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Bernoulli Wavelet Method for Numerical Solutions of System of Fuzzy Integral Equations

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Abstract In this paper, we have proposed an efficient numerical method to solve a system linear fuzzy Fredholm integral equations of the second kind based on Bernoulli wavelet method (BWM). Bernoulli wavelets have been generated by dilation and translation of Bernoulli polynomials. The aim of this paper is to apply Bernoulli wavelet method to obtain approximate solutions of a system of linear Fredholm fuzzy integral equations. First, we introduce properties of Bernoulli wavelets then we used it to transform the integral equations to the system of algebraic equations, the error estimates of the proposed method are given and compared by solving some numerical examples.

Keywords. Parametric form of a Fuzzy number, Bernoulli wavelets, Fuzzy integral equations, Approximate solution, product matrix, Error estimation.

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1. Introduction

Many different powerful methods have been proposed to obtain approximate solutions of integral equations [12, 14]. This study is an effort to propose a new method for solving linear and nonlinear Fredholm integral and integro-differential equations of the second kind and the systems which consist of the mentioned equations.

It is well-known that an important class of fuzzy integral equations is fuzzy Fredholm integral equations. Existence and uniqueness of solutions of such equations has been studied by several authors [6, 11]. In this paper, we will introduce a simple method for solving fuzzy Fredholm integral equations of the second kind (FFIE-2).

The structure of paper is organized as follows; In section 2, some basic definitions of fuzzy integral equation. In section 3, we expand a variable function by Bernoulli wavelet method. In section 4, we propose Bernoulli wavelet method to solve fuzzy Fredholm integral equation. In section 5, we obtain error estimation, and then the

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proposed methods are implemented for solving three illustrative examples in section 6 and finally, conclusion is drawn in section 7.

2. Preliminaries of fuzzy integral equation

In this section the basic notations used in fuzzy calculus are introduced. We start by defining the fuzzy number.

Definition 2.1. (See Ref. [8].) A fuzzy number is a fuzzy set $u : R^1 \to [0, 1]$ which satisfying the following properties.

I. u is upper semi continuous.
II. $u(x) = 0$ outside some interval $[c, d]$.
III. There are real numbers $a$and b,$c \leq a \leq b \leq d$, for which
   a. $u(x)$is monotonically increasing on $[c, a]$,
   b. $u(x)$is monotonically decreasing on $[b, d]$,
   c. $u(x) = 1$for $a \leq x \leq b$.

The set of all fuzzy numbers, as given by definition (2.1) is denoted by E^1. Parametric form of a fuzzy number which yields the same E^1 is given by Kaleva [4].

Definition 2.2. (See Ref. [8].) A fuzzy number u is represented by an ordered pair of $(\underline{u}(r), \bar{u}(r))$,of functions $\underline{u}(r)$and $\bar{u}(r)$, $0 \leq r \leq 1$ satisfying the following properties.

I. $\underline{u}(r)$ is a bounded monotonic increasing left continuous function.
II. $\bar{u}(r)$ is a bounded monotonic decreasing left continuous function.
III. $\underline{u}(r) \leq \bar{u}(r)$, $0 \leq r \leq 1$.

For arbitrary $u(r)=(\underline{u}(r),\bar{u}(r))$, $v(r)=(\underline{v}(r),\bar{v}(r))$, and $k > 0$ we define addition $(u + v)$ and scalar multiplication by k as:

a.
$$\underline{(u + v)}(r) = \underline{u}(r) + \underline{v}(r) \tag{2.1}$$
b.
$$\overline{(u + v)}(r) = \bar{u}(r) + \bar{v}(r) \tag{2.2}$$
c.
$$\underline{(k\,u)}\,(r) = k\,\underline{u}(r), \overline{(k\,u)}(r) = k\,\bar{u}(r) \tag{2.3}$$

The collection of all the fuzzy numbers with addition and multiplication as de?ned by Eqs. (2.1,2.2) and (2.3) is denoted by E^1 and is u convex cone. It can be shown that Eqs. (2.1,2.2) and (2.3) are equivalent to the addition and multiplication as defined by using the α-*cut* approach [3] and the extension principles [10]. We will next de?ne the fuzzy function notation and a metric D in E^1 [3].

Definition 2.3. For arbitrary numbers $u = (\underline{u}(r), \bar{u}(r))$ and $v = (\underline{v}(r), \bar{v}(r))$

$$D(u, v) = \max \left\{ \begin{array}{ll} \sup & |\bar{u}(r) - \bar{v}(r)| \quad \sup \quad |\underline{u}(r) - \underline{v}(r)| \\ 0 \leq r \leq 1 & \qquad\quad 0 \leq r \leq 1 \end{array} \right\}, \tag{2.4}$$

in the distance between u and v[15].

