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Exact solutions of diffusion equation on sphere

Yadollah AryaNejad*
Department of Mathematics, Payame Noor University, 19395-4697, Tehran, Iran.

Abstract

We examine the diffusion equation on the sphere. In this sense, we answer the question of the symmetry classification. We provide the algebra of symmetry and build the optimal system of Lie subalgebras. We prove for one-dimensional optimal systems of Eq.1.4, space is expanding Ricci solitons. Reductions of similarities related to subalgebras are classified, and some exact invariant solutions of the diffusion equation on the sphere are presented.

Keywords. Ricci soliton, Lie Subalgebras, Reduction equations, Diffusion equation.
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1. Introduction

Suppose (G, g) is a (pseudo-)Riemannian manifold, V a smooth vector field on (G, g), and ϱ the Ricci tensor of g. The space (G, g) is *Ricci soliton*, if g, ϱ, and V satisfy the equation;

$$\mathcal{L}_V g + \varrho = \alpha g, \tag{1.1}$$

where $\mathcal{L}$ and α are the Lie derivative and a real constant, respectively. According to α ($\alpha > 0, \alpha = 0$, or $\alpha < 0$), the space (G, g) can also be a *shrinking*, *steady* or *expanding* Ricci soliton. The Ricci solitons are important for researchers to interpret equations. The second-order linear partial differential equation

$$\kappa u_{xx} + \kappa u_{yy} - u_t = 0. \tag{1.2}$$

is called the diffusion equation, where κ is a real constant called the diffusivity. Condensation of a diffuser or unstable temperature in an area without a heat source is ruled by this equation. It is well known that the metric on $S^2 \times R$ is:

$$ds^2 = dt^2 - dx^2 - sin^2x dy^2 \qquad f \in C^\infty(G). \tag{1.3}$$

Adjusting the metric (1.3) on $S^2 \times R$ and rewriting Equation (1.2), the diffusion differential equation on the sphere would be:

$$u_t = \kappa u_{xx} + \kappa(cotx)u_x + \kappa(csc^2x)u_{yy}. \tag{1.4}$$

where κ is a real constant. Performing the Lie symmetry group procedure, the problem of symmetry classification for different equations is widely considered in various spaces [1, 2, 5, 8, 9, 11, 13]. On the other hand, the symmetry group approach or Lei's approach itself, which is a computational method algorithmic for finding group-invariant solutions, is significantly used in the resolution of differential equations. Using this procedure, one can find appropriate solutions through known ones, study the invariant solutions, and even decrease the order of ODEs [3, 4, 7, 10, 12, 15].

In this paper, using Lei's method, we earn symmetries of the diffusion differential equation on the sphere. Then, an optimal subalgebras system linked to the symmetries Lie algebra is given. The article is organized as follows. The symmetry algebra infinitesimal generators of Eq.(1.4) are characterized, and several effects are obtained in section

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∗ Corresponding author. Email: y.aryanejad@pnu.ac.ir .

2. In section 3, we construct the optimal systems of subalgebras. In the next section, we prove for one-dimensional optimal systems of Eq.(1.4), the space is expanding Ricci solitons. In section 4, we find the Lie invariants, similarity solutions, and similarity reduction corresponding to the infinitesimal symmetries of Eq.(1.4). Finally, in section 5, some exact invariant solutions of the diffusion equation on the sphere are presented.

2. The symmetry algebra of Eq.(1.4)

Generally,

$$\Delta_\alpha(X, U^{(p)}) = 0, \qquad \alpha = 1, ..., t, \tag{2.1}$$

is a system of PDE of order pth, where $X = (x^1, ..., x^m)$ and $U = (u^1, ..., u^n)$ are m independent and n dependent variables respectively, and $U^{(i)}$ is the $i-$ order derivative of U with respect to x, $0 \leq i \leq p$. Infinitesimal transformations Lie group acts on both X and U, is:

$$\tilde{x}^i = x^i + \varepsilon\xi^i(X, U) + o(\varepsilon^2), \qquad i = 1, ..., m, \tag{2.2}$$

$$\tilde{u}^j = u^j + \varepsilon\phi_j(X, U) + o(\varepsilon^2), \qquad j = 1, ..., n, \tag{2.3}$$

where ξ^i and ϕ^j represent the infinitesimal transformations for $\{x^1, ..., x^p\}$ and $\{u^1, ..., u^q\}$, respectively. An arbitrary infinitesimal generator corresponding to the group of transformations (2.2) is

$$V = \sum_{i=1}^{p} \xi^i(X, U)\partial_{x^i} + \sum_{j=1}^{q} \phi_j(X, U)\partial_{u^j}. \tag{2.4}$$

