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Regularized Prabhakar derivative for partial differential equations

Ahmed Bokhari[1,2,*], Dumitru Baleanu[3,4], and Rachid Belgacem[1,2]

[1]Department of Mathematics, Faculty of Exact Sciences and Informatics, Hassiba Benbouali University of Chlef, Algeria.

[2]Laboratory of Mathematics and its Applications (LMA).

[3]Department of Mathematics, Faculty of Arts and Sciences, Cankaya University, TR-06530 Ankara, Turkey.

[4]Institute of Space Sciences, Magurele-Bucharest, Romania.

Abstract

Prabhakar fractional operator was applied recently for studying the dynamics of complex systems from several branches of sciences and engineering. In this manuscript, we discuss the regularized Prabhakar derivative applied to fractional partial differential equations using the Sumudu homotopy analysis method(PSHAM). Three illustrative examples are investigated to confirm our main results.

Keywords. Regularized Prabhakar derivative, Homotopy analysis method, Sumudu transform, Mittag-Leffler function.

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1. Introduction

Over the last few decades, fractional differential equations (FDEs) have been shown to have great strength and precision in modeling a broad category of phenomena: viscoelasticity, fluid mechanics, biology, chemistry, acoustics, control theory, economics, electronics, finance, psychology and other fields of science and engineering [1, 2, 17, 18, 20–23, 27–29].

Several definitions of fractional order derivatives have been proposed, in particular the derivatives of Caputo and Riemann - Liouville are studied and used in various fields of science. the modeling of natural phenomena requires different fractional derivation operators, for this reason, it was necessary to generalize the derivatives of Riemann-Liouville and thus of Caputo.

The integral of Prabhakar is defined by modifying the Riemann-Liouville integral operator by extending its kernel with the three-parameter Mittag-Leffler function also known as the Prabhakar function [26]. The temporal evolution of polarization processes in Havriliak-Negami models and other phenomena can be appropriately described by integral and differential operators based on Prabhakar's function [14].

Roberto Garra et. al [15] have defined the regularized version of the fractional derivatives of Prabhakar, which is a generalization of the derivative of Caputo, they also calculated their transformation of Laplace, while Panchal et. al [24] calculated their transformation of Sumudu. In this work, we will treat fractional differential equations in the sense of regularized Prabhakar, where the homotopy analysis method coupled with the transformation of Sumudu is used. The basic definitions and results of fractional calculus and Sumudu transform are given in section 2, several test problems to demonstrate the effectiveness of the proposed method are presented in section 3, and the conclusion is finally presented in section 4.

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* Corresponding author. Email: bokhari.ahmed@ymail.com.

2. Preliminaries

In this section, we mainly present the main features of fractional calculus and the Sumudu transform.

Definition 2.1. [4, 8, 23] Let $f \in L^1(a,b)$ If $\alpha \geq 0$ then left sided Riemann–Liouville fractional integral of order α is defined by

$$I_t^{\alpha} f(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(x,\tau)\, d\tau, \quad \alpha > 0, \quad t > 0, \tag{2.1}$$

$$I_t^0 f(x,t) = f(x,t).$$

Definition 2.2. [4, 8, 23] Let $f \in L_1(a,b)$, and $m-1 < \alpha \leq m$. The Caputo fractional derivative of order α $(\alpha > 0)$ is defined as

$$\begin{aligned} {}_0^C\mathrm{D}_t^{\alpha} f(x,t) &= \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-x)^{m-\alpha-1} f^{(m)}(x,\tau)\, dx, \quad m-1 < \alpha \leq m. \\ &= \frac{\partial^m}{\partial t^m} f(x,t) \ \text{ if } \alpha = m. \end{aligned} \tag{2.2}$$

Definition 2.3. [4, 8, 23] Let $f \in L_1(a,b)$, and $m-1 < \alpha \leq m$. The Liouville fractional derivative of order, $\alpha > 0$ is defined in the form

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} f(x,t) = \frac{1}{\Gamma(m-\alpha)} \frac{\partial^m}{\partial t^m} \left(\int_{-\infty}^t \frac{f(x,\tau)}{(t-\tau)^{\alpha-m+1}} d\tau \right), \quad m = [\alpha] + 1. \tag{2.3}$$

Definition 2.4. [13] The one parameter Mittag-Leffler function (M-L function) is defined by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^n}{\Gamma(\alpha k + 1)}, \qquad z, \alpha \in \mathbb{C}, \ Re(\alpha) > 0 \tag{2.4}$$

The two parameter Mittag-Leffler function reads [31]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^n}{\Gamma(\alpha k + \beta)}, \qquad z, \alpha, \beta \in \mathbb{C}, \quad Re(\alpha) > 0 \tag{2.5}$$

such that $E_{\alpha,1}(z) = E_{\alpha}(z)$.