Remark 2.4. If the fuzzy function $f(t)$ is continuous in the metric D, its de?nite integral exists. Also

$$\underline{(\int_a^b f(t;r)dt\,)} = \int_a^b \underline{f}\,(t;r)dt, \tag{2.5}$$

$$\overline{(\int_a^b f(t,r)dt)} = \int_a^b \bar{f}(t;r)dt, \tag{2.6}$$

where $\underline{f}(t,r), \bar{f}(t,r)$ is the parametric form of $f(t)$. It should be noted that the fuzzy integral can be also de?ned by using the Lebesgue-type approach [5]. However, if $f(x)$ is continuous, both approaches yield the same value. Moreover, the representation of the fuzzy integral by using Eq. (2.3) is more convenient for numerical calculations. More details about the properties of the fuzzy integral are given in [3, 4].

Definition 2.5. ([10]) Suppose $\tilde{u}$ is a fuzzy number and $r \in [0, 1]$. Then the r-cut representation of $\tilde{u}$ is the pair of functions $L(r)$ and $R(r)$ both form $[0, 1]$ to R defined respectively, by

$$L(r) = \inf\{\langle x \mid x\rangle \in\ [u]_r\};\ if\, r\ \in (0,1]$$

$$= \inf\{\langle x \mid x\rangle \in \sup\ p(\tilde{u})\};\ if\, r = 0 \tag{2.7}$$

and

$$R(r) = \sup\{\langle x \mid x\rangle \in\ [u]_r\};\ if\, r\ \in (0,1]$$

$$= \sup\{\langle x \mid x\rangle \in \sup\ p(\tilde{u})\};\ if\, r = 0 \tag{2.8}$$

Lemma 2.6. [2] *If f and $g : [a, b] \subseteq R \to E$ are fuzzy continuous function, then the function $F : [a, b] \to R$ by $F(t) = D(f(t), g(t))$ is continuous on $[a, b]$ and*

$$D(\int_a^b f(t)\,dx, \int_a^b g(t)\,dt) \leq\ D(\int_a^b f(t)\ ,\, g(t))\,dt \tag{2.9}$$

3. Wavelets and Bernoulli wavelets

wavelets constitute a family of functions constructed from dilation and translation of a single function called mother wavelet. When the dilation parameter a and the translation parameter **b** vary continuously, we have the following family of continuous wavelets as

$$\psi_{a,b}(t) = |a|^{\frac{-1}{2}}\ \psi(\frac{t-b}{a}), a, b\ \in \mathrm{R},\ a \neq 0 \tag{3.1}$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}, b = n\, b_0\, a_0^{-k}, a_0 > 1,\ b_0 > 0$ and n and k are positive integers, we have the family of discrete wavelets:

$$\psi_{k,n}(t) = |a_0|^{\frac{k}{2}}\ \psi(a_0^k t - nb_0), n, k\ \in Z^+ \tag{3.2}$$

where $\psi_{k,n}(t)$ forms a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2,\ b_0 = 1,$ then $\psi_{k,n}(t)$ form an orthonormal basis.

Bernoulli wavelets $\psi_{n,m}(t) = \psi(k, n, m, t)$ have four arguments, where $n = 1, 2, ..., 2^{k-1}$, $k \in Z^+, m$ is the order of Bernoulli polynomials and t is normalized time. They are defined on the interval [0,1) as [2].

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k-1}{2}}\, \tilde{\beta}_m(2^{k-1}t - n + 1) & \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}} \\ 0 & otherewise \end{cases} \tag{3.3}$$

with

$$\tilde{\beta}_m(t) = \begin{cases} 1 & m = 0 \\ \dfrac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}}}\,\beta_m(t) & m > 0 \end{cases} \tag{3.4}$$

where $n = 1, 2, ..., 2^{k-1}$ and $m = 0, 1, ..., M - 1$.

The coefficient $\dfrac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}}}$ is for the orthonormality, the dilation parameter is $a = 2^{-(k-1)}$ and translation parameter is $b = (n - 1)\, 2^{-(k-1)}$.

Here $\beta_m(t)$ are the well-known m^{th} order Bernoulli polynomials which are defined on the interval [0, 1], and can be determined with the aid of the explicit formula [3].

$$\beta_m(t) = \sum_{i=0}^{m} \begin{pmatrix} m \\ i \end{pmatrix} \alpha_{m-i}\, t^i, \tag{3.5}$$

where are α_i. $i = 0, 1, ..., m$ are Bernoulli numbers.