Now in order to apply the Lie group procedure for Eq.(1.4), an infinitesimal transformation's one parameter Lie group is considered: (we use x, y and t instead of x^1, x^2 and x^3 respectively in order not to use index. So, $x^1 = x, x^2 = y, x^3 = t, u^1 = u$),

$$\begin{aligned}
\tilde{x} &= x + \varepsilon\xi^1(x, y, t, u, f) + o(\varepsilon^2), \\
\tilde{y} &= y + \varepsilon\xi^2(x, y, t, u, f) + o(\varepsilon^2), \\
\tilde{t} &= t + \varepsilon\xi^3(x, y, t, u, f) + o(\varepsilon^2), \\
\tilde{u} &= u + \varepsilon\phi_1(x, y, t, u, f) + o(\varepsilon^2).
\end{aligned} \tag{2.5}$$

The corresponding symmetry generator is as follows:

$$\begin{aligned}
V = \xi^1(x, y, t, u, f)\partial_x + \xi^2(x, y, t, u, f)\partial_y + \xi^3(x, y, t, u, f)\partial_t + \\
\phi_1(x, y, t, u, f)\partial_u.
\end{aligned} \tag{2.6}$$

The proviso of being invariance corresponds to the equations:

$$\begin{aligned}
Pr^{(2)}V[\kappa u_{xx} + \kappa(cotx)u_x + \kappa(csc^2x)u_t - u_t] = 0, \quad \text{whenever} \\
\kappa u_{xx} + \kappa(cotx)u_x + \kappa(csc^2x)u_t - u_t = 0.
\end{aligned}$$

Since ξ^1, ξ^2, ξ^3, and ϕ_1 are only dependent on x,y, t and u, setting the individual coefficients equal to zero, we have the following system of equations:

$$\begin{cases}
-\kappa^2\xi^3_{uu} = 0, \\
\quad \kappa cos(x)^2\xi^2_u - \kappa\xi^2_u = 0, \\
\quad 2\kappa cos(x)^2\xi^3_u - 2\kappa\xi^3_u = 0, \\
\quad \vdots
\end{cases}$$

The total number of these equations is 26. By solving these PDE equations, we earn the following result:

TABLE 1. Lie algebra for Eq.(1.4).

[,]	X_1	X_2	X_3	X_4	X_5
X_1	0	0	X_4	$-X_3$	0
X_2	$*$	0	0	0	0
X_3	$*$	$*$	0	X_1	0
X_4	$*$	$*$	$*$	0	0
X_5	$*$	$*$	$*$	$*$	0

Theorem 2.1. *The point symmetries Lie group of equation (1.4) possesses a Lie algebra generated by (2.6), whose coefficients are the following infinitesimals:*

$$\begin{aligned} &\xi^1 = c_3 siny - c_2 cosy, \\ &\xi^2 = cotx(c_3 cosy + c_2 siny) + c_4, \\ &\xi^3 = c_1, \\ &\phi_1 = c_5 u + \alpha(u), \end{aligned} \tag{2.7}$$

where $c_i \in R$, $i = 1, ..., 5$ and $\alpha(u)$ is a function satisfying Eq.(1.4).

Corollary 2.2. *Every point symmetry's one-parameter Lie group of Eq.(1.4) has the infinitesimal generators as follows:*

$$\begin{aligned} &X_1 = \partial_y, \\ &X_2 = \partial_t, \\ &X_3 = -cosy \quad \partial_x + cotx \quad siny \quad \partial_y, \\ &X_4 = siny \quad \partial_x + cotx \quad cosy \quad \partial_y, \\ &X_5 = u\partial_u, \\ &X_\alpha = \alpha\partial_u. \end{aligned} \tag{2.8}$$

We provide Lie algebra for Eq.(1.4) by Table (1). The expression $[X_i, X_j] = X_i X_j - X_j X_i$ determines the entry in row i^{th} and column j^{th}, $i, j = 1, ..., 5$.