Definition 2.5. [26] The three parameter M-L function, also called Prabhakar function is given by

$$E_{\alpha,\beta}^{\gamma}(z) = \frac{1}{\Gamma(\gamma)} \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+k)}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \qquad z, \alpha, \beta, \gamma \in \mathbb{C}, \quad Re(\alpha) > 0. \tag{2.6}$$

In applications, it is usually used a further generalization of Eq. (2.6) given by [7]

$$e_{\alpha,\beta,\omega}^{\gamma}(t;\omega) = t^{\beta-1} E_{\alpha,\beta}^{\gamma}(\omega t^{\alpha}), \tag{2.7}$$

where $\omega \in \mathbb{C}$ is a parameter and $t > 0$ the independent real variable.

Definition 2.6. [11, 26] Let $f \in L^1[0,b]$, $0 < t < b \leq \infty$. The Prabhakar integral is defined by

$$\mathbf{E}_{\alpha,\beta,\omega,0^+}^{\gamma} f(t) = \int_0^t (t-\tau)^{\beta-1} E_{\alpha,\beta}^{\gamma}\left[\omega (t-\tau)^{\alpha}\right] f(\tau)\, d\tau = \left(f * e_{\alpha,\beta,\omega}^{\gamma}\right)(t), \tag{2.8}$$

where $\alpha, \beta, \gamma, \omega \in \mathbb{C}$, $\quad \mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$

We also recall that the left-inverse to the operator Eq. (2.8) is the Prabhakar derivative. We define it below in a slightly different form.

Definition 2.7. [26] Let $f \in L^1[0,b]$, $0 < t < b < \infty$, and $f * e^{-\gamma}_{\alpha,m-\beta,\omega}(.) \in W^{m,1}[0,b]$, $m = \lceil \beta \rceil$. The Prabhakar derivative is defined as

$$\mathbf{D}^{\gamma}_{\alpha,\beta,\omega,0^+} f(t) = \frac{d^m}{dx^m} \mathbf{E}^{-\gamma}_{\alpha,m-\beta,\omega,0^+} f(t). \tag{2.9}$$

where $\alpha, \beta, \gamma, \omega \in \mathbb{C}$, $\operatorname{Re}(\alpha), \operatorname{Re}(\beta) > 0$.

The Riemann–Liouville integrals in Eq. (2.1) can be expressed in terms of Prabhakar integrals as

$$I^{m-(\beta+\theta)}_{0^+} f(t) = \mathbf{E}^{0}_{\alpha,m-(\beta+\theta),\omega,0^+} f(t), \tag{2.10}$$

we have that,

$$\begin{aligned} \mathbf{D}^{\gamma}_{\alpha,\beta,\omega,0^+} f(t) &= \frac{d^m}{dt^m} \mathbf{E}^{-\gamma}_{\alpha,m-\beta,\omega,0^+} f(t) \\ &= \frac{d^m}{dt^m} I^{m-(\beta+\theta)}_{0^+} \mathbf{E}^{-\gamma}_{\alpha,\theta,\omega,0^+} f(t) \\ &= D^{\beta+\theta}_{0^+} \mathbf{E}^{-\gamma}_{\alpha,\theta,\omega,0^+} f(t). \end{aligned} \tag{2.11}$$

($D^{\beta+\theta}_{0^+}$ the Riemann–Liouville derivative).

Definition 2.8. [15] Let $f \in AC[0,b]$, $0 < t < b < \infty$, and $m = \lceil \beta \rceil$. The regularized Prabhakar derivative is given

$${}^{C}\mathbf{D}^{\gamma}_{\alpha,\beta,\omega,0^+} f(t) = \mathbf{E}^{-\gamma}_{\alpha,m-\beta,\omega,0^+} \frac{d^m}{dx^m} f(t). \tag{2.12}$$

where $\alpha, \beta, \gamma, \omega \in \mathbb{C}$, $\operatorname{Re}(\alpha), \operatorname{Re}(\beta) > 0$.

Remark 2.9. When $\gamma = 0$, The regularized Prabhakar derivative becomes the Caputo fractional derivative.

Definition 2.10. [30] The Sumudu transform is defined by,

$$G(u) = \mathbb{S}[f(t)] = \int_0^\infty f(ut) e^{-t} dt, \quad u \in (-\tau_1, \tau_2),$$

Over the set of functions

$$\mathcal{A} = \left\{ f(t) \,/\, \exists \tau_1, \tau_2 > 0, \ |f(t)| < M e^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty) \right\}.$$

Hence, $G(u)$, is referred to as the Sumudu of $f(t)$.