The first four such polynomials, respectively, are

$$\beta_0(t) = 1,\ \beta_1(t) = t - \frac{1}{2}, \beta_2(t) = t^2 - t + \frac{1}{6}, \beta_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t\,.$$

Bernoulli polynomials satisfy the following formula [3].

$$\int_0^1 \beta_n(t)\, \beta_m(t)\, dt = (-1)^{n-1}\frac{m!\, n!}{(m+n)!}\,\beta_{m+n} \ \ m, n \ge 1.$$

we can approximate function with this base. for example for $k = 2$and $M = 2$

$$\psi_{1,0}(t) = \begin{cases} \sqrt{2} & 0 \le x < \frac{1}{2} \\ 0 & otherewise \end{cases}, \psi_{2,0}(x) = \begin{cases} \sqrt{2} & \frac{1}{2} \le x < 1 \\ 0 & otherewise \end{cases}$$

$$\psi_{1,1}(x) = \begin{cases} \sqrt{6}(4t-1) & 0 \le x < \frac{1}{2} \\ 0 & otherewise \end{cases}, \psi_{2,1}(x) = \begin{cases} \sqrt{6}(4t-3) & \frac{1}{2} \le x < 1 \\ 0 & otherewise \end{cases}$$

Any function $y(t)$ which is square integrable in the interval [0, 1) can be expanded in a Bernoulli wavelet method.

$$y(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} y_{n,m}\psi_{n,m}(t),\ n = 1, 2, ..., \infty,\ m = 0, 1, 2, ..., \infty,\ t \in [0, 1), \tag{3.6}$$

$$y_{n,m} = \frac{(y(t), \psi_{n,m}(t))}{(\psi_{n,m}(t), \psi_{n,m}(t))} \tag{3.7}$$

In (3.7), $(. \ , .)$ denotes the inner product.

If the infinite series in (3.6) is truncated, then (3.6) can be rewritten as:

$$y(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} y_{n,m}\, \psi_{n,m}(t) \ = Y^T\, \Psi(t) \tag{3.8}$$

$$Y = [y_{1,0}, y_{1,1}, ..., y_{1,M-1}, y_{2,0}, ..., y_{2,M-1}, ..., y_{2^{k-1},0}, ..., y_{2^{k-1},M-1}]^T$$

$$\Psi(t) = [\psi_{1,0}(t), \psi_{1,1}(t), ..., \psi_{1,M-1}(t), \psi_{2,0}(t), ..., \psi_{2,M-1}(t), \\ ...., \psi_{2^{k-1},0}(t), ..., \psi_{2^{k-1},M-1}(t)]^T$$

Therefore, we have

$$Y^T < \Psi(t), \Psi(t) >=< u(t), \Psi(t)) >,$$

then $Y = D^{-1} < u(t), \Psi(t) >$, where,

$$D = \int_0^1 \Psi(t)\, \Psi^T(t)\, dt$$

$$= \begin{pmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & D_n \end{pmatrix} \tag{3.9}$$

then by using (3.8) $D_i (i = 1, 2, ..., M)$ is defined as follow:

$$(D_n)_{i,j+1} = \int_{\frac{i-1}{2^{(k-1)}}}^{\frac{i}{2^{(k-1)}}} \Psi_{i,n}(2^{k-1}t - i + 1)\, \Psi_{j,n}(2^{k-1}t - i + 1) dt$$

$$= \frac{1}{2^{k-1}} \int_0^1 \Psi_{i,n}(t)\, \Psi_{j,n}(t)\, dt$$

We can also approximate the function$k(x, t) \in L[0, 1]$ as follow:

$$k(x, t) \approx \Psi^T(x)\, K\, \Psi(t),$$

where K is an $2^{k-1} \cdot M \times 2^{k-1} \cdot M$ matrix that we can obtain as follows:

$$K = D^{-1} < \Psi(x) < k(x, t), \Psi(t) >> D^{-1} \tag{3.10}$$

4. Solving Linear Fuzzy Fredholm Integral Equation by Bernoulli wavelet method

4.1. In this section, we present Bernoulli wavelet method to solve Fredholm integral equation. First, consider the following equation:

$$y(\mathrm{x}) = f(x) + \lambda \int_0^1 k(\mathrm{x},\mathrm{t}) y(\mathrm{t}) dt \tag{4.1}$$

where $f(\mathrm{x}) \in L^2([0,1)),\ k(\mathrm{x},\mathrm{t}) \in L^2([0,1)) \times L^2([0,1))$ is an arbitrary kernel function and $y(\mathrm{x})$ is the unknown fuzzy real valued function. In [16], the authors presented sufficient conditions for the existence and unique solution of (4.1) as the following theorem:

Theorem 4.1. [16] *Let$k(x,t)$ be continuous for $a = x\ t = b$, and$f(x)$ a fuzzy continuous of x, $a = x = b$. If $\lambda < \frac{1}{M(b-a)}$ where$M = \max_{a \le x,t \le 1} |k(x,t)|$, then the iterative procedure $y_0(x) = f(x)$, Converges to the unique solution of* (4.1). *Specially,* $\sup\limits_{a \le x \le b} D(y(x), y_k(x)) \le \frac{L^k}{1-L} \sup\limits_{a \le x \le b} D(y(x)_0, y_1(x))$, *where* $L = \lambda M(b-a)$.