For example, the flow of vector field X_1 in Corollary 2.2 is shown by

$$\Phi_\epsilon = (x, y + \epsilon, t).$$

3. Classification of one-dimensional subalgebras

Using the symmetry group, we can determine the one-parameter optimal system of Eq.(1.4). It's important to obtain those subgroups which present different kinds of solutions. Thus, we need to search for invariant solutions that are not linked by a transformation in the full symmetry group. This subject leads to the notion of an optimal set of subalgebras. The problem of classifying one-dimensional subalgebras would be the same as the question of classifying the adjoint representation orbits. An optimal set of subalgebras problem is solved by considering one representative from every group of corresponding subalgebras [14] and [12]. The definition of the adjoint representation of each X_t, $t = 1, ..., 5$ would be:

$$\mathrm{Ad}(\exp(s.X_t).X_r) = X_r - s.[X_t, X_r] + \frac{s^2}{2}.[X_t, [X_t, X_r]] - \cdots , \tag{3.1}$$

where s is a parameter and $[X_t, X_r]$ is defined in Table (1) for $t, r = 1, \cdots, 5$ ([12],page 199). Let $\mathfrak{g}$, be the Lie algebra that produced by (2.8). We obtain the adjoint action for $\mathfrak{g}$ in Table (2).

TABLE 2. Adjoint representation of the Lie algebra

Ad	X_1	X_2	X_3	X_4	X_5
X_1	X_1	X_2	$cos(s)X_3 - sin(s)X_4$	$cos(s)X_4 + sin(s)X_3$	X_5
X_2	X_1	X_2	X_3	X_5	X_5
X_3	$cos(s)X_1 + sin(s)X_4$	X_2	X_3	$cos(s)X_4 - sin(s)X_1$	X_5
X_4	$cos(s)X_1 - sin(s)X_3$	X_2	$cos(s)X_3 + sin(s)X_1$	X_4	X_5
X_5	X_1	X_2	X_3	X_4	X_5

Theorem 3.1. *One-dimensional subalgebras of Eq.(1.4) are as follows:*

$$\begin{aligned} &1)\ X_1 + c_1X_2 + c_2X_5,\\ &2)\ X_3 + c_1X_2 + c_2X_5,\\ &3)\ X_4 + c_1X_2 + c_2X_5,\\ &4)\ X_2 + c_1X_5, \end{aligned}$$

where $c_i \in \mathbb{R}$ are arbitrary numbers for $i = 1, \cdots, 5$.

Proof. From Table (1), it is clear that the center of Lie algebra is $\langle X_2, X_5\rangle$. Hence, it would be sufficient to determine the sub-algebras of

$$\langle X_1, X_3, X_4\rangle.$$

For $t = 1, \cdots, 5$, the map:

$$\begin{cases} F_t^s : \mathfrak{g} \to \mathfrak{g}\\ X \mapsto \mathrm{Ad}(\exp(sX_t).X) \end{cases}$$

is a linear function. Considering basis $\{X_1, \cdots, X_5\}$, the matrixes M_t^s of F_t^s, $t = 1, \cdots 5$ are given by:

$$M_1^s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & cos(s_1) & -sin(s_1) & 0\\ 0 & 0 & sin(s_1) & cos(s_1) & 0\\ 0 & 0 & -s & 0 & 1 \end{bmatrix} \quad M_2^s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$M_3^s = \begin{bmatrix} cos(s_2) & 0 & 0 & sin(s_2) & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ -sin(s_2) & 0 & 0 & cos(s_2) & 0\\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad M_4^s = \begin{bmatrix} cos(s_4) & 0 & -sin(s_4) & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ sin(s_4) & 0 & cos(s_4) & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$M_5^s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

By applying these matrixes on a vector field $X = \sum_{i=1}^{5} a_iX_i$ alternatively, we can simplify X as follows:

For $a_1 \neq 0$, the coefficients of X_3 and X_4 can be disappeared by setting $s_4 = -tan^{-1}(a_3/a_1)$ and $s_3 = tan^{-1}(a_4/a_1)$ respectively. If needed, by scaling X, we suppose $a_1 = 1$. Thus, X turns into (1).

