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Shifted Jacobi collocation method for Volterra-Fredholm integral equation

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Abstract

In this paper, we compute the approximate numerical solution for the Volterra-Fredholm integral equation (V-FIE) by using the shifted Jacobi collocation (SJC) method which depends on the operational matrices. Some properties of the shifted Jacobi polynomials are introduced. These properties allow us to transform the Volterra-Fredholm integral equation into a system of algebraic equations in a nice form with the expansion coefficients of the solution. Also, the convergence and error analysis are studied extensively. Finally, some examples which verify the efficiency of the given method are supplied and compared with other methods.

Keywords. Shifted Jacobi polynomials; Collocation method; Volterra-Fredholm integral equation; Convergence and error analysis.
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1. Introduction

Of course, it is not an easy target to compute an exact solution of a wide class of differential equations of fractional order. However, in the past years; many numerical techniques have been introduced to find an approximate solution of this class of fractional models of differential equations, such as variational iteration method for fractional partial differential equations, the time-fractional Fornberg, and fractional initial-boundary value problems [9, 28, 46]. Finite difference method for fractional sub-diffusion equations [20], finite element method for the space and time fractional Fokker-Planck equation [12], A domain decomposition method [25, 39, 40]. The well-known methods are called the spectral methods in which the solution can be expressed as the expansion of polynomials, and one of special interest is the solution of space fractional diffusion equation using Jacobi operational matrix [15]. The shifted Jacobi method is used in solving multidimensional time-fractional order telegraph equation, linear multi-term fractional differential equations, and variable-order fractional reaction-subdiffusion equation [14, 16, 21, 23]. The solution of hyperbolic partial differential equations by using the exponential Jacobi method can be found in [49]. In literature, there are many papers that deal with the spectral methods for solving differential equations, such as shifted Legendre polynomial for variable-order fractional functional differential equations [22], solutions of third and fifth-order differential equations by using Petro-Galerkin methods [1]. The most used spectral methods are the Galerkin, Collocation, and Tau methods. These are used in many articles such as shifted Chebyshev polynomials of the first kind in time and the Sinc function in space, and solutions of time-fractional Telegraph equation by using Legendre-Galerkin algorithm [41, 47], solution for telegraph equation of space fractional order by using Legendre Wavelets spectral tau algorithm [31]. Use sinc-Legendre collocation method for solving diffusion equations with distributed-order in time [32]. Solve hyperbolic partial differential equations using Shifted Jacobi Galerkin method [17], solution of eigenvalue problems using tau method [37], solution of the space fractional advection-dispersion problem using tau-Jacobi algorithm [29]. [8] solving a general class of multi-order fractional pantograph equations using Galerkin schemes, solving Riemann-Liouville and Riesz fractional advection-dispersion problems using shifted Chebyshev method [13], solving linear and nonlinear fractional-order differential equations using shifted fifth-kind Chebyshev polynomials [3], solving multi-term fractional differential

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equations and a system of high-order linear differential equations with variable coefficients using Lucas tau method [6, 33], solving multi-term fractional differential equations using generalized Lucas tau method [4, 30], solutions for the connection problems between generalized Lucas polynomial sequence, third and fourth kinds of Chebyshev polynomials [2], solutions for a certain coupled systems of fractional differential equations using the generalized Fibonacci tau method [7].

Spectral methods can also solve the integral equations such as solving Volterra integral equations and system of Volterra integral equations using Legendre collocation method and Taylor-collocation method [34, 36, 43]. Solving Volterra-Fredholm integral equations using Block pulse functions [10], expansion method [11], Legendre collocation method [19, 35], shifted Legendre and shifted Chebyshev polynomials [24], shifted Chebyshev collocation method [48], the second kind Chebyshev polynomials [5], Lagrange collocation method [44]. We compared the obtained results with the Lagrange collocation (LC) method [44], the Taylor collocation (TC) method [45], and the shifted Chebyshev collocation (SCC) method [48]. To the best of my knowledge, our work in this paper is the first to use the shifted Jacobi collocation method for solving Volterra-Fredholm integral equations. This method is certainly will provide highly accurate results but unfortunately takes a longer times. The paper consists of six sections and is organized as follows: the first section contains a brief history of the subject of our work. Section 2 deals with the basic definitions and properties of the shifted Jacobi polynomials and the V-FIE, which will be used in the following sections. Section 3 explores the algorithm of the method for solving Volterra-Fredholm integral equation. Section 4 is devoted to the study of convergence and error analysis. Some numerical examples with remarks are given in section 5. Finally, section 6 contains the conclusion, and we end up with the list of used references.