In this paper, we consider fuzzy Fredholm integral equation (4.1) with $a = 0,\ b = 1$ and $\lambda > 0$, where $u(x)$ and $f(x)$ are in $L^2([0,1))$ and $k(x,t)$ belongs to $L^2([0,1) \times [0,1))$.

Our problem is to determine TFs pair coef?cients of $y(x)$ in the interval $[0,1)$ from the know functions $f(x)$ and kernel $k(x,t)$. So, we introduce the parametric form of a FFIE-2 with respect to definition 2.2 let $(\underline{f}(x,r), \bar{f}(x,r))$ and $\underline{y}(x,r), \bar{y}(x,r)$ $0 \le r \le 1$ and $x \in [0,\ 1)$ be parametric forms of $f(x)$,$y(x)$, respectively. Therefore, we rewrite system (4.1) in the following form

$$\begin{array}{l} \underline{y}(x,r) = \underline{f}(x,r) + \int_0^1 k(x,t)\, \underline{y}(t,r)\, dt \\ \bar{y}(x,r) = \bar{f}(x,r) + \int_0^1 k(x,t)\, \bar{y}(t,r)\, dt \end{array} \tag{4.2}$$

where

$$k(x,t)\, \underline{y}(t,r) = \begin{cases} k(x,t)\, \underline{y}(t,r) & k(x,t) \ge 0 \\ k(x,t)\, \bar{y}(t,r) & k(x,t) < 0 \end{cases}$$

and

$$k(x,t)\, \bar{y}(t,r) = \begin{cases} k(x,t)\, \bar{y}(t,r) & k(x,t) \ge 0 \\ k(x,t)\, \underline{y}(t,r) & k(x,t) < 0 \end{cases}$$

4.2. In this section, we present Bernoulli wavelet method to solve the system of Fredholm integral equation. The second kind fuzzy Fredhoolm integral equations system is in the form:

$$\begin{cases} y_1(\mathrm{x}) = f_1(x) + \sum_{j=1}^m (\lambda_{1,j} \int_a^b k_{1,j}(\mathrm{x},\mathrm{t}) y_j(\mathrm{t})\, dt\,) \\ \vdots \\ y_i(\mathrm{x}) = f_i(x) + \sum_{j=1}^m (\lambda_{i,j} \int_a^b k_{i,j}(\mathrm{x},\mathrm{t}) y_j(\mathrm{t})\, dt\,) \end{cases} \tag{4.3}$$

where $a \le x,\ t \le\ b$, and $\lambda_{ij} \ne 0$for $(i,j = 1,...,m)$ are real constants.

Let $(\underline{y_i}(x,r),\bar{y}_i(x,r))$ and $(\underline{f_i}(x,r),\bar{f}_i(x,r))$ $(0 \le r \le 1,\ a \le x \le 1)$ be parametric form of $y_i(x)$ and $f_i(x)$ respectively. In order to design a numerical scheme for solving (4.3), we write the parametric form of the given fuzzy integral equations system as follows:

$$\bar{y}_i(\mathrm{x},\mathrm{r}) = \bar{f}_i(x,r) + \sum_{j=1}^{m}(\lambda_{i,j}\int_a^b k_{i,j}(\mathrm{x},\mathrm{t})\bar{y}_j(\mathrm{t})\,dt\,) \tag{4.4}$$

$$\underline{y}_i(\mathrm{x},\mathrm{r}) = \underline{f}_i(x,r) + \sum_{j=1}^{m}(\lambda_{i,j}\int_a^b k_{i,j}(\mathrm{x},\mathrm{t})\underline{y}_j(\mathrm{t})\,dt\,),\ \ i=1,....,m \tag{4.5}$$

where

$$k_{i,j}(\mathrm{x},\mathrm{t})\bar{y}_j(\mathrm{t}) \ = \begin{cases} k_{i,j}(\mathrm{x},\mathrm{t})\bar{y}_j(\mathrm{t}) & ,k_{i,j}(\mathrm{x},\mathrm{t}) \ge 0 \\ k_{i,j}(\mathrm{x},\mathrm{t})\,\underline{y}_j(\mathrm{t}) & ,k_{i,j}(\mathrm{x},\mathrm{t}) < 0 \end{cases}$$

