Combining the reproducing kernel method with a practical technique to solve the system of nonlinear singularly perturbed boundary value problems

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Abstract
In this paper, a reliable new scheme is presented based on combining Reproducing Kernel Method (RKM) with a practical technique for the nonlinear problem to solve the System of Singularity Perturbed Boundary Value Problems (SSPBVP). The Gram-Schmidt orthogonalization process is removed in the present RKM. However, we provide error estimation for the approximate solution and its derivative. Based on the present algorithm in this paper, can also solve linear problem. Several numerical examples demonstrate that the present algorithm does have higher precision.

Keywords. Reproducing kernel method, Singularly perturbed BVPs, Convergence analysis, Error analysis, System of differential equations.

2010 Mathematics Subject Classification. 65L04, 65L11.

1. INTRODUCTION

The systems of singularly perturbed BVPs are used in various fields of engineering sciences. They have applications in modeling electrochemical reactions, turbulence in water wave when they interact with current, electroanalytical chemistry when investigating diffusion processes complicated by chemical reactions, equations of predator—prey population dynamics, control theory[19, 24, 28, 30, 27]. Many authors have presented various mathematical methods to solve SSPBVP [29, 26, 34, 15]. In [2, 3, 4, 5, 6, 7, 38, 16, 23, 18, 13, 25, 17, 21] RKM is provided to solve some system of
the differential and Volterra integral equations. Some applications of reproducing kernel method without using the orthogonalization process are given in [31, 32, 1, 8, 9]. Consider the following system of singularly perturbed differential equations:

\[
\begin{align*}
L_1(u_1(t)) &= N_1(t, U(t)) + F_1(t), \\
L_2(u_2(t)) &= N_2(t, U(t)) + F_2(t), \\
& \vdots \\
L_k(u_k(t)) &= N_k(t, U(t)) + F_k(t), \\
u_d(a) &= u_d(b) = \gamma_d,
\end{align*}
\]

(1.1)

where \( t \in \Omega = [a, b] \) and \( \gamma_d \) is constant,

\[
L_d(\varepsilon_d(t)) = \varepsilon_d u_d''(t) + \frac{1}{P_d(t)} u_d'(t) + \frac{1}{Q_d(t)} u_d(t),
\]

and \( N_d(t, U(t)) \) for \( d = 1, \ldots, k \) are linear and nonlinear differential operators respectively,

\[
F_d(t), \quad P_d(t), \quad Q_d(t),
\]

are sufficiently smooth functions and \( U(t) = (u_1(t), u_2(t), \ldots, u_k(t))^T \) is unknown vector functions which must be determined. \( \varepsilon_d \) is perturbation parameter and \( 0 < \varepsilon_d \ll 1 \). The present method solve the equations in system of differential equations (1.1), simultaneously. In this paper, regardless of whether the system of differential equations is linear or nonlinear, we solve the system of differential equations with general technique for nonlinear problem. In fact, for solving any equation in the system of differential equations, we suppose only one of the functions and its derivatives as unknown that we call the linear operator \((L_d(\varepsilon_d(t)))\), and the remaining unknown functions are assumed as part of nonlinear term that we call the nonlinear operator \((N_d(t, U(t)))\), where \( d = 1, 2, \ldots, k \) and eventually the problem is converted to nonlinear form and using general technique for the nonlinear problems we solve the system of differential equations. In fact, nonlinear problem turns into a number of iterations to solve its corresponding linear problem. However, based on fundamental concept of general technique for nonlinear problems, the number of iterations for the nonlinear terms are very important to solve the system of differential equations (1.1) by using present method. Obviously, if the number of iteration is higher, the approximate solution is more accurate.

2. Preliminaries

**Definition 2.1.** Consider the reproducing kernel space \( W^m_2[a, b] \) such that \( y^{m-1}(t) \) is absolutely continuous and \( y^{(m)}(t) \in L^2[a, b] \) and \( y(a) = y(b) = 0 \) where \( y(t) \in W^m_2[a, b] \). The inner product and norm are given as follows:

\[
<y_1(t), y_2(t)>_{W^m_2[a, b]} = \sum_{i=0}^{m-1} y_1^{(i)}(a)y_2^{(i)}(a) + \int_{a}^{b} y_1^{(m)}(t)y_2^{(m)}(t)dt,
\]

