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The convergence of exponential Euler method for weighted fractional stochastic equations

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Abstract

In this paper, we propose an exponential Euler method to approximate the solution of a stochastic functional differential equation driven by weighted fractional Brownian motion $B^{a,b}$ under some assumptions on a and b. We obtain also the convergence rate of the method to the true solution after proving an L^2-maximal bound for the stochastic integrals in this case.

Keywords. Malliavin calculus, Stochastic differential equations, Weighted fractional Brownian motion, Exponential Euler scheme.

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1. Introduction

Many researchers are interested in fractional Brownian motion (fBm) because of some compact properties like, stationary increments, long/short range dependence, self-similarity and Holder's continuity and also because of its applications in diverse scientific areas containing finance, telecommunications, image processing and turbulence. There are more informations about it in [4, 8, 9, 11, 12]. As models for different physical phenomena, long-range dependence (or long memory) stochastic processes with self-similarity have been intensively used. We look for information in work of Taqqu [18] for a guide on the appearance of the self-similarity in many applications and the monographs in [5, 14, 15], and by Sheluhin et al. [16] for complete expositions on self-similar processes. In the meantime, a type of generalization of fractional Brownian motion (fBm), that is, the weighted fractional Brownian motion (wfBm) can be also used for modeling. The weighted fractional Brownian motion in the rang of time fluctuations is a system of independent and symmetric particles that moving in $\mathbb{R}^d$.
We consider the following SDE:

$$\begin{aligned} &dx(t) = \left(Ax(t) + f(x(t))\right)dt + \sigma(t)dB^{a,b}(t),\ t \in [0,T],\\ &x(0) = \xi_0, \end{aligned} \tag{1.1}$$

where $B_t^{a,b}$ be a weighted fractional Brownian motion with parameters a, b satisfying $a > -1,\ |b| < 1,\ |b| < a+1$ and A is the generator of a strongly continuous analytic semigroup $S = S(t)_{t\geq 0}$ on a Banach space [19]. In order to get main results, it is necessary to put some restrictions on f and σ.

Assumption 1.1. For some positive constant L, K_1 and K_2, for every $x, y \in \mathbb{R}^n$ and $t, s \in [0,T]$.

$$\begin{aligned} |f(x) - f(y)| \leq L|x-y|, \quad |f(x)|^2 \leq K_1\left(1+|x|^2\right),\\ |\sigma(t) - \sigma(s)| \leq L|t-s|, \quad |\sigma(t)| \leq K_2, \end{aligned} \tag{1.2}$$

where σ is a deterministic function.

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Numerical methods for fractional brownian motion are applied extensively, whereas these methods are not considered vastly for weighted fractional Brownian motion. We are interested to use the exponential Euler scheme to approximate the solution of Eq.(1.1). It is worth menthion that in this case there is no estimation for stochastic integrals driven by weighted fractional Brownian motion, we done in section 3. We apply this powerful inequality to show the convergence of the method to the true solution in L^2 sense.
In section 4, we show that Eq. (1.1) has the exact solution as follows

$$x(t) = e^{At}\xi_0 + \int_0^t e^{A(t-s)} f(x(s))ds + \int_0^t e^{A(t-s)}\sigma(s)dB^{a,b}(s). \tag{1.3}$$

Next, we consider the exponential Euler method for Eq. (1.1). Given a step size $h > 0$, the exponential Euler approximate solution [10]

$$y_{k+1} = e^{Ah}\left(y_k + hf(y_k) + \sigma(t_k)\Delta B_k^{a,b}\right), \tag{1.4}$$

where y_k is an approximation to $x(t_k)$ with $t_k = kh$, for $k = 1, \cdots N$, also $\Delta B_k^{a,b} = B^{a,b}(t_{k+1}) - B^{a,b}(t_k)$ is the weighted fractional Brownian motion increment. It is convenient to use the continuous exponential Euler approximate solution and hence $y(t)$ is defined by

$$y(t) = e^{At}\xi_0 + \int_0^t e^{A(t-\bar{s})} f(z(s))ds + \int_0^t e^{A(t-\bar{s})}\sigma(\bar{s})dB^{a,b}(s), \tag{1.5}$$