and

$$k_{i,j}(\mathrm{x},\mathrm{t})\,\underline{y}_j(\mathrm{t}) \ = \begin{cases} k_{i,j}(\mathrm{x},\mathrm{t})\,\underline{y}_j(\mathrm{t}) & ,k_{i,j}(\mathrm{x},\mathrm{t}) \ge 0 \\ k_{i,j}(\mathrm{x},\mathrm{t})\,\bar{y}_j(\mathrm{t}) & ,k_{i,j}(\mathrm{x},\mathrm{t}) < 0 \end{cases}$$

We can approximate the function $\underline{y_i}(x,r)$, $\bar{y}_i(x,r)$, $\underline{f_i}(x,r)$, $\bar{f}_i(x,r)$ and $k_{i,j}(x,t)$ by Bernoulli wavelet method as follows:

$$\underline{y_i}(x,r) \approx \Psi^T(x)\,Y_{1,i}\,\Psi(r), \bar{y}_i(x,r) \approx \Psi^T(x)\,Y_{2,i}\,\Psi(r) \tag{4.6}$$

$$\underline{f}_i(x,r) \approx \Psi^T(x)\,F_{1,i}\,\Psi(r), \bar{f}_i(x,r) \approx \Psi^T(x)\,F_{2,i}\,\Psi(r), k_{i,j}(x,t) \approx \Psi^T(x)\,K_{i,j}\,\Psi(t)$$

After substituting the approximate equations(4.6) into equations (4.4) and (4.5), we get

$$\Psi^T(x)\,Y_{1,i}\,\Psi(r) = \Psi^T(x)\,F_{1,i}\,\Psi(r) + \textstyle\sum_{j=1}^{m}\lambda_{i,j}\int_0^1 \Psi^T(x)\,K_{i,j}\,\Psi(t)\ \Psi^T(t)\,Y_{1,i}\,\Psi(r)\,dt$$
$$\Psi^T(x)\,Y_{2,i}\,\Psi(r) = \Psi^T(x)\,F_{2,i}\,\Psi(r) + \textstyle\sum_{j=1}^{m}\lambda_{i,j}\int_0^1 \Psi^T(x)\,K_{i,j}\,\Psi(t)\ \Psi^T(t)\,Y_{2,i}\,\Psi(r)\,dt$$

We have:

$$\Psi^T(x)\,Y_{1,i}\,\Psi(r) = \Psi^T(x)\,F_{1,i}\,\Psi(r) + \Psi^T(x)\ \textstyle\sum_{j=1}^{m}\lambda_{i,j}\,K_{i,j}\int_0^1\ \Psi(t)\ \Psi^T(t)\,Y_{1,i}\,\Psi(r)\,dt$$
$$\Psi^T(x)\,Y_{2,i}\,\Psi(r) = \Psi^T(x)\,F_{2,i}\,\Psi(r) + \Psi^T(x)\ \textstyle\sum_{j=1}^{m}\lambda_{i,j}\,K_{i,j}\int_0^1\ \Psi(t)\ \Psi^T(t)\,Y_{2,i}\,\Psi(r)\,dt$$

with the powerful properties of equation (3.9) we get:

$$\Psi^T(t)\,Y_{1,i} = \Psi^T(t)\,F_{1,i} + \sum_{j=1}^{m}\lambda_{i,j}\,\Psi^T(x)\,K_{i,j}\,D\,Y_{1,i}$$

$$\Psi^T(t)\,Y_{2,i} = \Psi^T(t)\,F_{2,i} + \sum_{j=1}^{m}\lambda_{i,j}\,\Psi^T(x)\,K_{i,j}\,D\,Y_{2,i}$$

Therefore

$$Y_{1,i} = F_{1,i} + \sum_{j=1}^{m}\lambda_{i,j}\ K_{i,j}\,D\,Y_{1,i}$$

$$Y_{2,i} = F_2 + \sum_{j=1}^{m}\lambda_{i,j}\ K_{i,j}\,D\,Y_{2,i} \tag{4.7}$$

Where, the dimensional subscripts have been dropped to simplify the notation. Rewriting (4.7), we have

$$Y_{1,i} = (I - \sum_{j=1}^{m} \lambda_{i,j}\, K_{i,j}\, D\,)^{-1}\, F_{1,i}$$

$$Y_{2,i} = (I - \sum_{j=1}^{m} \lambda_{i,j}\, K_{i,j}\, D\,)^{-1}\, F_{2,i} \tag{4.8}$$

From Eqs. (4.8), we have a system of $2^{k-1}M$linear equations and $2^{k-1}M$ unknowns. After solving above linear system using finite iterative algorithm [15]. We can achieve the unknown vectors $Y_{1,i}$ and $Y_{2,i}$. The required approximated solution $\underline{y_i}(x, r) \approx B^T(x)\, Y_{1,i}\, B(r)$, $\bar{y}_i(x, r) \approx B^T(x)\, Y_{2,i}\, B(r)$ for Fredholm integral Eq. (4.2).