\[
\|y(t)\|_{W^m_2[a, b]} = \sqrt{<y, y>_{W^m_2[a, b]}},
\]
Theorem 2.2. Reproducing kernels $W^3_2[a, b]$ and $W^3_2[a, b]$ for $a = 0$, $b = 1$ are given respectively as [14],

$$
\mathcal{R}(t, x) = \begin{cases} 
50tx^4 + 13t^5 - 15t^4x + 105t^3x^2 - 25t^2x^3 & t \leq x \\
13x^5 + 50t^4x - 25t^3x^2 + 105t^2x^3 - 15tx^4 & x < t 
\end{cases}
$$

$$
\mathcal{K}(t, x) = \begin{cases} 
1 + t, & t \leq x, \\
1 + x, & x < t.
\end{cases}
$$

Consider one of the equation in the SSPBVP (1.1) in iterative scheme as

$$
\mathcal{L}_{d}(u_{n,0}(t)) = \mathcal{N}_{d}(t, U_{n-1}(t)) + \mathcal{F}_{d}(t),
$$

where $\mathcal{L}_{d} : W^3_2[a, b] \rightarrow W^3_2[a, b]$ is a bounded linear operator, and $\mathcal{N}_{d}$ is a continuous nonlinear operator, and

$$
U_{n-1}(t) = [u_{1,n-1}(t), u_{2,n-1}(t), \ldots, u_{k,n-1}(t)]^T.
$$

For $n = 1$, $U_0(t) = [u_{1,0}(t), u_{2,0}(t), \ldots, u_{k,0}(t)]^T$ that satisfies boundary conditions of the problem (1.1), therefore $U_0(t) = [\gamma_1, \gamma_2, \ldots, \gamma_k]^T$, and in each iteration $n = 2, 3, \ldots$, we obtain $U_{n-1}(t)$. We choose a dense set $\{t_i\}_{i=1}^{\infty}$ on $\Omega$ and define, $\phi_{d,i}(t) = \mathcal{K}_x(t)|_{x=t_i}$, and $\psi_{d,i}(t) = \mathcal{L}_{d}\phi_{d,i}(t)$, where $\mathcal{L}_{d}$ is adjoint operator of $\mathcal{L}_{d}$ and suppose $\mathcal{L}_{d}^{-1}$ exists. Consider one of the following complete function systems that are obtained from reproducing kernel $\mathcal{R}_x(t)$ of the space $W^3_2[a, b]$,

$$
\psi_{d,i}(t) = \mathcal{L}_{d,x}\mathcal{R}_x(t)|_{x=t_i}, \quad \text{or} \quad \mathcal{R}_x(t)|_{x=t_i} = \mathcal{R}_x(t)|_{x=t_i}.
$$

Theorem 2.3. The Exact solution of equation

$$
\mathcal{L}_{d}(u_{n,0}(t)) = \mathcal{N}_{d}(t, U_{n-1}(t)) + \mathcal{F}_{d}(t)
$$

can be represented as,

$$
u_{d,n}(t) = \sum_{i=1}^{\infty} c_{d,i,n} \psi_{d,i}(t), \quad n = 1, 2, \ldots, \tag{2.1}\n$$

where the unknown coefficients $c_{d,i,n}$ must be determined.

Proof. See [35, 36, 33]. \hfill \Box

3. Construction of the numerical method

Suppose the approximate solution with $N$ collocation points throughout the interval $\Omega$ is,

$$
u_{d,n,N}(t) = \sum_{i=1}^{N} c_{d,i,n} \psi_{d,i}(t), \quad \tag{3.1}\n$$

Uncorrected Proof
and \( n \) must be sufficiently large. Now determine the unknown coefficients \( c_{d,i,n} \) where \( i = 1, 2, \ldots, N \) and \( n = 1, 2, \ldots \). First consider following equation,

\[
\mathbb{R}_{d,N}(t) = \mathcal{L}_d(u_{d,n,N}(t)) - \mathcal{N}_d(t, U_{n-1,N}(t)) - \mathcal{F}_d(t),
\]