in which $\bar{s} = [\frac{s}{h}]h$ in which $[x]$ denote the greatest integer less than or equal to x, and $z(t)$ is the step function which defined by [7]

$$z(t) = \sum_{k=0}^{n-1} I_{[t_k,t_{k+1})}(t)y_{t_k}, \tag{1.6}$$

in which I_C is the indicator function of the set C. Note that, for any integer is $k \geq 0$ we have $y(t_k) = z(t_k) = y_k$.
The organization of this paper is as follows. In section 2, we state the weighted fractional Brownian motion and Malliavin approach on it. In section 3, we prove some L^2-maximal bound for the stochastic integrals of weighted Brownian motion. In section 4, we first show the existence and uniquness of the solution of the SDE (1.1) and then the outcomes of the former sections are employed to prove the convergence rate of the method for the stochastic functional differential equations driven by weighted fractional Brownian motion. In section 5, we illustrate and justify our theoretical results by numerical examples. In section 6, conclusions are given.

2. Weighted fractional Brownian motion

Consider a weighted fractional Brownian motion $B^{a,b}$ with parameters a, b that $a > -1$, $|b| < 1$ and $|b| < a+1$ on the complete probability space (Ω, F, P) for $t \in [0, T]$.
This is a mean zero Guassian process and with simple covariance function [17]

$$R^{a,b}(t, s) = \mathbb{E}[B^{a,b}(t)B^{a,b}(s)] = \int_0^{s\wedge t} u^a[(t-u)^b + (s-u)^b]du, \ s, t \geq 0.$$

The process with weighted fractional Brownian motion (wfBm) is self-similar, long-range dependence with Hölder paths. Some surveys and references could be found in [2, 12].
Let ε be the space of indicator functions $\{\mathbf{1}_{[0,t]}, t \in [0, T]\}$ and H be the Hilbert space defined as the closure of the linear space ε with respect to the inner product $\left\langle \mathbf{1}_{[0,s]}\mathbf{1}_{[0,t]}\right\rangle_H = R^{a,b}(t, s)$. Then

$$\|u\|_H^2 = \int_0^T \int_0^T u(s)u(t)\phi(t, s)dtds,$$

where $\phi(t, s) = b(t \wedge s)^a(t \vee s - t \wedge s)^{b-1}$. The mapping $1_{[0,t]} \to B^{a,b}(t)$ can be extended to an isometry between H and the Gaussian space associated with $B^{a,b}$. For every $u \in H$, we denote $B^{a,b}(u) = \int_0^T u(s)dB^{a,b}(s)$.

Also, let us consider the subspace $|H|$ of H defined as the set of measurable function u on $[0,T]$ such that

$$\|u\|^2_{|H|} = \int_0^T \int_0^T |u(s)||u(t)|\phi(t,s)dtds < \infty. \tag{2.1}$$

It was shown that $|H|$ is a Banach space with the norm $\|u\|_{|H|}$ and ε is dense in $|H|$, see [13, 17].
Moreover,

$$L^2([0,T]) \subset L^{2/(a+b+1)} \subset |H| \subset H. \tag{2.2}$$

When $b>0$, we denote by $\mathbf{S}$ the set of smooth functionals of the form

$$F = f\left(B^{a,b}(u_1), B^{a,b}(u_2), \cdots B^{a,b}(u_n)\right),$$

where $f \in C_b^\infty(\mathbb{R}^n)$ (f and all its derivatives are bounded) and $u_i \in H,\ i=1,2,\cdots,n$. Denote by $D^{a,b}$ and $\delta^{a,b}$ the Malliavin operator and its adjoint operator associated with the wfBm. We have the following properties to the adjoint operator $\delta^{a,b}$.