Theorem 2.11. *[3] In the complex plane* $\mathbb{C}$, *for any* $\operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0$ *and* $\omega \in \mathbb{C}$

$$\mathbb{S}^{-1}\left[\frac{u^{\beta-1}}{(1 - \omega u^{\alpha})^{\gamma}} \right] = t^{\beta-1} E^{\gamma}_{\alpha,\beta}(\omega t^{\alpha}). \tag{2.13}$$

where $\mathbb{S}^{-1}$ *is the inverse Sumudu operator.*

Theorem 2.12. *[24] The Sumudu transform of regularized Prabhakar fractional derivative* (2.12) *is*

$$\mathbb{S}\left(\mathbf{D}^{\gamma}_{\alpha,\beta,\omega,0^+} f(x) \right)(u) = u^{-\beta} (1 - \omega u^{\alpha})^{\gamma} G(u) - \sum_{k=0}^{m-1} u^{k-\beta} (1 - \omega u^{\alpha})^{\gamma} f^{(k)}(0^+). \tag{2.14}$$

For further detail and properties of Sumudu transform (see [5, 6] and [30]).

3. Description of the method using the regularized Prabhakar operator

Consider the following equation in the Prabhakar sense:

$$\mathbf{D}^{\gamma}_{\alpha,\beta,\omega,0+}f\left(x,t\right)=a\left(x\right)\frac{\partial^{2}f\left(x,t\right)}{\partial x^{2}}+b\left(x\right)\frac{\partial f\left(x,t\right)}{\partial x}+c\left(x\right)f\left(x,t\right)+d\left(x,t\right),\tag{3.1}$$

with initial conditions

$$\frac{\partial^{m}f\left(x,0\right)}{\partial t^{m}}=f_{m}\left(x\right),\qquad m=0,1,...,n-1.$$

Let $\mathbb{S}\left[f\left(x,t\right)\right]\left(u\right)=G\left(x,u\right)$. Applying the Sumudu transform to Eq. (3.1) using Eq. (2.14) we have,

$$u^{-\beta}\left(1-\omega u^{\alpha}\right)^{\gamma}G\left(x,u\right)-\sum_{k=0}^{m-1}u^{k-\beta}\left(1-\omega u^{\alpha}\right)^{\gamma}f^{(k)}\left(x,0^{+}\right)=\left(a\left(x\right)\frac{\partial^{2}}{\partial x^{2}}+b\left(x\right)\frac{\partial}{\partial x}+c\left(x\right)\right)G\left(x,u\right)+\mathbb{S}\left[d\left(x,t\right)\right],$$

equivalent to

$$G\left(x,u\right)=\frac{u^{\beta}}{\left(1-\omega u^{\alpha}\right)^{\gamma}}\left(a\left(x\right)\frac{\partial^{2}}{\partial x^{2}}+b\left(x\right)\frac{\partial}{\partial x}+c\left(x\right)\right)G\left(x,u\right)+\frac{u^{\beta}}{\left(1-\omega u^{\alpha}\right)^{\gamma}}\left(\sum_{k=0}^{m-1}u^{k-\beta}\left(1-\omega u^{\alpha}\right)^{\gamma}f^{(k)}\left(x,0^{+}\right)+\mathbb{S}\left[d\left(x,t\right)\right]\right),$$

or

$$G\left(x,u\right)=\frac{u^{\beta}}{\left(1-\omega u^{\alpha}\right)^{\gamma}}\left(a\left(x\right)\frac{\partial^{2}}{\partial x^{2}}+b\left(x\right)\frac{\partial}{\partial x}+c\left(x\right)\right)G\left(x,u\right)+\sum_{k=0}^{m-1}u^{k}f^{(k)}\left(x,0^{+}\right)+\frac{u^{\beta}}{\left(1-\omega u^{\alpha}\right)^{\gamma}}\mathbb{S}\left[d\left(x,t\right)\right].\tag{3.2}$$

The homotopy for Eq. (3.2) is constructed as follows[10] :

$$G\left(x,u\right)=z\frac{u^{\beta}}{\left(1-\omega u^{\alpha}\right)^{\gamma}}\left(a\left(x\right)\frac{\partial^{2}}{\partial x^{2}}+b\left(x\right)\frac{\partial}{\partial x}+c\left(x\right)\right)G\left(x,u\right)+\sum_{k=0}^{m-1}u^{k}f^{(k)}\left(x,0^{+}\right)+\frac{u^{\beta}}{\left(1-\omega u^{\alpha}\right)^{\gamma}}\mathbb{S}\left[d\left(x,t\right)\right].\tag{3.3}$$