(3.2)
determine the unknown coefficients \( c_{d,i,n} \) such that \( \langle \mathbb{R}_{d,N}(t), \psi_{d,j}(t) \rangle = 0 \) for \( j = 1, 2, \ldots, N \), therefore we have,

\[
< \mathbb{R}_{d,N}(t), \psi_{d,j}(t) > = < \mathcal{L}_1(u_{d,n,N}(t)) - \mathcal{N}_d(t, U_{n-1,N}(t)) - \mathcal{F}_d(t), \psi_{d,j}(t) > =
\]

\[
< \mathcal{L}_1(u_{d,n,N}(t)), \psi_{d,j}(t) > - < \mathcal{N}_d(t, U_{n-1,N}(t)) + \mathcal{F}_d(t), \psi_{d,j}(t) > =
\]

\[
< \mathcal{L}_d \sum_{i=1}^{N} c_{d,i,n} \psi_{d,i}(t), \psi_{d,j}(t) > - < \mathcal{N}_d(t, U_{n-1,N}(t)) + \mathcal{F}_d(t), \psi_{d,j}(t) > =
\]

\[
\sum_{i=1}^{N} c_{d,i,n} < \mathcal{L}_1(\psi_{d,i}(t)), \psi_{d,j}(t) > - < \mathcal{N}_d(t, U_{n-1,N}(t)) + \mathcal{F}_d(t), \psi_{d,j}(t) > = 0,
\]

and therefore we have following system of linear algebraic equations and solve it to obtain the unknown coefficients \( c_{d,i,n} \) for \( n = 1, 2, \ldots \) and \( j = 1, 2, \ldots, N \) as follows

\[
\sum_{i=1}^{N} c_{d,i,n} \mathcal{L}_d \psi_{d,i}(t)|_{t=t_j} = \mathcal{N}_d(t, U_{n-1,N}(t))|_{t=t_j} + \mathcal{F}_d(t_j),
\]

(3.3)

and \( \psi_{d,i}(t) = \mathcal{L}_{d,x} \mathcal{R}_x(t)|_{x=x_i} \) and \( i = 1, 2, \ldots, N \) and \( \mathcal{L}_{d,x} \) is the linear operator in equation (1.1). We obtain a approximate solution for the SSPBVP (1.1), define the approximate solution in the form,

\[
U_{n,N}(t) = [u_{1,n,N}(t), u_{2,n,N}(t), \ldots, u_{k,n,N}(t)]^T, \ n = 1, 2, \ldots,
\]

where \( n \) is the number of iterations for nonlinear term \( \mathcal{N}_d(t, U_{n-1,N}(t)) \) and,

\[
\begin{cases}
\ u_{1,n,N}(t) = \sum_{i=1}^{N} c_{1,i,n} \psi_{1,i}(t), \\
\ u_{2,n,N}(t) = \sum_{i=1}^{N} c_{2,i,n} \psi_{2,i}(t), \\
\ u_{k,n,N}(t) = \sum_{i=1}^{N} c_{k,i,n} \psi_{k,i}(t),
\end{cases}
\]

(3.4)

and we determine the coefficients \( c_{d,i,n} \) where \( d = 1, 2, \ldots, k \) by solving the following linear system of algebraic equations,

\[
\begin{cases}
\sum_{i=1}^{N} c_{1,i,n} \mathcal{L}_1 \psi_{1,i}(t)|_{t=t_j} = \mathcal{N}_1(t, U_{n-1,N}(t))|_{t=t_j} + \mathcal{F}_1(t_j), \\
\sum_{i=1}^{N} c_{2,i,n} \mathcal{L}_2 \psi_{2,i}(t)|_{t=t_j} = \mathcal{N}_2(t, U_{n-1,N}(t))|_{t=t_j} + \mathcal{F}_2(t_j), \\
\vdots \\
\sum_{i=1}^{N} c_{k,i,n} \mathcal{L}_k \psi_{k,i}(t)|_{t=t_j} = \mathcal{N}_k(t, U_{n-1,N}(t))|_{t=t_j} + \mathcal{F}_k(t_j),
\end{cases}
\]

(3.5)

where \( n = 1, 2, \ldots, j = 1, 2, \ldots, N \) and for \( n = 1 \) we choose the initial vector function \( U_{0,N}(t) = [\gamma_1, \gamma_2, \ldots, \gamma_k]^T \) and we obtain \( U_{n-1,N}(t) \) for each iteration, \( n = 2, 3, \ldots \).
Remark 3.1. Based on the mentioned algorithm, obviously all of the equations in the system of differential equations (1.1) are solved simultaneously. This Property makes the present method easy to apply linear and nonlinear system of differential equations.

