pandemic.

TABLE 1. The estimated parameters of Italy and Iran

Country	Infected people	Recovered people	Death people
Italy	$\hat{\lambda}_I = 2.8621$	$\hat{\lambda}_R = 2.8051$	$\hat{\lambda}_D = 2.5489$
	$\hat{\mu}_I = 0.0134$	$\hat{\mu}_R = 0.0065$	$\hat{\mu}_D = 0.0254$
	$\hat{\sigma}_I = 0.2849$	$\hat{\sigma}_R = 0.1293$	$\hat{\sigma}_D = 0.4043$
Iran	$\hat{\lambda}_I = 2.4866$	$\hat{\lambda}_R = 4.8640$	$\hat{\lambda}_D = 2.1892$
	$\hat{\mu}_I = 0.0203$	$\hat{\mu}_R = 0.0044$	$\hat{\mu}_D = 0.0287$
	$\hat{\sigma}_I = 0.2948$	$\hat{\sigma}_R = 0.2229$	$\hat{\sigma}_D = 0.2382$

TABLE 2. The root mean square error for Italy and Iran

RMSE	**Italy**	**Iran**
For infected people	704.13	220.28
For recovered people	860.12	521.86
For death people	66.58	16.51

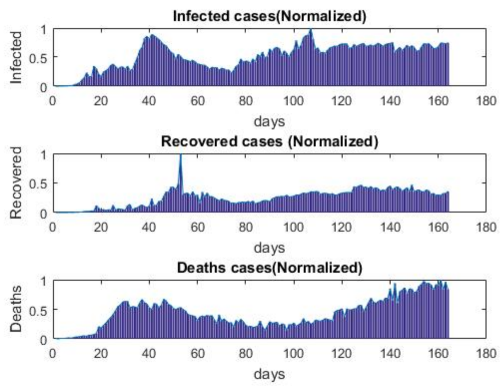

FIGURE 1. Stationary time series plot of dataset for Iran.

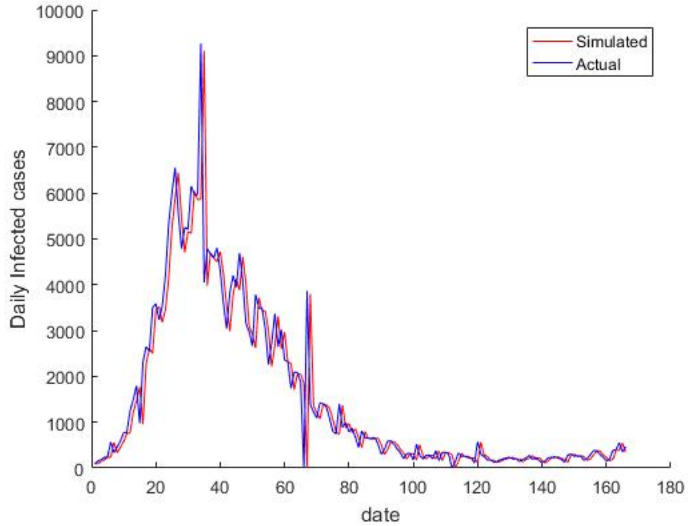

FIGURE 2. Simulated and real repoted data of Infected people in Italy .

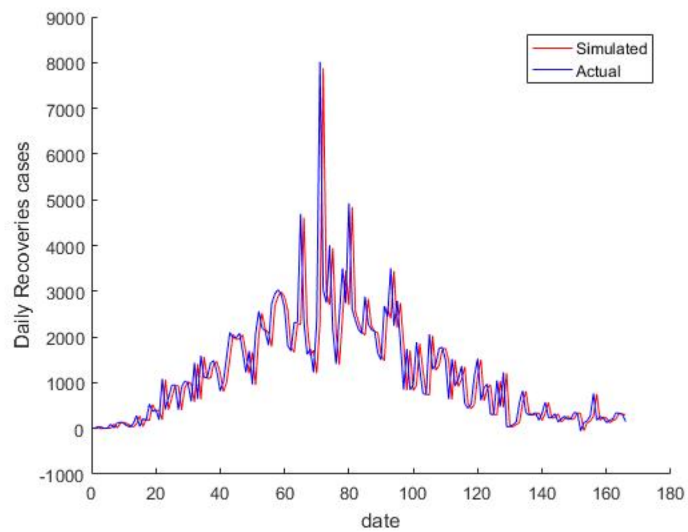

FIGURE 3. Simulated and real repoted data of recovered people in Italy.