The solution of Eq. (3.3) is obtained by applying the hypothesis that the solution $G\left(x,u\right)$ is expressed as

$$G\left(x,u\right)=\sum_{n=0}^{\infty}z^{n}G_{n}\left(x,u\right),\quad m=0,1,2,...,\tag{3.4}$$

substituting Eq. (3.4) into Eq. (3.3), we get

$$\sum_{n=0}^{\infty}z^{n}G_{n}\left(x,u\right)=z\frac{u^{\beta}}{\left(1-\omega u^{\alpha}\right)^{\gamma}}\left(a\left(x\right)\frac{\partial^{2}}{\partial x^{2}}+b\left(x\right)\frac{\partial}{\partial x}+c\left(x\right)\right)\sum_{n=0}^{\infty}z^{n}G_{n}\left(x,u\right)+\sum_{k=0}^{m-1}u^{k}f^{(k)}\left(0^{+}\right)+\frac{u^{\beta}}{\left(1-\omega u^{\alpha}\right)^{\gamma}}\mathbb{S}\left[d\left(x,t\right)\right],\tag{3.5}$$

which, on comparing the coefficients of powers of z, yields

$$\begin{aligned}
z^{0}\quad &:\quad G_{0}\left(x,u\right)=\sum_{k=0}^{m-1}u^{k}f^{(k)}\left(0^{+}\right)+\frac{u^{\beta}}{\left(1-\omega u^{\alpha}\right)^{\gamma}}\mathbb{S}\left[d\left(x,t\right)\right],\\
z^{1}\quad &:\quad G_{1}\left(x,u\right)=-\frac{u^{\beta}}{\left(1-\omega u^{\alpha}\right)^{\gamma}}\left(a\left(x\right)\frac{\partial^{2}}{\partial x^{2}}+b\left(x\right)\frac{\partial}{\partial x}+c\left(x\right)\right)G_{0}\left(x,u\right)\\
z^{2}\quad &:\quad G_{2}\left(x,u\right)=-\frac{u^{\beta}}{\left(1-\omega u^{\alpha}\right)^{\gamma}}\left(a\left(x\right)\frac{\partial^{2}}{\partial x^{2}}+b\left(x\right)\frac{\partial}{\partial x}+c\left(x\right)\right)G_{1}\left(x,u\right)\\
&\quad\vdots\\
z^{n}\quad &:\quad G_{n+1}\left(x,u\right)=-\frac{u^{\beta}}{\left(1-\omega u^{\alpha}\right)^{\gamma}}\left(a\left(x\right)\frac{\partial^{2}}{\partial x^{2}}+b\left(x\right)\frac{\partial}{\partial x}+c\left(x\right)\right)G_{n-1}\left(x,u\right),
\end{aligned}\tag{3.6}$$

and when $z \to 1$, Eq. (3.6) becomes the approximate solution of Eq. (3.2) and Eq. (3.3), and this solution implies that

$$H_n(x,u) = \sum_{i=0}^{n} G_i(x,u). \tag{3.7}$$

Applying the inverse of the Sumudu transform of Eq. (3.7), we obtain the approximate solution of Eq. (3.1),

$$f_n(x,t) = \mathbb{S}^{-1}(H_n(x,u)).$$

The series solution, Eq. (3.7) generally converge very rapidly [12, 25]

3.1. **Examples.** In this section, some test problems are presented using the regularized Prabhakar fractional operators

Example 3.1. Consider the following equation in the regularized Prabhakar sense:

$$\mathbf{D}^{\gamma}_{\alpha,\beta,\omega,0^+} f(x,t) = -2\frac{\partial f(x,t)}{\partial x} - \frac{\partial^3 f(x,t)}{\partial x^3} \tag{3.8}$$

$\alpha, \beta, \gamma, \omega \in \mathbb{C}$, $\operatorname{Re}(\alpha) > 0$, $0 < \beta \leq 1$, $(x,t) \in [0,1] \times [0,1]$,
with the initial conditions

$$f(x,0) = \sin(x),$$

Let $\mathbb{S}[f(x,t)](u) = G(x,u)$, Eq. (3.8), can be written

$$u^{-\beta}(1-\omega u^{\alpha})^{\gamma} G(x,u) - \sum_{k=0}^{m-1} u^{k-\beta}(1-\omega u^{\alpha})^{\gamma} f^{(k)}(x,0^+) = \left(-2\frac{\partial}{\partial x} - \frac{\partial^3}{\partial x^3}\right) G(x,u),$$

equivalent to

$$G(x,u) = \frac{u^{\beta}}{(1-\omega u^{\alpha})^{\gamma}}\left(-2\frac{\partial}{\partial x} - \frac{\partial^3}{\partial x^3}\right) G(x,u) + \frac{u^{\beta}}{(1-\omega u^{\alpha})^{\gamma}} \sum_{k=0}^{m-1} u^{k-\beta}(1-\omega u^{\alpha})^{\gamma} f^{(k)}(0^+),$$

or

$$G(x,u) = \frac{u^{\beta}}{(1-\omega u^{\alpha})^{\gamma}}\left(-2\frac{\partial}{\partial x} - \frac{\partial^3}{\partial x^3}\right) G(x,u) + f(0^+). \tag{3.9}$$

The homotopy for Eq. (3.9) is constructed as follows:

$$G(x,u) = z\frac{u^{\beta}}{(1-\omega u^{\alpha})^{\gamma}}\left(-2\frac{\partial}{\partial x} - \frac{\partial^3}{\partial x^3}\right) G(x,u) + \sin x, \tag{3.10}$$