4. Convergence and error analysis

Lemma 4.1. \( \mathcal{E} = \{ u_{d,n,N}(t) \mid \| u_{d,n,N}(t) \|_{W_2^2[a,b]} \leq \varrho_d, \ d = 1, 2, \ldots, k \} \) is compact in \( C[a,b] \) and \( \varrho_d \) is a constant.

Proof. See [37, 22, 10, 33]. \( \square \)

Theorem 4.2. The approximate solution \( U_{n,N}(t) \) converge to the exact solution \( U(t) \).

Proof. From [33], \( \| u_{d,n,N}(t) - u_{d}(t) \|_{W_2^2} \rightarrow 0 \) when \( N \rightarrow \infty \). Now we prove that \( u_{d,n,N}(t) \rightarrow u_{d}(t) \) uniformly convergent when, \( N \rightarrow \infty, l \rightarrow \infty \), from the reproducing properties we have,

\[
|u_{d,n,N}(t) - u_{d}(t)| = |< u_{d,n,N}(x) - u_{d}(x), R(x) >_{W_2^2}| \\
\leq \| u_{d,n,N}(x) - u_{d}(x) \|_{W_2^2} \| R(x) \|_{W_2^2} \\
\leq m \| u_{d,n,N}(x) - u_{d}(x) \|_{W_2^2},
\]

therefore \( u_{d,n,N}(t) \rightarrow u_{d}(t) \) uniformly convergent when, \( N \rightarrow \infty \). Similar to above equation we have \( u^{(j)}_{d,n,N}(t) \rightarrow u^{(j)}_{d}(t) \) uniformly convergent when, \( N \rightarrow \infty \) for \( j = 1, 2 \).

\( \square \)

Theorem 4.3. Let \( U_{n,N}^{(j)}(t) \) be approximate solution in \( W_2^2[a,b] \) and \( U^{(j)}(t) \) where \( j = 0, 1 \), be exact solution for the system of differential equations (1.1), then

\[
\| u_{d,n,N}(t) - u_{d}(t) \|_{\infty} = \max_{t \in [a,b]} | u_{d,n,N}(t) - u_{d}(t) | \leq \hat{c}_d \ h_d^2,
\]

\[
\| u_{d,n,N}'(t) - u_{d}'(t) \|_{\infty} = \max_{t \in [a,b]} | u_{d,n,N}'(t) - u_{d}'(t) | \leq \hat{c}_d \ h_d,
\]

where \( h_d = \max_{i+1 \rightarrow i} t_i \) and \( d = 1, 2, \ldots, k, \ i = 1, 2, \ldots, N \) and \( \hat{c}_d, \hat{c}_d \) are positive constants.

Proof. See [20, 11, 12, 33]. \( \square \)

5. Numerical results

In this section several numerical examples are presented to show the validity of the theoretical results. Let \( \mathcal{S}_1 = \{(\varepsilon_1, \varepsilon_2) \mid \varepsilon_1 = 1, 10^{-2}, 10^{-4}, \ldots, 10^{-30}; \ \varepsilon_2 = 1, 10^{-2}, 10^{-4}, \ldots, 10^{-30} \} \), \( \mathcal{S}_2 = \{(\varepsilon_1, \varepsilon_2, \varepsilon_3) \mid \varepsilon_1 = 1, 10^{-2}, 10^{-4}, \ldots, 10^{-30}; \varepsilon_2 = 1, 10^{-2}, 10^{-4}, \ldots, 10^{-30}; \varepsilon_3 = 1, 10^{-2}, 10^{-4}, \ldots, 10^{-30} \} \). Absolute error for the approximate solution and its derivative are given with, \( E_{u,N}(t) = | u_{N}(t) - u(t) | \), \( E_{v,N}(t) = | v_{N}(t) - v(t) | \), \( E_{z,N}(t) = | z_{N}(t) - z(t) | \) and \( E_{u',N}(t) = | u'_{N}(t) - u'(t) | \), \( E_{v',N}(t) = | v'_{N}(t) - v'(t) | \), \( E_{z',N}(t) = | z'_{N}(t) - z'(t) | \) respectively, and collocation points are \( t_i = \frac{N}{N+1}, \ i = 1, 2, \ldots, N \) and for the Figures 1,2,\ldots,10 suppose \( \varepsilon_1, \varepsilon_2, \varepsilon_3 = 10^{-20} \).
Algorithm