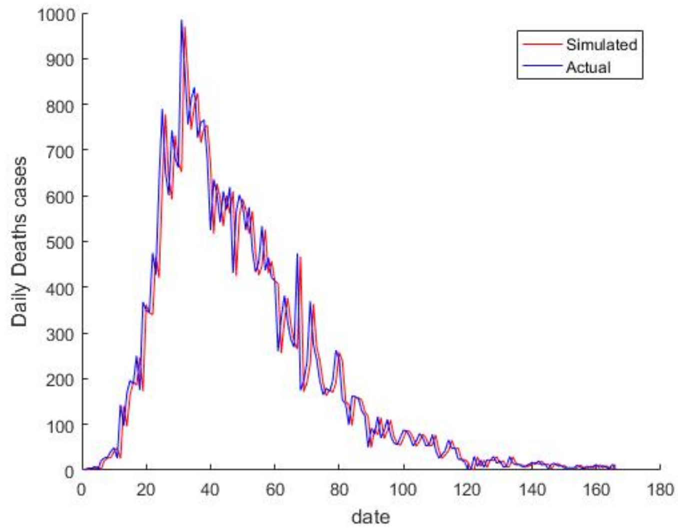

FIGURE 4. Simulated and real repoted data of death people in Italy.

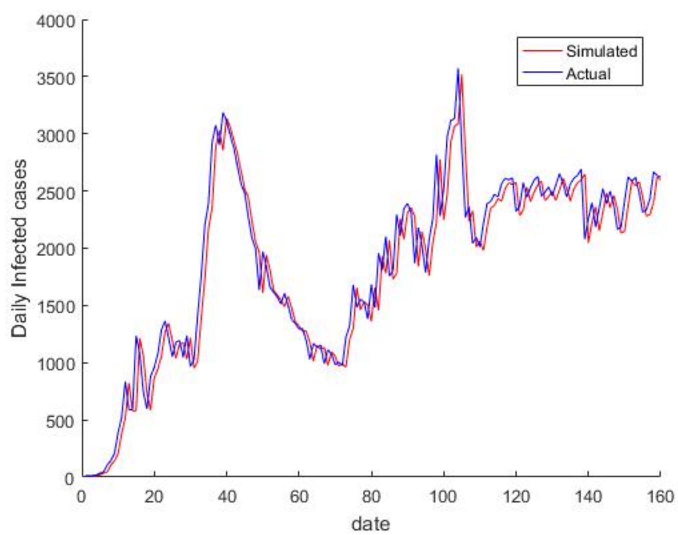

FIGURE 5. Simulated and real repoted data of Infected people in Iran.

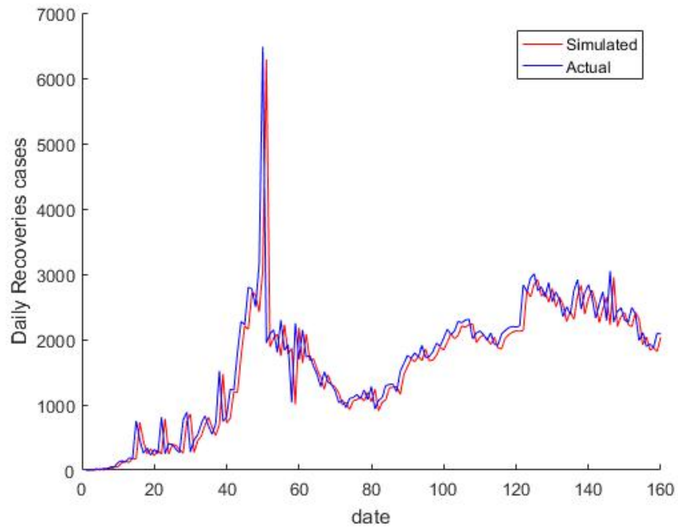

FIGURE 6. Simulated and real repoted data of recovered people in Iran.

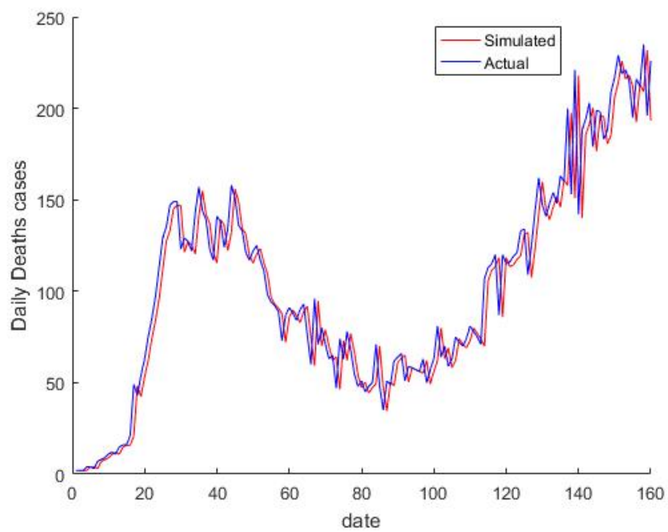

FIGURE 7. Simulated and real repoted data of death people in Iran.

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