For $a_1 = 0$ and $a_3 \neq 0$, the coefficients of X_4 can be disappeared by setting $s_1 = -tan^{-1}(a_4/a_3)$. If needed, by scaling X, we suppose $a_3 = 1$. Thus, X turns into (2).

For $a_1 = a_3 = 0$ and $a_4 \neq 0$, if needed, by scaling X, we suppose $a_4 = 1$. Thus, X turns into (3).

Table 3. Lie invariants and similarity solution.

i	H_i	ξ_i	η_i	w_i	u_i
1	X_1	x	t	u	$h(\xi,\eta)$
2	X_2	x	y	u	$h(\xi,\eta)$
3	X_1+cX_2	x	$y-\frac{t}{c}$	u	$h(\xi,\eta)$
4	X_1+cX_5	x	t	$lnu-cy$	$e^{cy+h(\xi,\eta)}$
5	$X_1+c_1X_2+c_2X_5$	x	$y-\frac{t}{c_1}$	$lnu-\frac{c_2}{c_1}t$	$e^{\frac{c_2}{c_1}t+h(\xi,\eta)}$
6	$c_1X_2+c_2X_5$	x	y	$lnu-\frac{c_2}{c_1}t$	$e^{\frac{c_2}{c_1}t+h(\xi,\eta)}$

Table 4. Reduced equations regarding infinitesimal symmetries.

i	Reduction of equations
1	$\kappa h_{\xi\xi}+\kappa(cot\xi)h_\xi-h_\eta=0$
2	$\kappa h_{\xi\xi}+\kappa(cot\xi)h_\xi+\kappa(csc^2\xi)h_{\eta\eta}=0$
3	$\kappa h_{\xi\xi}+\kappa(cot\xi)h_\xi+\kappa(csc^2\xi)h_{\eta\eta}+\frac{1}{a}h_\eta=0$
4	$\kappa h_{\xi\xi}+\kappa h_\xi^2+\kappa(cot\xi)h_\xi+c^2\kappa(csc^2\xi)-h_\eta=0$
5	$\kappa h_{\xi\xi}+\kappa h_\xi^2+\kappa(cot\xi)h_\xi+\kappa(csc^2\xi)(h_{\eta\eta}+h_\eta^2)+\frac{1}{c_1}h_\eta-c_2=0$
6	$\kappa h_{\xi\xi}+\kappa h_\xi^2+\kappa(cot\xi)h_\xi+\kappa(csc^2\xi)(h_{\eta\eta}+h_\eta^2)-\frac{c_2}{c_1}=0$

For $a_1=a_3=0$ and $a_4=0$, X turns into (4). □

4. Ricci soliton with one-dimensional optimal system

We are now reporting some essential concepts on Ricci solitons [6]. The Ricci soliton spaces are an inherent generalization of Einstein field spaces. According to Theorem 6.1, we characterize the vector fields which are satisfied with Equation (1.1) in the following theorem.

Theorem 4.1. *For one-dimensional optimal systems of Eq.(1.4), $(S^2\times R, g)$ is expanding Ricci soliton as follows:*

1) $X_1+\Phi X_2$,
2) $X_3+\Phi X_2$,
3) $X_4+\Phi X_2$,
4) ΦX_2,

where $\Phi=-\frac{1}{2}t+C_1$ for all four cases.