1. Choose $N$ points in the interval $\Omega$;
2. Put $\psi_{d,i}(t) = L_{d,i} R_x(t)|_{x=t_i}$ and $d = 1, 2, \ldots, k; i = 1, 2, \ldots, N$;
3. Put $G_d = \left[ L_{d,i} \psi_{d,j}(t)|_{x=t_i} \right]_{i,j=1,2,\ldots,N}$ and $d = 1, 2, \ldots, k$;
4. Choose a proper value of $n$, such that $n$ is the number of iterations for the nonlinear terms;
5. Put $\ell = 0$;
6. Choose the initial functions $U_{\ell,N}(t) = [u_1, u_2, \ldots, u_k]^T$ (for the problem (1.1), suppose $U_{0,N}(t) = [0, 0, \ldots, 0]^T$);
7. Set $\ell = \ell + 1$;
8. For $d = 1, 2, \ldots, k$ compute $F_d = \left[ N_d(t, U_{\ell-1,N}(t)) |_{t=t_i} + F_d(t_i) \right]_{i=1,2,\ldots,N}$;
9. For $d = 1, 2, \ldots, k$ solve system $G_d A_d = F_d$, where
   \[ A_d = [c_{d,1}, c_{d,2}, \ldots, c_{d,N}]^T \]
10. Put $u_{d,\ell,N}(t) = \sum_{j=1}^{N} c_{d,j} \psi_{d,j}(t)$ where $d = 1, 2, \ldots, k$;
11. Put $U_{\ell,N}(t) = [u_{1,\ell,N}(t), u_{2,\ell,N}(t), \ldots, u_{k,\ell,N}(t)]^T$;
12. If $\ell < n$ then go to step 7 else stop the algorithm.

Example 5.1

\[ \begin{align*}
\varepsilon_1 u''(t) + \frac{1}{t^2}u'(t) + t^2 v(t) &= f_1(t), & 0 \leq t \leq 1, \\
\varepsilon_2 v''(t) + \frac{1}{t^2}v'(t) + tu'(t) &= f_2(t), \\
u(0) &= u(1) = 0, & v(0) = v(1) = 0.
\end{align*} \]

The exact solutions for Example 5.1 are $u(t) = t(t-1)e^{-t}$, $v(t) = t^2 - t^3$.

Example 5.2

\[ \begin{align*}
\varepsilon_1 u''(t) + \frac{1}{(t-1)^2}u'(t) + e^{-t}v'(t) &= f_1(t), & 0 \leq t \leq 1, \\
\varepsilon_2 v''(t) + \frac{1}{t^2}v'(t) + \sin(\sqrt{t})u'(t) &= f_2(t), \\
u(0) &= u(1) = 0, & v(0) = v(1) = 0.
\end{align*} \]

The exact solutions for Example 5.2 are $u(t) = \sin(\pi t)e^{-t}$, $v(t) = \cos(\pi t(t-1)) - 1$.

Example 5.3

\[ \begin{align*}
\varepsilon_1 u''(t) + \frac{1}{\sin(\pi t)}u'(t) + u'(t)^2 \sqrt{v(t)} &= f_1(t), & 0 \leq t \leq 1, \\
\varepsilon_2 v''(t) + \frac{1}{(t-1)^2}v'(t) + tv'(t)u'(t)^2 &= f_2(t), \\
u(0) &= u(1) = 0, & v(0) = v(1) = 0.
\end{align*} \]
Max absolute errors with $N = 100$

<table>
<thead>
<tr>
<th>$(\varepsilon_1, \varepsilon_2) \in S_1$</th>
<th>Present Method</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Example 5.1</td>
</tr>
<tr>
<td>$E^{uN}(t)$</td>
<td>$1.50 \times 10^{-6}$</td>
</tr>
<tr>
<td>$E^{vN}(t)$</td>
<td>$8.50 \times 10^{-7}$</td>
</tr>
<tr>
<td>$E^{uN'}(t)$</td>
<td>$3.50 \times 10^{-4}$</td>
</tr>
<tr>
<td>$E^{vN'}(t)$</td>
<td>$2.70 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

The exact solutions for Example 5.3 are

$$u(t) = \sin\left(\frac{\pi}{2} t \right)(t - 1), \quad v(t) = (t^2 - t^3)e^{-\sin(t(t-1))}.$$  

Example 5.4.