5. Error Estimation

First, we obtain the error estimation for given FFIE-2 (4.1) by Bernoulli wavelet method. Suppose that $\tilde{y}(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} {}^{*} y_{n,m}\, \psi_{n,m}(x)$ is an approximate solution of $y(x)$, where $\sum^{*}$ denotes the fuzzy summation Therefore, we get:

$$D(y(x), \tilde{y}(x)) = D(\int_0^1 k(x,t)\, y(t)\, dt, \int_0^1 (k(x,t) \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} {}^{*} y_{n,m}\, \psi_{n,m}(t)\ )\, dt)$$

$$\leq M \int_0^1 D(\, y(t), \int_0^1 (k(x,t) \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} {}^{*} y_{n,m}\, \psi_{n,m}(t)\ )\, dt) \tag{5.1}$$

where

$$M = \max_{0 \leq x,\, t \leq 1} |k(x,t)|\,,$$

we have:

$$D(y(x), \tilde{y}(x)) \quad \leq M \int_0^1 D(\, y(t),\, \tilde{y}(t))\, dt, \tag{5.2}$$

$$\sup_{x \in [0,1)} D(y(x), \tilde{y}(x)) \quad \leq M \sup_{x \in [0,1)} D(y(x), \tilde{y}(x)) \tag{5.3}$$

Therefore, if $M < 1$, we will have:

$$\lim_{m \to \infty} \sup_{x \in [0,1)} D(y(x), \tilde{y}(x)) = 0. \tag{5.4}$$

6. Numerical Examples

We consider three examples to illustrate the Bernoulli wavelet method for system of FFIE-2. In this case, fuzzy approximate solutions using [15] is calculated at five iterations and are given in Table1 the Bernoulli wavelet method isless absolute error than TFs method and DFs method. In Table 2 Bernoulli wavelet method and TFs have been compared. Bernoulli wavelet method and the exact solutions have been compared In Table 3. Bernoulli wavelet method has less error compared with TFs method. It is worth mentioning that we used $2^{k-1}M \times 2^{k-1}M$ and $(2m) \times (2m)$ matrix for Bernoulli wavelet method and TFs methods, respectively.

Example 6.1. Consider the following fuzzy Fredholm integral equation [7] with

$$\underline{f}(x,r) = (r^2+r)\,(\sin(\frac{x}{2}) - 0.05\,\sin(x)(1-\sin(1)),$$

$$\bar{f}(x,r) = (4-r^3-r)(\sin(\frac{x}{2}) - 0.05\,\sin(x)(1-\sin(1)), \tag{6.1}$$

and $k(x,t) = 0.1\sin(x)\,\sin(\frac{t}{2})\,,\quad 0 \le x,\ t < 1$ and $\lambda = 1$

The exact solution in this case in given by

$$\underline{y}(x,r) = (r^2+r)\,(\sin(\frac{x}{2}), \bar{y}(x,r) = (4-r^3-r)(\sin(\frac{x}{2}), \tag{6.2}$$

Comparison between the exact and the numerical solutions for the fuzzy integral equation using the BWM, TFs method and DFs method at $x = 0.1$ are shown in table 1.

Example 6.2. Consider the following fuzzy Fredholm integral equation [7] with

$$\underline{f}(x,r) = r\,x\,(r^4+2)(3\cos(1-x)+5\,\sin(1-x)-6\,\cos(x)+x^2),$$

$$\bar{f}(x,r) = -3\,x\,(r^3-2)(3\cos(1-x)+5\,\sin(1-x)-6\,\cos(x)+x^2), \tag{6.3}$$

and $k(x,t) = x\,\cos(t-x),\quad 0 \le x, t < 1$ and $\lambda = 1$.

The exact solution in this case in given by

$$\underline{y}(x,r) = x^3(r^5+2r), \bar{y}(x,r) = x^3(6-3\,r^3), \tag{6.4}$$

In this example we calculate the approximate solution using BWM and the resulted system is calculated using MAPLE software.