Proof. The above statement is obtained from a case-by-case argument. For instance, we detail the computations for case (1). Let g be the metric which is described by the Eq. (1.3). We apply $\{\partial_i=\frac{\partial}{\partial x^i}: i=1...4\}$ for local basis of the tangent space. Considering

$$\begin{aligned}2g(\nabla_UV,Z)&=Xg(V,Z)+Yg(Z,U)-Zg(U,V)\\&\quad-g(X,[V,Z])+g(Y,[Z,U])+g(Z,[U,V]),\end{aligned}\tag{4.1}$$

the non-vanishing elements of the connection can be obtained:

$$\begin{aligned}&\nabla_{\partial_1}\partial_2=cot(x)\partial_2,\\&\nabla_{\partial_2}\partial_2=-sin(x)cos(x)\partial_1,\end{aligned}\tag{4.2}$$

The tensor of curvature is determined by using $R(U,V) = [\nabla_U, \nabla_V] - \nabla_{[U,V]}$. The non-vanishing elements of R are determined by relations:

$$\begin{aligned} &R(\partial_1, \partial_2) = sin^2(x)\partial_1 dy, \\ &R(\partial_2, \partial_1) = \partial_2 dx, \end{aligned} \tag{4.3}$$

Putting $R(\partial_k, \partial_l)\partial_j = R^i_{jkl}\partial_i$, we can create the Ricci tensor ϱ by contracting on the second and third indices of R. The Ricci tensor matrix is as follows;

$$\varrho(\partial_i, \partial_j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & sin^2(x) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.4}$$

From (4.4), it is not hard to see that $(S^2 \times R, g)$ is not Einstein. For the four cases in Theorem 6.1, we separately assume that:

$$\begin{aligned} &1)\ X^1 = X_1 + \Phi X_2 + \Psi X_5, \\ &2)\ X^2 = X_3 + \Phi X_2 + \Psi X_5, \\ &3)\ X^3 = X_4 + \Phi X_2 + \Psi X_5, \\ &4)\ X^4 = \Phi X_2 + \Psi X_5, \end{aligned}$$

where Φ and Ψ are functions. We consider $X^1 = X_1 + \Phi X_2 + \Psi X_5$, for the other cases we have the same result. Because the meter g is in three-dimensional mode, the Lie derivative of g with respect to a four-dimensional vector field X is impossible, so we assume $\Psi = 0$, although if we generalize the meter eventually again $\Psi = 0$. The Lie derivative of g with respect to $X = X_1 + \Phi X_2$ is:

$$\mathcal{L}_X g = \partial_1 \Phi dxdt + \partial_2 \Phi dydt + 2\partial_3 \Phi (dt)^2$$

Applying Eqs. (1.3) and (4.4) in the Eq. (1.1), we earn the following set of PDEs:

$$\begin{cases} (\lambda + 1) sin^2(x) = 0, \\ 2\partial_3 \Phi - \lambda = 0, \\ \lambda + 1 = 0, \\ \partial_1 \Phi = 0, \\ \partial_2 \Phi = 0, \end{cases}$$

which admits the following solution

$$\begin{cases} \lambda = -1, \\ \Phi = -\dfrac{1}{2} t + C_1, \end{cases}$$

for a real constant C_1.

□

For example, the flow of vector field X^1 is shown by

$$\Phi_\epsilon = (x, y + \epsilon, 2 + e^{-\frac{1}{2}\epsilon}(t - 2)).$$

The flow Φ_ϵ is plotted in Figure 1.

5. Similarity reduction of Equation (1.4)

Here, we want to classify symmetry reduction of Eq.(1.4) concerning subalgebras of Theorem 6.1. We need to search for a new form of Eq. (1.4) in specific coordinates so that it would reduce. Such a coordinate will be constructed by finding independent invariant ξ, η, k, h regarding the infinitesimal generator. So, expressing the equation in new coordinates applying the chain rule reduces the system. For 1-dimensional subalgebras in the Theorem 6.1 the similarity variables ξ_i, η_i, k_i and h_i are listed in Table 3. Each similarity variable is applied to find the reduced PDE of Eq.(1.4) which, they are listed in Table 4.

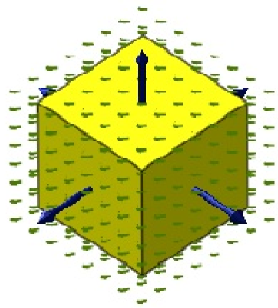

FIGURE 1. The plot of flow Φ_ϵ.

For instance, we compute the invariants associated with subalgebra $H_5 := X_1 + c_1X_2 + c_2X_5$ by integrating the following characteristic equation.