$$\begin{aligned}
\varepsilon_1 u''(t) &+ \frac{1}{4^{\pi^2(t-1)^2}} u'(t) + v'(t)^3 u'(t) + e^{\sin(v'(t))v'(t)^2} = f_1(t), \\
\varepsilon_2 v''(t) &+ \frac{1}{4^{\pi^2(t-1)^3}} v'(t) - \sin(v(t)u'(t)^2) + e^{v''(t)}v'(t) = f_2(t), \\
u(0) = u(1) = 0, &\quad v(0) = v(1) = 0, \quad 0 \leq t \leq 1.
\end{aligned}$$

The exact solutions for Example 5.4 are

$$u(t) = \sin(\pi t^2(t - 1)^2), \quad v(t) = \cos(\sin(\pi t(t - 1) + \pi) + 1.$$  

Example 5.5.

$$\begin{aligned}
\varepsilon_1 u''(t) &+ \frac{1}{4^{\pi^2(t-1)^2}} u'(t) + z'(t)^2 v'(t)^3 u'(t) + e^{\sin(z'(t)v'(t))u'(t)^3} = f_1(t), \\
\varepsilon_2 v''(t) &+ \frac{1}{4^{\pi^2(t-1)^3}} v'(t) - \sin(v(t)u'(t)^2) + e^{v''(t)}v'(t)^3 u'(t) + \\
v(z(t)^2 v(t)^3 u(t)) = f_2(t), &\quad v(0) = v(1) = 0, \quad 0 \leq t \leq 1,
\end{aligned}$$

The exact solutions for Example 5.5 are

$$u(t) = \cos(\sin(\pi t(t - 1) + \pi) + 1, \quad v(t) = \sin(\pi t^2(t - 1)^2), \quad z(t) = t^2 \sin(e^{t(t-1)} - 1).$$

Remark 5.6. All numerical examples are designed for the first time. A great deal of effort has been made to solve these examples using NDSolve and DSolve commands of Wolframe Mathematica software, and even all of the options in the NDSolve and DSolve commands are tested for these examples. Since, numerical examples have a strong singularity, so they are not solvable with the numerical commands in Wolframe Mathematica software, and we only compare the results with the exact solutions. We solved five numerical examples to illustrate the efficiency of the present method.
Table 2. Max absolute error and convergence order \( (\log_2 E^N/E^{2N} ) \)

<table>
<thead>
<tr>
<th>((\varepsilon_1, \varepsilon_2 = 10^{-30}))</th>
<th>Present Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N = 16)</td>
<td>(N = 32)</td>
</tr>
<tr>
<td>Example 5.1</td>
<td></td>
</tr>
<tr>
<td>(n = 10)</td>
<td></td>
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<tr>
<td>(E^N(t))</td>
<td>(E^{vN}(t))</td>
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<tr>
<td>2.00 (\times 10^{-4})</td>
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<tr>
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<td>2.93198</td>
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<tr>
<td>3.80 (\times 10^{-6})</td>
<td>2.98489</td>
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<tr>
<td>4.80 (\times 10^{-7})</td>
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</tr>
<tr>
<td>Example 5.2</td>
<td></td>
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<tr>
<td>(n = 15)</td>
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</tr>
<tr>
<td>(E^N(t))</td>
<td>(E^{vN}(t))</td>
</tr>
<tr>
<td>3.40 (\times 10^{-4})</td>
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<td>Example 5.3</td>
<td></td>
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<td>(n = 20)</td>
<td></td>
</tr>
<tr>
<td>(E^N(t))</td>
<td>(E^{vN}(t))</td>
</tr>
<tr>
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<tr>
<td>2.70 (\times 10^{-5})</td>
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<td>4.60 (\times 10^{-7})</td>
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</tr>
<tr>
<td>Example 5.4</td>
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<td>(n = 25)</td>
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<tr>
<td>(E^N(t))</td>
<td>(E^{vN}(t))</td>
</tr>
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<td>2.77761</td>
</tr>
<tr>
<td>1.75 (\times 10^{-4})</td>
<td>2.92765</td>
</tr>
<tr>
<td>2.30 (\times 10^{-5})</td>
<td>2.9386</td>
</tr>
<tr>
<td>3.00 (\times 10^{-6})</td>
<td>2.98489</td>
</tr>
</tbody>
</table>
Figure 1. Absolute error for Example 5.1 with $N = 100$ and $n = 10$ (Left: $E_{u N}(t)$; Right: $E_{v N}(t)$).