2. Basic properties

In this section, we present some properties of the shifted Jacobi polynomials [26, 27, 38] and the V-FIE [48].

The orthogonality of the shifted Jacobi polynomials with the weight function $\omega_L^{(a,b)}(y) = (L-y)^a y^b$, over $[0, L]$ is,

$$\int_0^L P_{L,i}^{(a,b)}(y)\; P_{L,k}^{(a,b)}(y)\; \omega_L^{(a,b)}(y)\; dy = \; g_{L,k}^{(a,b)}, \tag{2.1}$$

where

$$g_{L,k}^{(a,b)} \;= \frac{L^{a+b+1}\Gamma\,(k+a+1)\,\Gamma\,(k+b+1)}{(2k+a+b+1)\;k!\;\Gamma\,(k+a+b+1)}.$$

The shifted Jacobi polynomials have the form

$$P_{L,k}^{(a,b)}(y) = \sum_{i=0}^{k} \frac{(-1)^{k-i}\;\Gamma\,(k+b+1)\;\;\Gamma\,(k+i+a+b+1)}{\Gamma\,(i+b+1)\;\;\Gamma\,(k+a+b+1)\;\;(k-i)!\;i!\;L^i} \tag{2.2}$$

where

$$P_{L,k}^{(a,b)}(0) = \frac{(-1)^k\Gamma\,(k+b+1)}{\Gamma\,(b+1)\;\;k!}, \quad P_{L,k}^{(a,b)}(L) = \frac{\Gamma\,(k+a+1)}{\Gamma\,(a+1)\;\;k!}.$$

The function $W(y)$ can be expanded as terms of shifted Jacobi polynomials

$$W(y) \;= \sum_{k=0}^{\infty} c_k\; P_{L,k}^{(a,b)}(y), \tag{2.3}$$

so

$$c_k = \frac{1}{g_{L,k}^{(a,b)}} \int_0^L W(y)\; P_{L,k}^{(a,b)}(y)\; \omega_L^{(a,b)}(y)\; dy. \quad k = 0, 1, 2, ... \tag{2.4}$$

suppose that we approximate $W(y)$ by using only the first $(M+1)$ terms

$$W(y) \approx W_M(y) \ = \sum_{k=0}^{M} c_k \ P_{L,k}^{(a,b)}(y) \ = C^T \ \psi_{L,M}(y), \tag{2.5}$$

where the coefficients

$$C^T = [c_0, c_1, ..., c_M],$$

are unknowns and must be determined, while

$$\psi_{L,M}(y) \ = \left[P_{L,0}^{(a,b)}(y), P_{L,1}^{(a,b)}(y), ..., P_{L,M}^{(a,b)}(y)\right]^T.$$

Consider the Volterra-Fredholm integral equation [48]:

$$A(y) \ W(y) + B(y) \ W(E(y)) = h(y) + \alpha_1 \int_0^{E(y)} d_1(y,t) \ W(t) \ dt + \alpha_2 \int_0^{\ell} d_2(y,t) \ W(E(t)) \ dt, \tag{2.6}$$

where $W(y)$ is an unknown function. $A(y)$, $B(y)$, $E(y)$ and $h(y)$, are known and defined on the interval $[0,\ell], 0 \leq E(y) < \infty$. $d_1(y,t)$ and $d_2(y,t)$ are known kernel functions on $[0,\ell] \times [0,\ell]$. α_1 and α_2 are real constants such that $\alpha_1^2 + \alpha_2^2 \neq 0$.

3. The algorithm of the shifted Jacobi collocation method

In this section, we approximate the solution of Equation (2.6) using the shifted Jacobi polynomials. For this proposal, let $0 \leq E(y) < \ell$. And from the approximation (2.5), we have

$$W(E(y)) \approx \sum_{k=0}^{M} c_k \ P_{L,k}^{(a,b)}(E(y)). \tag{3.1}$$

Using Equations (2.5) and (3.1), Equation (2.6) becomes

$$\begin{aligned} A(y) \sum_{k=0}^{M} c_k \ \ P_{L,k}^{(a,b)}(y) + B(y) \sum_{k=0}^{M} c_k \ \ P_{L,k}^{(a,b)}(E(y)) = h(y) + \alpha_1 \int_0^{E(y)} d_1(y,t) \sum_{k=0}^{M} c_k \ P_{L,k}^{(a,b)}(t)\, dt \\ + \alpha_2 \int_0^{\ell} d_2(y,t) \sum_{k=0}^{M} c_k \ P_{L,k}^{(a,b)}(E(t))\, dt. \end{aligned} \tag{3.2}$$