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dt}{c_1} = \frac{du}{c_2 u}.$$

Hence, the similarity variables would be:

$$\xi = x, \qquad \eta = lnu - \frac{t}{c_1}, \qquad h = lnu - \frac{c_2}{c_1}t,$$

Substituting the similarity variables in Eq.(1.4) and applying the chain rule it results that, the solution of Eq.(1.4) is:

$$u = e^{\frac{c_2}{c_1}t + h(\xi,\eta)}$$

where $h(\xi, \eta)$ satisfies a reduced PDE with two variables as follows:

$$-\frac{1}{c_1}h_\eta + c_2 = \kappa h_{\xi\xi} + \kappa h_\xi^2 + \kappa cot\xi h_\xi + \kappa csc^2\xi(h_{\eta\eta} + h_\eta^2). \tag{5.1}$$

Subalgebra $X_1 + c_1X_2 + c_2X_5$ and the reduced Eq. (5.1) are shown in Tables 3 and 4, by the case (5).

6. The Diffusion solutions on the sphere

Consider the reduced equations cases (1) and (2) and thier 2 independent variables in Tables 3 and 4:

$$\kappa h_{\xi\xi} + \kappa(cot\xi)h_\xi - h_\eta = 0,$$
$$\kappa h_{\xi\xi} + \kappa(cot\xi)h_\xi + \kappa(csc^2\xi)h_{\eta\eta} = 0,$$

We rewrite these Equations by substituting u and the variables x, y and y as follows:

$$\kappa u_{xx} + \kappa(cotx)u_x - u_t = 0, \tag{6.1}$$

$$\kappa u_{xx} + \kappa(cotx)u_x + \kappa(csc^2x)u_{yy} = 0. \tag{6.2}$$

In the Eq. (6.1), $u = u(x, t)$ and in the Eq. (6.2), $u = u(x, y)$. Compering Equation (1.4) with these two equations, we find out the term $\kappa(csc^2x)u_{yy}$ in the Eq. (6.1) vanished, namely $u_{yy} = 0$ and u_t in the Eq. (6.2) vanished, namely

$u_t = 0$. Note that, u is independent of y in the Eq. (6.1) and independent of t in the Eq. (6.2). Therefore in the Eq. (1.4), we can consider $u = u(x, y, t)$ as $u = f(x)g(y)h(t)$ where

$$\begin{cases} h_t = c_1 h \\ g_{yy} = c_2 g \\ f_{xx} = (cotx)f_x - c_2(csc^2x)f + \dfrac{c_1}{\kappa}f. \end{cases} \tag{6.3}$$

Thus by solving these three equations, we have

$$\begin{cases} h(t) = C_1 e^{c_1 t}, \\ g(y) = C_2 e^{\sqrt{c_2}y} + C_3 e^{-\sqrt{c_2}y}, \end{cases} \tag{6.4}$$

which the third equation of (6.3) is a Legendre's differential equation. Hence we discuss f.

(1) If $c_1 = 0$, then

$$\begin{aligned} f(x) =& C_4 sin\left(\sqrt{c_2} arctanh(\frac{1}{cos(x)})\right) \\ &+ C_5 cos\left(\sqrt{c_2} arctanh(\frac{1}{cos(x)})\right), \end{aligned} \tag{6.5}$$

Which f includes complex values.

(2) If $c_2 = 0$, then the Eq. (6.3)III turns into a second-order linear ODE:

$$\begin{aligned} f(x) = & C_6 \textbf{hypergeom}\left([\frac{\sqrt{\kappa} + \sqrt{-4c_1 + \kappa}}{4\sqrt{\kappa}}, \right. \\ & \left. \frac{\sqrt{\kappa} - \sqrt{-4c_1 + \kappa}}{4\sqrt{\kappa}}], [\frac{1}{2}], cos^2(x) \right) \\ & + C_7 cos(x) \textbf{hypergeom}\left([\frac{3\sqrt{\kappa} - \sqrt{-4c_1 + \kappa}}{4\sqrt{\kappa}}, \right. \\ & \left. \frac{\sqrt{3\kappa} + \sqrt{-4c_1 + \kappa}}{4\sqrt{\kappa}}], [\frac{1}{2}], cos^2(x) \right), \end{aligned} \tag{6.6}$$

which in this case, for proper c_1 and κ, f is a real function.