Figure 2. Absolute error for Example 5.1 with $N = 100$ and $n = 10$ (Left: $E_{u N}^\prime(t)$; Right: $E_{v N}^\prime(t)$).

Table 3. Max absolute error for Example 5.5 with $N = 100$ and $n = 30$ and $(\varepsilon_1, \varepsilon_2, \varepsilon_3) \in S_2$

<table>
<thead>
<tr>
<th>Present Method</th>
<th>$E_{u N}(t)$</th>
<th>$E_{v N}(t)$</th>
<th>$E_{z N}(t)$</th>
<th>$E_{u N}^\prime(t)$</th>
<th>$E_{v N}^\prime(t)$</th>
<th>$E_{z N}^\prime(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$1.00 \times 10^{-5}$</td>
<td>$6.20 \times 10^{-6}$</td>
<td>$3.20 \times 10^{-6}$</td>
<td>$3.00 \times 10^{-3}$</td>
<td>$2.00 \times 10^{-3}$</td>
<td>$1.00 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

6. Conclusion

In this paper, we have been solved SSPBVP using a reliable new technique based on the RKM and general method for the nonlinear problems. The convenient implementation for the linear or nonlinear system of differential equations is one of the advantages of the present method. Numerical examples demonstrated the accuracy of the present method is high and it is practical for the different types of linear or nonlinear systems of differential equations. However, we provide error estimations for the approximate solution and its derivative. Therefore from the theoretical results, convergence order for the approximate solution and its derivative are at least $O(h^2)$ and $O(h)$ respectively. It is important to note here that many of the singularly perturbed problems with the severe boundary layer behaviors are solvable with high precision using numerical methods in mathematical software which are freely available.
Figure 3. Absolute error for Example 5.2 with $N = 100$ and $n = 15$ (Left: $E_u^N(t)$; Right: $E_v^N(t)$).

Figure 4. Absolute error for Example 5.2 with $N = 100$ and $n = 15$ (Left: $E_u'{}^N(t)$; Right: $E_v'{}^N(t)$).

Table 4. Max absolute error for Example 5.5 and convergence order ($\log_2 E^N/E^{2N}$)

<table>
<thead>
<tr>
<th>$(\varepsilon_1, \varepsilon_2, \varepsilon_3 = 10^{-30})$</th>
<th>Present Method</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$N = 16$</td>
</tr>
<tr>
<td></td>
<td>$n = 15$</td>
</tr>
<tr>
<td>$E_u^N(t)$</td>
<td>$2.30 \times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td>2.9386</td>
</tr>
<tr>
<td>$E_v^N(t)$</td>
<td>$1.8 \times 10^{-3}$</td>
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<tr>
<td></td>
<td>3.40439</td>
</tr>
<tr>
<td>$E_u'{}^N(t)$</td>
<td>$5.60 \times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>2.8255</td>
</tr>
<tr>
<td>$E_v'{}^N(t)$</td>
<td>$2.00 \times 10^{-1}$</td>
</tr>
<tr>
<td></td>
<td>2.78588</td>
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<tr>
<td>$E_u^N(t)$</td>
<td>$6.00 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>1.81943</td>
</tr>
<tr>
<td>$E_v^N(t)$</td>
<td>$2.90 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>1.85798</td>
</tr>
</tbody>
</table>

(such as NDSolve command in Wolfram Mathematica). Since the considered problem (1.1) has very severe singularity without boundary layer behavior, this software is not...
Figure 5. Absolute error for Example 5.3 with $N = 100$ and $n = 20$ (Left: $E^{u_N}(t)$; Right: $E^{w_N}(t)$).

Figure 6. Absolute error for Example 5.3 with $N = 100$ and $n = 20$ (Left: $E^{u_N}(t)$; Right: $E^{w_N}(t)$).

Figure 7. Absolute error for Example 5.4 with $N = 100$ and $n = 25$ (Left: $E^{u_N}(t)$; Right: $E^{w_N}(t)$).

Figure 8. Absolute error for Example 5.4 with $N = 100$ and $n = 25$ (Left: $E^{u_N}(t)$; Right: $E^{w_N}(t)$).
currently able to solve the problem (1.1), therefore for very severe singularities in singularly perturbed problem without layer behavior, we recommend using the present method.
ACKNOWLEDGMENT

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REFERENCES