Let

$$f_k(y) = A(y) \ P_{L,k}^{(a,b)}(y) + B(y) \ P_{L,k}^{(a,b)}(E(y)) - \alpha_1 \int_0^{E(y)} d_1(y,t) \ P_{L,k}^{(a,b)}(t) \ \ dt \ - \alpha_2 \int_0^{\ell} d_2(y,t) \ P_{L,k}^{(a,b)}(E(t)) \ \ dt.$$

Then Eqaution (3.2) can be written as

$$\sum_{k=0}^{M} c_k \ f_k(y) = h(y). \tag{3.3}$$

Obviously Equation (3.3) has $M+1$ roots; consequently, we have a system of equations

$$\sum_{k=0}^{M} c_k \ f_k(y_i) = h(y_i), \ i = 0, 1, ..., M. \tag{3.4}$$

So the matrix form of Equation (3.4) will be

$$F^T \ C = H,$$

where

$$F = (f_{ki}), \quad i, k = 0, 1, ...M,$$

and

$$H = [h(y_0), h(y_0), ..., h(y_M)]^T.$$

Now, we can determine the unknown constants by the following equation

$$C = (F^T)^{-1} H.$$

4. Convergence and error analysis

In this section, we investigate the convergence and error analysis of the shifted Jacobi polynomials for V-FIE. The following Lemmas and theorems will be used in the sequence:

Lemma 1. *i) If $a, b > -1$ then $\int\limits_0^1 (1-y)^a \, P_k^{(a,b)}(y) dy = O\left(\frac{1}{\sqrt{k}}\right)$,*

ii) If $a, b > -1$ then $\left|P_k^{(a,b)}(y)\right| = O\left(k^r\right)$, $r = \max(a, b, \frac{-1}{2})$.

Proof. See SzegŌ (1937), p. 163 [42] □

Lemma 2. $\Gamma\left(n + \lambda\right) \; = O\left(n^{\lambda - 1} \; n!\right)$.

Proof. See Rainville (1971)[38] □

Theorem 4.1. *If $\xi_k = \frac{(2k+a+b+2)(2k+a+b++3)}{(k+a+1)(k+b+1)}$ then $\phi_k(y) = \frac{\xi_k}{L^2}(L-y) P_{L,k}^{(a+1,b+1)}(y)$.*

Proof. See Doha (2004)[18] □

Theorem 4.2. *The following orthogonality is valid:*

$$\int\limits_0^L \phi_k(y) \; \phi_i(y) \widetilde{\omega}^{(a,b)}(y) \; dy = \delta_{ik} \; R_k,$$

where $\widetilde{\omega}^{(a,b)}(y) \; = y^{b-1} \left(L - y\right)^{a-1}$, $R_k = \xi_k^2 \, \frac{L^{a+b-1} \Gamma(k+a+2) \Gamma(k+b+2)}{(2k+a+b+3) \; k! \; \Gamma(k+a+b+3)}$.

Proof. See R. Hafez and Y. Youssri (2018)[21] □

Now we can prove the following:

Theorem 4.3. *If $W(y) = y \; (L-y) \; v(y)$ and $\left|v^{'''}(y)\right| \leq e$, then $|c_k| = O\left(k^{\frac{-5}{2}}\right)$ $\forall k > 3$.*

Proof. Using Equation (2.5) and Theorem (4.1), one can have:

$$c_k = \frac{1}{R_k} \int\limits_0^L W(y) \; \phi_k(y) \; \widetilde{\omega}^{(a,b)}(y) \; dy.$$

By the given assumptions, it is not hard to prove that

$$c_k = \frac{\xi_k}{L^2 R_k} \int\limits_0^L y^{b+1} \left(L - y\right)^{a+1} \; v(y) \; P_{L,k}^{(a+1,b+1)}(y) \, dy.$$