Now, using the PSHAM, we have

$$\begin{aligned}
z^0 &: G_0(x,u) = \sin x \\
z^1 &: G_1(x,u) = \frac{u^{\beta}}{(1-\omega u^{\alpha})^{\gamma}}(-\cos x) \\
z^2 &: G_2(x,u) = \frac{u^{2\beta}}{(1-\omega u^{\alpha})^{2\gamma}}(-\sin x) \\
z^3 &: G_3(x,u) = \frac{u^{3\beta}}{(1-\omega u^{\alpha})^{3\gamma}}\cos x \\
&\vdots \\
z^n &: G_{n+1}(x,u) = \frac{u^{n\beta}}{(1-\omega u^{\alpha})^{n\gamma}}\sin(x - n\frac{\pi}{2}),
\end{aligned}$$

the approximate solution is

$$H_n(x,u) = \sum_{i=0}^{n} G_i(x,u).$$

Therefore

$$\begin{aligned} H_n(x,u) &= \sin(x)[1 - \frac{u^{2\beta}}{(1-\omega u^{\alpha})^{2\gamma}} + \frac{u^{4\beta}}{(1-\omega u^{\alpha})^{4\gamma}} - \frac{u^{6\beta}}{(1-\omega u^{\alpha})^{6\gamma}} + ... + (-1)^n \frac{u^{2n\beta}}{(1-\omega u^{\alpha})^{2n\gamma}}] \\ &\quad + \cos(x)[-\frac{u^{\beta}}{(1-\omega u^{\alpha})^{2\gamma}} + \frac{u^{3\beta}}{(1-\omega u^{\alpha})^{3\gamma}} - \frac{u^{5\beta}}{(1-\omega u^{\alpha})^{5\gamma}} + ... + (-1)^{n+1} \frac{u^{(2n+1)\beta}}{(1-\omega u^{\alpha})^{(2n+1)\gamma}}] \end{aligned} \tag{3.11}$$

Using relation (2.13), we have

$$S^{-1}\left[\frac{u^{i\beta}}{(1-\omega u^{\alpha})^{i\gamma}}\right] = t^{i\beta+1} E^{i\gamma}_{\alpha,i\beta+1}(\omega t^{\alpha}).$$

Applying the inverse Sumudu transform to Eq. (3.11), we get

$$\begin{aligned} f_n(x,t) &= \sin(x) + \sin(x)\sum_{i=1}^{n}(-1)^i t^{2i\beta} E^{2i\gamma}_{\alpha,2i\beta+1}(\omega t^{\alpha}) \\ &\quad + \cos(x)\sum_{i=0}^{n}(-1)^{i+1} t^{(2i+1)\beta} E^{(2i+1)\gamma}_{\alpha,(2i+1)\beta+1}(\omega t^{\alpha}), \end{aligned}$$

when $n \to \infty$, we obtain the solution of Eq. (3.8),

$$\begin{aligned} f(x,t) &= \sin(x) + \sin(x)\sum_{i=1}^{\infty}(-1)^i t^{2i\beta} E^{2i\gamma}_{\alpha,2i\beta+1}(\omega t^{\alpha}) \\ &\quad - \cos(x)\sum_{i=0}^{\infty}(-1)^{i} t^{(2i+1)\beta} E^{(2i+1)\gamma}_{\alpha,(2i+1)\beta+1}(\omega t^{\alpha}). \end{aligned} \tag{3.12}$$

Remark 3.2. When $\gamma = 0$, Eq. (3.12) is equivalent to

$$f(x,t) = \sin(x)\sum_{i=0}^{\infty}(-1)^i \frac{1}{\Gamma(2i\beta+1)} t^{2i\beta} - \cos(x)\sum_{i=0}^{\infty}(-1)^i \frac{1}{\Gamma((2i+1)\beta+1)} t^{(2i+1)\beta},$$

which is the solution of Eq. (3.8) in the Caputo sens.

Furthermore, in the subcase $\beta = 1$, the exact solution is

$$\begin{aligned} f(x,t) &= \sin(x)\sum_{i=0}^{\infty}(-1)^i \frac{1}{(2i)!} t^{2i} - \cos(x)\sum_{i=0}^{\infty}(-1)^i \frac{1}{(2i+1)!} t^{(2i+1)} \\ &= \sin(x)\cos(t) - \cos(x)\sin(t), \end{aligned} \tag{3.13}$$

which is the solution of Eq. (3.8) in the ordinary sens.

$$f(x,t) = \sin(x-t).$$

Example 3.3. Consider the following time fractional model with homogeneous boundary conditions

$$\begin{cases} \mathbf{D}^{\gamma}_{\alpha,\beta,\omega,0^+} f(x,y,t) = -2\frac{\partial^3}{\partial x^3} f(x,y,t) - 2\frac{\partial^3}{\partial y^3} f(x,y,t) \\ f(x,y,0) = \cos(x+y), \end{cases} \tag{3.14}$$