(3) If $c_1 \neq 0$ and $c_2 \neq 0$, then

$$\begin{aligned} f(x) = & C_8 \textbf{LegendreP}\left(\frac{-\sqrt{\kappa} + \sqrt{-4c_1 + \kappa}}{2\sqrt{\kappa}}, i\sqrt{c_2}, cos(x) \right) \\ & + C_9 cos(x) \textbf{LegendreQ}\left(\frac{-\sqrt{\kappa} + \sqrt{-4c_1 + \kappa}}{2\sqrt{\kappa}}, i\sqrt{c_2}, cos(x) \right), \end{aligned} \tag{6.7}$$

Which f includes complex values.

LegendreP and **LegendreQ** are Legendre functions of the first and second types, respectively, a Maple method for expressing the solutions of Legendre equation. The Legendre equation can be turned into a hypergeometric differential equation by changing variables, and its solutions can be found applying **hypergeometric** functions. We always deal with functions with real value. Therefore we have to select c_2 so that it allows us to find solutions with real value. If c_2 is chosen as a zero, we are able to select the real function (6.6). According to Eqs. (6.4), (6.5), (6.6), and (6.7), we earn the result:

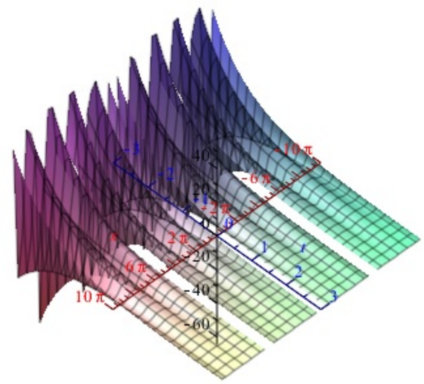

FIGURE 2. The plot of u .

Theorem 6.1. *Solutions of* (1.4) *are:*

$$
\begin{aligned}
&u = u(x,y,t) = \\
&f(x)g(y)h(t) = (C_6\mathbf{hypergeom}\Big([\frac{\sqrt{\kappa}+\sqrt{-4c_1+\kappa}}{4\sqrt{\kappa}}, \\
&\qquad \frac{\sqrt{\kappa}-\sqrt{-4c_1+\kappa}}{4\sqrt{\kappa}}],[\frac{1}{2}],cos^2(x)\Big) \\
&+C_7cos(x)\mathbf{hypergeom}\Big([\frac{3\sqrt{\kappa}+\sqrt{-4c_1+\kappa}}{4\sqrt{\kappa}}, \\
&\qquad \frac{3\sqrt{\kappa}-\sqrt{-4c_1-\kappa}}{4\sqrt{\kappa}}],[\frac{3}{2}],cos^2(x)\Big)) \\
&\qquad C_1e^{c_1t}(C_2e^{\sqrt{c_2}y}+C_3e^{-\sqrt{c_2}y}),
\end{aligned}
$$

where $c_1, c_2, C_1, C_2, C_3, C_6, C_7 \in \mathbb{R}$.

For $c_2 = 0, c_1 = -1, \kappa = C_1 = C_2 = C_3 = C_6 = C_7 = 1$ we have

$$
\begin{aligned}
u = 2e^{-t}(&\mathbf{hypergeom}\Big([\frac{1+\sqrt{5}}{4}, \\
&\qquad \frac{1-\sqrt{5}}{4}],[\frac{1}{2}],cos^2(x)\Big) \\
&+cos(x)\mathbf{hypergeom}\Big([\frac{3+\sqrt{5}}{4}, \\
&\qquad \frac{3-\sqrt{5}}{4}],[\frac{3}{2}],cos^2(x)\Big)),
\end{aligned}
$$

which is plotted in Figure 2.

7. Conclusion

In this paper, by applying the method of Lie symmetries, we find the Lie point symmetries group of Eq. (1.4). Also, we have obtained the one-parameter optimal system of sub-algebras for Eq. (1.4). We provide an initial classification of group invariant solutions. The Lie invariants and similarity reduced equations corresponding to infinitesimal symmetries are obtained.

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