Then by integrating three times, we have

$$|c_k| = O\left(\left|\frac{\xi_k}{L^2 R_k \; k^3} \int\limits_0^L y^{b+4} \left(L-y\right)^{a+4} \; v^{'''}(y) \; P_{L,k-3}^{(a+4,b+4)}(y) \, dy\right|\right),$$

and from Theorem (4.1), we see

$$\frac{\xi_k}{R_k} = \frac{k!\,\Gamma\left(k+a+b+3\right)}{L^{a+b-1}\Gamma\left(k+a+1\right)\Gamma\left(k+b+1\right)\ \left(2k+a+b+2\right)}.$$

Therefore using Lemmas (1) and (2) the proof is completed ☐

Theorem 4.4. *The estimation*

$$\left|c_k\ \phi_k(y)\right| = O\left(k^{b-\frac{3}{2}}\right).$$

holds

Proof. From Theorem (4.3), one we have

$$\left|c_k\ \phi_k(y)\right| = \left|\left(O\left(k^{b-\frac{3}{2}}\right)\right) y\ \ (L-y)\, P_{L,k}^{(a+1,b+1)}\left(y\right)\right|.$$

From Lemma (1) (ii) and by supposing $a<b$, the estimation holds. ☐

Furthermore, to deal with the global error we suppose that $\varepsilon_M(y) = \left|W(y) - W_M(y)\right|$, $\varepsilon_M^E(y) = \varepsilon_M(E(y))$, $\epsilon_M = \max\limits_{0\le y\le \ell} \varepsilon_M(y)$ and $\epsilon_M^E = \max\limits_{0\le y\le \ell} \varepsilon_M^E(y)$.

Theorem 4.5. *Let*

$$T_M(y) = \mid A(y)\ W_M(y) + B(y)\ W_M(E(y)) - \alpha_1 \int_0^{E(y)} d_1(y,t)\ W_M(t)\ dt - \alpha_2 \int_0^{\ell} d_2(y,t)\ W_M(E(t))\ dt\ \ - h(y) \mid,$$

$$\check{\mathrm{T}}_M = \max_{0\le y\le \ell} T_M(y),$$

and if $|A(y)| \le A_1, |B(y)| \le B_1, |d_1(y,t)| \le D_1, |d_2(y,t)| \le D_2$ and $|E(y)| \le q$, where A_1, B_1, D_1, D_2, and q are positive constants. Then we have the following global error:

$$\left|\check{\mathrm{T}}_M\right| = O\left(k^{b-\frac{3}{2}}\right),$$

where

$$\rho = \max\left\{A_1, B_1, |\alpha_1|\ \ D_1 q, |\alpha_2|\ \ D_2 \ell\right\}.$$

Proof. From Equation (2.6), we have

$$h(y) = A(y)\ W(y) + B(y)\ W(E(y)) - \alpha_1 \int_0^{E(y)} d_1(y,t)\ W(t)\ dt - \alpha_2 \int_0^{\ell} d_2(y,t)\ W(E(t))\ dt.$$

So

$$T_M(y) \le \left|A(y)\ \varepsilon_M(y)\right| + \left|B(y)\ \varepsilon_M^E(y)\right| + \left|\alpha_1 \int_0^{E(y)} d_1(y,t)\ \varepsilon_M(t) dt\right| + \left|\alpha_2 \int_0^{\ell} d_2(y,t)\ \varepsilon_M^E(t)\ dt\ \right|,$$

$$\check{\mathrm{T}}_M \le \left\{A_1 + B_1 + |\alpha_1|\ \ D_1 q + |\alpha_2|\ \ D_2 \ell\right\}\ \ \max\left(\varepsilon_M(y), \varepsilon_M^E(y)\right).$$

Consequently, from Theorem (4.4), the theorem is proved. ☐

5. Numerical examples

In this section, we will borrow the examples given in [44, 45, 48] and solve them by using the method suggested in the previous sections i.e. SJC. Then we show the effectiveness of our method compared with the others.

Example 1. *Let us consider the following V-FIE [44, 45] and [48]*

$$(\sin y)\, W(y) + (\cos y)\, W(e^y) = h(y) + \int_0^{e^y} e^{y+t} W(t)dt - \int_0^1 e^{y+t} W(e^t)dt. \tag{5.1}$$

The exact solution of this equation is $W(y) = y^2$, where

$$h(y) = \frac{1}{3} e^y \left(-1 + e^3\right) + e^y \left\{2 - e^{e^y} \left[2 + e^y \left(-2 + e^y\right)\right]\right\} + e^{2y} \cos y + y^2 \sin y.$$

Table 1 shows the comparison between the absolute errors of our suggested method with the Lagrange collocation (LC) method [44], the Taylor collocation (TC) method [45], and the shifted Chebyshev collocation (SCC) method [48]. Notice that the last two columns represent the time used for the running program (CPU time) and the difference between two consecutive errors (C_N).