Let $S\left[f\left(x,y,t\right)\right](u)=G\left(x,y,t\right)$, Eq. (3.14), can be written

$$u^{-\beta}\left(1-\omega u^{\alpha}\right)^{\gamma}G\left(x,y,u\right)-\sum_{k=0}^{m-1}u^{k-\beta}\left(1-\omega u^{\alpha}\right)^{\gamma}f^{(k)}\left(0^{+}\right)=\left(-2\frac{\partial^{3}}{\partial x^{3}}f\left(x,y,t\right)-2\frac{\partial^{3}}{\partial y^{3}}f\left(x,y,t\right)\right)G\left(x,y,u\right),$$

α,β,γ, and ω using the PSHAM, we have

$$\begin{aligned}
z^{0} &: G_{0}\left(x,y,u\right)=\cos(x+y)\\
z^{1} &: G_{1}\left(x,y,u\right)=-3\frac{u^{\beta}}{\left(1-\omega u^{\alpha}\right)^{\gamma}}\sin(x+y)\\
z^{2} &: G_{2}\left(x,y,u\right)=9\frac{u^{2\beta}}{\left(1-\omega u^{\alpha}\right)^{2\gamma}}\cos(x+y)\\
z^{3} &: G_{3}\left(x,y,u\right)=-27\frac{u^{3\beta}}{\left(1-\omega u^{\alpha}\right)^{3\gamma}}\sin(x+y)\\
&\vdots\\
z^{n} &: G_{n}\left(x,y,u\right)=\frac{\left(-1\right)^{n}-1}{2}\frac{3^{n}u^{n\beta}}{\left(1-\omega u^{\alpha}\right)^{n\gamma}}\sin(x+y)+\frac{\left(-1\right)^{n}+1}{2}\frac{3^{n}u^{n\beta}}{\left(1-\omega u^{\alpha}\right)^{n\gamma}}\cos(x+y).
\end{aligned}$$

The approximate solution is

$$H_{n}\left(x,y,u\right)=\sum_{i=0}^{n}G_{i}\left(x,y,u\right)$$

Therefore

$$\begin{aligned}
H_{n}\left(x,y,u\right)&=\sum_{i=0}^{n}\frac{\left(-1\right)^{i}-1}{2}\frac{3^{i}u^{i\beta}}{\left(1-\omega u^{\alpha}\right)^{i\gamma}}\sin(x+y)+\sum_{i=0}^{n}\frac{\left(-1\right)^{i}+1}{2}\frac{3^{i}u^{i\beta}}{\left(1-\omega u^{\alpha}\right)^{i\gamma}}\cos(x+y)\\
&=-\sin(x+y)\sum_{i=0}^{n}3^{2i+1}\frac{u^{(2i+1)\beta}}{\left(1-\omega u^{\alpha}\right)^{(2i+1)\gamma}}+\cos(x+y)\sum_{i=0}^{n}3^{2i+1}\frac{u^{2i\beta}}{\left(1-\omega u^{\alpha}\right)^{2i\gamma}}
\end{aligned}\tag{3.15}$$

Using relation (2.13), we have

$$S^{-1}\left[\frac{u^{i\beta}}{\left(1-\omega u^{\alpha}\right)^{i\gamma}}\right]=t^{i\beta}E_{\alpha,i\beta+1}^{i\gamma}\left(\omega t^{\alpha}\right).$$

Therefore, by applying the inverse Sumudu transform, we get

$$\begin{aligned}
f_{n}\left(x,y,t\right)&=-\sin(x+y)\sum_{i=0}^{n}3^{2i+1}t^{(2i+1)\beta}t^{i\beta}E_{\alpha,(2i+1)\beta+1}^{(2i+1)\gamma}\left(\omega t^{\alpha}\right)\\
&\quad+\cos(x+y)\sum_{i=0}^{n}3^{2i}t^{2i\beta}E_{\alpha,2i\beta+1}^{2i\gamma}\left(\omega t^{\alpha}\right)
\end{aligned}$$

,

and, when $n\to\infty$, this becomes the following solution:

$$\begin{aligned}
f\left(x,y,t\right)&=-\sin(x+y)\sum_{i=0}^{\infty}3^{2i+1}t^{(2i+1)\beta}t^{i\beta}E_{\alpha,(2i+1)\beta+1}^{(2i+1)\gamma}\left(\omega t^{\alpha}\right)\\
&\quad+\cos(x+y)\sum_{i=0}^{\infty}3^{2i}t^{2i\beta}E_{\alpha,2i\beta+1}^{2i\gamma}\left(\omega t^{\alpha}\right).
\end{aligned}\tag{3.16}$$