Example 6.3. Consider the system of fuzzy integral equations in [9].

$$\begin{cases} \underline{y}(x,r) + \int_0^1 x\,\underline{y}(t)dt + \int_0^1 2x^2\,\bar{y}(t)dt = \underline{f}(x,r) \\ \bar{y}(x,r) + \int_0^1 4xt\,\bar{y}(t)\,dt + \int_0^1 2x\,\underline{y}(t)dt = \bar{f}(x,r) \end{cases} \tag{6.5}$$

where $\underline{f}(x,r) = x^2\,(r^2+2\,r+2, 7-2r) + \frac{x}{3}(r^2+r+1, 4-r)$,

$$\bar{f}(x,r) = x\,(r^2+3r+3,\ 10-3r), 0 \le x,t \le 1\ for\ 0 \le r \le 1.$$

Table 1. Comparison of Absolute error of BWM at $k = 3$, $M = 4$, TFs method and DFs method at $x = 0.1$ in Example 6.1.

r	Absolute error of BWM at $x = 0.1$	Absolute error of TFs method at $x = 0.1$ and $m = 10$	Absolute error of DFs method at $x = 0.1$ and $m = 10$
0	$(0.000000, 0.000000)$	$(0.000000, 0.000000)$	$(0.000000, 0.000000)$
0.1	$(6.714 \times 10^{-8}$,$5.655 \times 10^{-7})$	$(1.764 \times 10^{-7}$,$6.253 \times 10^{-6})$	$(1.764 \times 10^{-7}$,$6.253 \times 10^{-6})$
0.2	$0.53.754 \times 10^{-4})$	$(3.849 \times 10^{-7}$,$6.082 \times 10^{-6})$	$(3.849 \times 10^{-7}$,$6.082 \times 10^{-6})$
0.3	$(7.312\times10^{-8}$,$8.8585\times 10^{-7})$	$(5.771 \times 10^{-6}$,$5.435 \times 10^{-5})$	$(5.771 \times 10^{-6}$, $5.435 \times 10^{-5})$
0.4	$(1.328 \times 10^{-7}$,$1.564 \times 10^{-6})$	$(8.981 \times 10^{-7}$, $5.671 \times 10^{-6})$	$(8.981 \times 10^{-7}$, $5.671 \times 10^{-6})$
0.5	$(2.860 \times 10^{-8}$,$8.847 \times 10^{-6})$	$(1.203 \times 10^{-6}$, $5.413 \times 10^{-6})$	$(1.203 \times 10^{-6}$,$5.413 \times 10^{-6})$
0.6	$(7.700 \times 10^{-7}$,$7.020 \times 10^{-6})$	$(1.611 \times 10^{-6}$, $5.343 \times 10^{-6})$	$(1.611 \times 10^{-6}$,$5.344 \times 10^{-6})$
0.7	$(5.533 \times 10^{-7}$,$8.173 \times 10^{-6})$	$(2.619 \times 10^{-6}$, $6.507 \times 10^{-6})$	$(2.619 \times 10^{-6}$, $6.507 \times 10^{-6})$
0.8	$(7.690 \times 10^{-7}$,$6.559 \times 10^{-6})$	$(2.310 \times 10^{-6}$, $4.311 \times 10^{-6})$	$(2.310 \times 10^{-6}$, $4.311 \times 10^{-6})$
0.9	$(3.700 \times 10^{-9}$,$6.476 \times 10^{-6})$	$(2.743 \times 10^{-6}$, $3.803 \times 10^{-6})$	$(2.743 \times 10^{-6}$, $3.803 \times 10^{-6})$

The exact solution in this case is given by

$$\underline{y}(x, r) = x^2(r^2 + r + 1, 4 - r),$$

$$\bar{y}(x, r) = x(r + 1,\, 3 - r), \tag{6.6}$$

Results are shown in Tables 6.3.

7. Conclusion

In this paper, we have worked out a computational method to approximating the solution of Fredholm fuzzy integral equations system of the second kind. Based on the embedding method, we applied triangular and delta basis functions for approximation of the unique solution of Fredholm fuzzy integral equations system. The matrix used in Bernoulli wavelet method was$2^{k-1}M \times 2^{k-1}M$. Also, we proved the error estimation for the approximated solution of Fredholm fuzzy integral equations system. The results show that the proposed method is a promising tool for this type of fuzzy integral equations. The main advantage of these methods are the ability, reliability, and low cost of setting up the equations without using any projection method.