Table 1. Comparison between absolute errors with different values of N.

N	SJC	LC[44]	TC[45]	SCC[48]	CPU time	C_N
2	1.6×10^{-15}	2.8×10^{-15}	7.6×10^{-15}	5×10^{-16}	21.234	9.4×10^{-15}
3	1.1×10^{-14}	1.4×10^{-14}	1.2×10^{-14}	1.4×10^{-15}	43.688	6.1×10^{-15}
4	4.9×10^{-15}	1.9×10^{-13}	3.4×10^{-14}	3.1×10^{-15}	101.672	4.9×10^{-15}

Example 2. *Let us consider the following V-FIE [44, 45] and [48]*

$$y^2 W(y) + e^y W(2y) = h(y) + \int_0^{2y} e^{y+t} W(t)dt - \int_0^1 e^{y-2t} W(2t)dt. \tag{5.2}$$

The exact solution of this Equation is $W(y) = \sin y$, where

$$h(y) = -\frac{1}{4} e^y - \frac{1}{4} e^{-2+y} \cos 2 + \frac{1}{2} e^{3y} \cos 2y - \frac{1}{4} e^{-2+y} \sin 2 + y^2 \sin y + e^y \sin 2y - \frac{1}{2} e^{3y} \sin 2y.$$

Table 2 compares our results with the others. Note that the absolute error of the proposed method is better than the others for small values N. The errors of this method are displayed at $N = 2$, 5, 8, and 9 in Figure 1. It is clear from this Figure that the absolute errors decrease drastically with decreasing the number of steps.

TABLE 2. Maximum absolute errors with various values of E.

N	SJC	LC[44]	TC[45]	SCC[48]
2	2×10^{-2}	7.9×10^{-2}	7.9×10^{-2}	3.4×10^{-2}
5	4.9×10^{-5}	6.2×10^{-5}	6.2×10^{-5}	5.5×10^{-5}
8	4.2×10^{-8}	1.8×10^{-7}	1.9×10^{-8}	8×10^{-9}
9	7.3×10^{-9}	7.2×10^{-6}	2.4×10^{-8}	5.5×10^{-10}

CPU time	C_N
22.015	2×10^{-2}
150.515	4.9×10^{-5}
208.781	3.5×10^{-8}
238.234	2.2×10^{-8}

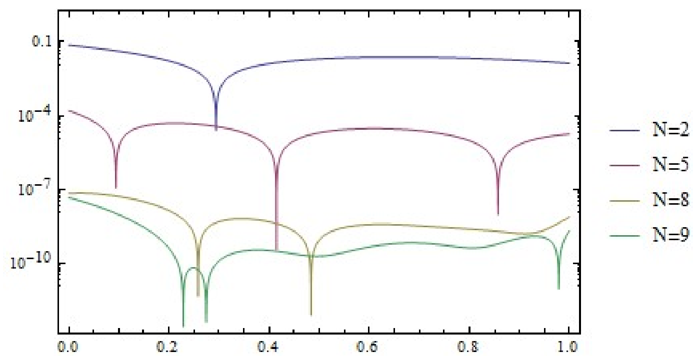

FIGURE 1. Graph of the error at N=2, 5, 8 and 9.

Example 3. *Let us consider the following V-FIE*

$$W(y) = h(y) + \int_0^y y\ tW(t)dt + \int_0^1 (y-t)\, W(t)dt. \tag{5.3}$$

The exact solution of this equation is $W(y) = y^{\frac{1}{2}}$, where

$$h(y) = \frac{-2}{5} y^{\frac{7}{2}} - \frac{2}{3} y + y^{\frac{1}{2}} + \frac{2}{5}.$$

Table 3 lists The numerical results obtained by the proposed method for N= 8, 12 and 16 for different values of a and b. The absolute errors of this method are plotted in Figure 2. We observe from the Figure that the convergence is exponential.