Remark 3.4. When $\gamma = 0$, Eq. (3.16) is equivalent to

$$\begin{aligned} f(x,y,t) &= -\sin(x+y)\sum_{i=0}^{\infty} 3^{2i+1}\frac{1}{\Gamma((2i+1)\beta+1)}t^{(2i+1)\beta} + \cos(x+y)\sum_{i=0}^{\infty} 3^{2i}\frac{1}{\Gamma(2i\beta+1)}t^{2i\beta} \\ &= -\sin(x+y)\sum_{i=0}^{\infty}\frac{1}{\Gamma((2i+1)\beta+1)}\left(3t^{\beta}\right)^{2i+1} + \cos(x+y)\sum_{i=0}^{\infty}\frac{1}{\Gamma(2i\beta+1)}\left(3t^{\beta}\right)^{2i}. \end{aligned}$$

Moreover, when $\beta = 1$, the exact solution is given by

$$\begin{aligned} f(x,y,t) &= -\sin(x+y)\sum_{i=0}^{\infty}\frac{1}{\Gamma(2i+2)}(3t)^{2i+1} + \cos(x+y)\sum_{i=0}^{\infty}\frac{1}{\Gamma(2i+1)}(3t)^{2i} \\ &= -\sin(x+y)\sum_{i=0}^{\infty}\frac{1}{(2i+1)!}(3t)^{2i+1} + \cos(x+y)\sum_{i=0}^{\infty}\frac{1}{(2i)!}(3t)^{2i} \\ &= -\sin(x+y)\sin(3t) + \cos(x+y)\cos(3t). \end{aligned}$$

Thus

$$f(x,y,t) = \cos(x+y+3t)$$

Example 3.5. [16, 19] Consider the the fractional Black–Scholes option pricing equation

$$\mathbf{D}^{\gamma}_{\alpha,\beta,\omega,0^+} f(x,t) = \frac{\partial^2 f(x,t)}{\partial x^2} + (q-1)\frac{\partial f(x,t)}{\partial x} - qf(x,t) \tag{3.17}$$

$\alpha, \beta, \gamma, \omega \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$, $0 < \beta \leq 1$, with the initial condition

$$f(x,0) = \max\left(e^{x} - 1, 0\right).$$

Now, using the PSHAM, we have

$$
\begin{aligned}
z^0 \quad &: \quad G_0(x,u) = u^0 f^{(0)}\left(x,0^+\right) = \max\left(e^x-1,0\right), \\
z^1 \quad &: \quad G_1(x,u) = \frac{u^\beta}{(1-\omega u^\alpha)^\gamma}\left(\frac{\partial^2}{\partial x^2}+(q-1)\frac{\partial}{\partial x}-q\right)G_0(x,u) \\
&= \frac{qu^\beta}{(1-\omega u^\alpha)^\gamma}\left(\max\left(e^x,0\right)-\max\left(e^x-1,0\right)\right) \\
z^2 \quad &: \quad G_2(x,u) = \frac{u^\beta}{(1-\omega u^\alpha)^\gamma}\left(\frac{\partial^2}{\partial x^2}+(q-1)\frac{\partial}{\partial x}-q\right)G_1(x,u) \\
&= \frac{u^\beta}{(1-\omega u^\alpha)^\gamma}\frac{qu^\beta}{(1-\omega u^\alpha)^\gamma}\left[-q\left(\max\left(e^x,0\right)-\max\left(e^x-1,0\right)\right)\right] \\
&= -\frac{q^2u^{2\beta}}{(1-\omega u^\alpha)^{2\gamma}}\left(\max\left(e^x,0\right)-\max\left(e^x-1,0\right)\right) \\
z^3 \quad &: \quad G_3(x,u) = \frac{u^\beta}{(1-\omega u^\alpha)^\gamma}\left(\frac{\partial^2}{\partial x^2}+(q-1)\frac{\partial}{\partial x}-q\right)G_2(x,u) \\
&= \frac{q^3u^{3\beta}}{(1-\omega u^\alpha)^{3\gamma}}\left(\max\left(e^x,0\right)-\max\left(e^x-1,0\right)\right) \\
&\vdots \\
z^n \quad &: \quad G_n(x,u) = -\frac{u^\beta}{(1-\omega u^\alpha)^\gamma}\left(\frac{\partial^2}{\partial x^2}+(q-1)\frac{\partial}{\partial x}-q\right)G_{n-1}(x,u) \\
&= (-1)^{n-1}\frac{q^nu^{n\beta}}{(1-\omega u^\alpha)^{n\gamma}},
\end{aligned}
$$

the approximate solution is

$$
\begin{aligned}
H_n(x,u) &= \sum_{i=0}^{n} G_i(x,u) \\
&= \max\left(e^x-1,0\right) + \frac{qu^\beta}{(1-\omega u^\alpha)^\gamma}\left(\max\left(e^x,0\right)-\max\left(e^x-1,0\right)\right) \\
&\quad -\frac{q^2u^{2\beta}}{(1-\omega u^\alpha)^{2\gamma}}\left(\max\left(e^x,0\right)-\max\left(e^x-1,0\right)\right) \\
&\quad +\frac{q^3u^{3\beta}}{(1-\omega u^\alpha)^{3\gamma}}\left(\max\left(e^x,0\right)-\max\left(e^x-1,0\right)\right) \\
&\quad +\ldots \\
&\quad +(-1)^{n-1}\frac{q^nu^{n\beta}}{(1-\omega u^\alpha)^{n\gamma}}\left(\max\left(e^x,0\right)-\max\left(e^x-1,0\right)\right)
\end{aligned}
$$