TABLE 2. Comparison of Absolute error of BWM at $k = 3$, $M = 4$, TFs method and DFs method at x=0.2 in Example 6.2.

r	Absolute error BWM at $x = 0.2$	Absolute error of TFs method at $x = 0.2$ and $m = 30$	Absolute error of DFs method at $x = 0.2$ and $m = 30$
0	$(0.0000000, 2.846 \times 10^{-7})$	$(0.00000000, 5.045 \times 10^{-4})$	(0.0000000, 0.04800000)
0.1	$(1.004 \times 10^{-7}, 1.187 \times 10^{-7})$	$(1.682 \times 10^{-5}, 5.043 \times 10^{-4})$	$(1.682 \times 10^{-5}, 5.043 \times 10^{-4})$
0.2	$(2.300 \times 10^{-6}, 4.166 \times 10^{-7})$	$(3.37 \times 10^{-5}, 5.025 \times 10^{-4})$	$(3.366 \times 10^{-5}, 5.025 \times 10^{-4})$
0.3	$(1.904 \times 10^{-6}, 5.896 \times 10^{-6})$	$(5.066 \times 10^{-5}, 4.977 \times 10^{-4})$	$(5.066 \times 10^{-5}, 4.977 \times 10^{-4})$
0.4	$(1.576 \times 10^{-7}, 2.401 \times 10^{-7})$	$(6.813 \times 10^{-5}, 4.884 \times 10^{-4})$	$(6.813 \times 10^{-5}, 4.884 \times 10^{-4})$
0.5	$(2.400 \times 10^{-7}, 4.020 \times 10^{-6})$	$(8.672 \times 10^{-5}, 4.730 \times 10^{-4})$	$(8.672 \times 10^{-5}, 4.730 \times 10^{-4})$
0.6	$(9.518 \times 10^{-7}, 1.560 \times 10^{-6})$	$(1.075 \times 10^{-4}, 4.500 \times 10^{-4})$	$(1.075 \times 10^{-4}, 4.500 \times 10^{-4})$
0.7	$(1.701 \times 10^{-7}, 3.430 \times 10^{-6})$	$(1.319 \times 10^{-4}, 4.180 \times 10^{-4})$	$(1.319 \times 10^{-4}, 4.180 \times 10^{-4})$
0.8	$(1.096 \times 10^{-6}, 2.590 \times 10^{-6})$	$(1.621 \times 10^{-4}, 3.754 \times 10^{-4})$	$(1.621 \times 10^{-4}, 3.754 \times 10^{-4})$
0.9	$(4.028 \times 10^{-7}, 2.400 \times 10^{-6})$	$(2.010 \times 10^{-4}, 3.206 \times 10^{-4})$	$(2.0101 \times 10^{-4}, 3.206 \times 10^{-4})$

TABLE 3. Comparison of Absolute error of BWM at $k = 4$, $M = 8$ and the exact solution at $x = 0.4$ in Example 6.3.

r	Absolute error of BWM at $x = 0.4$	Absolute error of BWM at $x = 0.4$
	$\left\|\underline{y_1}^e - \underline{y_1}^a\right\|, \|\bar{y}_1^e - \bar{y}_1^a\|$	$\left\|\underline{y}_1^e - \underline{y}_1^a\right\|, \|\bar{y}_1^e - \bar{y}_1^a\|$
0	$(1.1 \times 10^{-10}, 2 \times 10^{-10})$	$(4.51 \times 10^{-10}, 6.453 \times 10^{-10})$
0.1	$(1.2 \times 10^{-9}, 3.3 \times 10^{-9})$	$(6.41 \times 10^{-9}, 3.544 \times 10^{-9})$
0.2	$(1.4 \times 10^{-9}, 3.4 \times 10^{-9})$	$(5.125 \times 10^{-10}, 8.155 \times 10^{-10})$
0.3	$(1.5 \times 10^{-9}, 3.5 \times 10^{-9})$	$(4.523 \times 10^{-10}, 5.623 \times 10^{-10})$
0.4	$(1.70 \times 10^{-9}, 3.60 \times 10^{-9})$	$(2.546 \times 10^{-10}, 8.45 \times 10^{-10})$
0.5	$(1.80 \times 10^{-10}, 3.80 \times 10^{-10})$	$(3.21 \times 10^{-10}, 4.215 \times 10^{-10})$
0.6	$(1.90 \times 10^{-10}, 4.50 \times 10^{-10})$	$(5.125 \times 10^{-9}, 6.325 \times 10^{-9})$
0.7	$(1.40 \times 10^{-10}, 4.80 \times 10^{-10})$	$(6.28 \times 10^{-9}, 7.215 \times 10^{-9})$
0.8	$(1.50 \times 10^{-10}, 5.00 \times 10^{-10})$	$(1.565 \times 10^{-9}, 1.260 \times 10^{-9})$
0.9	$(2.70 \times 10^{-9}, 5.40 \times 10^{-9})$	$(1.70 \times 10^{-9}, 3.60 \times 10^{-9})$

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