TABLE 3. Results of absolute errors for various values of N, a and b.

a	b	N	E	N	E	N	E
1	1	8	3.4×10^{-3}	12	1.4×10^{-3}	16	7.7×10^{-4}
2	1		3.4×10^{-3}		1.4×10^{-3}		7.7×10^{-4}
$-\frac{1}{2}$	$-\frac{1}{2}$		3.4×10^{-3}		1.4×10^{-3}		7.7×10^{-4}

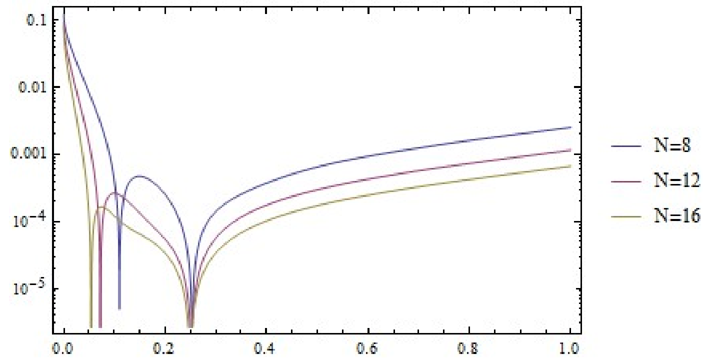

FIGURE 2. Graph of the absolute error at N=8, 12, 16 and different values of a and b.

Example 4. *Let us consider the following V-FIE [44, 45] and [48]*

$$W(y) = h(y) + \int_0^{\ln(y+1)} e^{y+t} W(t)dt - \int_0^1 e^{y+\ln(y+1)} W(\ln(t+1))dt. \tag{5.4}$$

The exact solution of this equation is $W(y) = e^{-y}$, where

$$h(y) = e^{-y} (\ln(y+1) - 1).$$

In Table 4, there is a comparison between the absolute errors of the present method with the Lagrange collocation (LC) method [44], the Taylor collocation (TC) method [45], and the shifted Chebyshev collocation (SCC) method [48]. In Figure 3, we illustrate the results of our suggested method at $N = 2$, 5, 8, and 9. The Figure shows that the convergence is exponential and the errors are better when the values of N are getting large.

TABLE 4. Comparison between the absolute errors with various values of N.

N	SJC	LC[44]	TC[45]	SCC[48]
2	1.3×10^{-2}	3.3×10^{-3}	3.3×10^{-3}	4.8×10^{-3}
5	3.5×10^{-7}	4.3×10^{-7}	4.3×10^{-7}	4.8×10^{-7}
8	1.1×10^{-11}	5.8×10^{-7}	6×10^{-8}	1.4×10^{-11}
9	2.8×10^{-13}	1.9×10^{-5}	8.8×10^{-8}	4.2×10^{-13}

CPU time	C_N
16.032	1.3×10^{-2}
197.5	3.5×10^{-7}
415.062	1.3×10^{-11}
512.736	3.2×10^{-8}

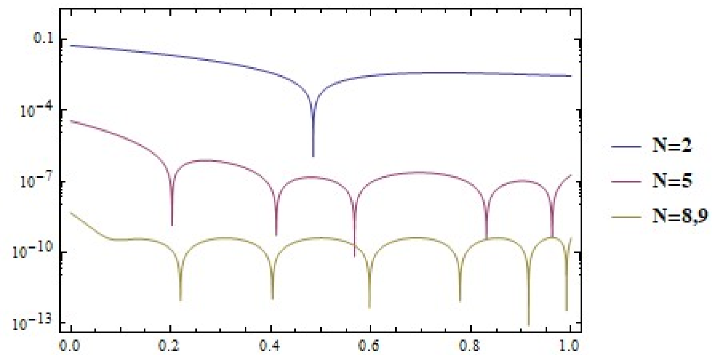

Figure 3. Graph of the absolute error at N=2, 5, 8 and 9

6. Conclusions

In this paper, the collocation method based on the shifted Jacobi polynomials is successfully implemented to compute numerical solutions for the V-FIEs. Four problems are examined to show the efficiency of this method by using Mathematica software. The analytical results have been given in terms of a system of linear algebraic equations. The proposed solutions declare better results outcome comparing with [44, 45] and [48]. Also, the spectral results of the proposed method are high adequacy, viable and easy to apply. Finally, the convergence and error analysis are given. And we conclude that the proposed method can be used to solve different types of fractional differential equations and integral equations.

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