Thus

$$
\begin{aligned}
H_n(x,u) &= \sum_{i=0}^{n} G_i(x,u) = \max\left(e^x-1,0\right) + \max\left(e^x-1,0\right)\sum_{i=1}^{n}(-1)^i\frac{q^iu^{i\beta}}{(1-\omega u^\alpha)^{i\gamma}} \\
&\quad +\max\left(e^x,0\right)\sum_{i=1}^{n}(-1)^{i-1}\frac{q^iu^{i\beta}}{(1-\omega u^\alpha)^{i\gamma}}.
\end{aligned} \tag{3.18}
$$

Using relation (2.13), we have

$$\mathbb{S}^{-1}\left[\frac{u^{i\beta}}{(1-\omega u^{\alpha})^{i\gamma}}\right]=t^{i\beta}E_{\alpha,i\beta+1}^{i\gamma}\left(\omega t^{\alpha}\right).$$

Therefore, by applying the inverse Sumudu transform to Eq. (3.18), we get

$$\begin{aligned} f_n(x,t) &= \max\left(e^x-1,0\right)+\max\left(e^x-1,0\right)\sum_{i=1}^{n}(-1)^i q^i t^{i\beta}E_{\alpha,\beta+1}^{i\gamma}\left(\omega t^{\alpha}\right) \\ &\quad +\max\left(e^x,0\right)\sum_{i=1}^{n}(-1)^{i-1} q^i t^{i\beta}E_{\alpha,i\beta+1}^{i\gamma}\left(\omega t^{\alpha}\right), \end{aligned}$$

and, when $n\to\infty$, this becomes the following solution:

$$\begin{aligned} f(x,t) &= \max\left(e^x-1,0\right)\left(1+\sum_{i=1}^{\infty}\left(-qt^{\beta}\right)^i E_{\alpha,i\beta+1}^{i\gamma}\left(\omega t^{\alpha}\right)\right) \\ &\quad -\max\left(e^x,0\right)\sum_{i=1}^{\infty}\left(-qt^{\beta}\right)^i E_{\alpha,i\beta+1}^{i\gamma}\left(\omega t^{\alpha}\right). \end{aligned} \tag{3.19}$$

Remark 3.6. When $\gamma=0$, Eq. (3.19) is equivalent to

$$\begin{aligned} f(x,t) &= \max\left(e^x,0\right)+\max\left(e^x-1,0\right)\sum_{i=1}^{n}\frac{1}{\Gamma(i\beta+1)}\left(-qt^{\beta}\right)^i \\ &\quad -\max\left(e^x,0\right)\sum_{i=1}^{n}\frac{1}{\Gamma(i\beta+1)}\left(-qt^{\beta}\right)^i \\ &= \max\left(e^x,0\right)\left(1-\sum_{i=0}^{n}\frac{1}{\Gamma(i\beta+1)}\left(-qt^{\beta}\right)^i\right)+\max\left(e^x-1,0\right)\sum_{i=0}^{n}\frac{1}{\Gamma(i\beta+1)}\left(-qt^{\beta}\right)^i. \end{aligned}$$

which gives

$$f(x,t)=\left(1-E_{\beta}\left(-qt^{\beta}\right)\right)\max\left(e^x,0\right)+E_{\beta}\left(-qt^{\beta}\right)\max\left(e^x-1,0\right).$$

Moreover, when $\beta=1$, the exact solution is given by

$$f(x,t)=\left(1-e^{-qt}\right)\max\left(e^x,0\right)+e^{-qt}\max\left(e^x-1,0\right).$$

4. Conclusion

In this work, PSHAM is proposed to solve fractional linear partial differential equations by considering regularized Prabhakar derivatives operators. On the basis of this method, a general scheme for estimating approximate analytical solutions of fractional equations has been developed. The methodology presented has become an important mathematical element. Since the choice of the fractional derivative depends on the problem studied and the phenomenological behaviour of the system, the known and widely used derivatives can not meet the growing and rapid needs of physicists and engineers in the study of various natural phenomena. Prabhakar derivatives are a good way to model these phenomena, due to its comprehensiveness. This work shows that PSHAM is a very efficient tool for solving partial linear differential equations considering regularized Prabhakar fractional operators.

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