- We have $\mathbb{D}^{1,2} \subset Dom(\delta^{a,b})$ and for any $\varphi \in \mathbb{D}^{1,2}$,

$$\begin{aligned}\mathbb{E}\left[\delta^{a,b}(\varphi)^2\right] &= \mathbb{E}\|\varphi\|^2_H + \mathbb{E}\int_{[0,T]^4} D^{a,b}_\xi\varphi(r)D^{a,b}_\eta\varphi(s)\phi(\eta,r)\phi(\xi,s)dsdrd\xi d\eta \\ &\le \mathbb{E}\|\varphi\|^2_{|H|} + \mathbb{E}\int_{[0,T]^4} |D^{a,b}_\xi\varphi(r)||D^{a,b}_\eta\varphi(s)|\phi(\eta,r)||\phi(\xi,s)|dsdrd\xi d\eta.\end{aligned}$$

- We note to Proposition 1.5.8 in [12] and employ the inclusion (2.2) to conclude that if $\varphi \in \mathbb{D}^{1,2}(|H|)$, the space of $|H|$-valued variables with derivative belongs to $L^2(|H|\otimes|H|)$, and $D^{a,b}\varphi = 0$, then for some constant C_0, we can write

$$\mathbb{E}\left|\delta^{a,b}(\varphi)\right|^2 \le C_0\|\mathbb{E}\varphi\|^2_{L^{2/(a+b+1)}([0,T])}, \tag{2.3}$$

  We refer to [3, 12] for more detailes.
- For every $\varphi \in \mathbb{D}^{1,2}(|H|)$, Shen and et.al. [17] have shown that the divergence operator $\delta^{a,b}(\varphi) = \int_0^T \varphi(s)\delta B^{a,b}(s)$ satisfy

$$\int_0^T \varphi(s)dB^{a,b}(s) = \delta^{a,b}(\varphi) + \int_0^T\int_0^T D^{a,b}_s\varphi(r)\phi(r,s)drds.$$

  In particular, if $D^{a,b}\varphi(r)=0$, then $\int_0^T \varphi(s)dB^{a,b}(s) = \delta^{a,b}(\varphi)$.

3. L^2-maximal estimates for the stochastic integral

Let $I=(a_0,b_0)$ with $0<a_0<b_0\le\infty$ and v be almost everywhere positive functions, which are locally integrable on the interval I.
Denote by $L_2(v,I)$ the set of all functions measurable on I such that

$$\|f\|_{2,v} = \left(\int_{a_0}^{b_0} |f(x)|^2 v(x)dx\right)^{\frac{1}{2}} < \infty.$$

For every $\alpha_0 \ge 0$ and $0\le\beta\le1$, we consider the Hardy type operator $T_{\alpha_0,\beta}$ defined by

$$T_{\alpha_0,\beta}f(x) = \int_{a_0}^x \frac{s^{\alpha_0}f(s)ds}{(x-s)^{1-\beta}} \qquad x\in[0,T].$$

According to Theorem 3.3 in [1] for every $1\le\alpha\le a+b+1 =: p$, $\frac{1}{p}+\frac{1}{q}=1$, if

$$A_{\alpha_0,\beta} := \sup_{z\in I}\Big(\int_{a_0}^z u^q(s)s^{q\beta}ds\Big)^{\frac{1}{p}}\Big(\int_z^{b_0} s^{p(\alpha_0-1)}ds\Big)^{\frac{1}{p}} < \infty,$$

then for some constant C_0'

$$\left(\int_a^b \left(T_{\alpha_0,\beta} f(x)\right)^p v(x)dx\right)^{\frac{1}{p}} \leq C_0' \left(\int_a^b \left(f(x)\right)^\alpha dx\right)^{\frac{1}{\alpha}}. \tag{3.1}$$

The following theorem will give us a maximal L^2-estimate for the indefinite integral $\int_0^t \varphi(s)\delta B^{a,b}(s)$.

Theorem 3.1. *Let $0 \leq a \leq \frac{1}{2}$, $0 < b < 1$ and $a+3b \geq 1$ and $u = \{u(t), t \in [0,T]\}$ be a stochastic process with $D^{a,b}u = 0$. Then for every $1 \leq \alpha \leq a+b+1$*

$$\begin{aligned}\mathbb{E}\left(\sup_{t\in[0,T]}\left|\int_0^t u(s)\delta B_s^{a,b}\right|^2\right) &\leq C\Big(\int_0^T |\mathbb{E}u(s)|^{\frac{2\alpha}{a+b+1}}\, ds\Big)^{\frac{a+b+1}{\alpha}}\\ &\leq C\int_0^T |\mathbb{E}u(s)|^2\, ds,\end{aligned}$$

where the constant C depends on a, b and T.

Proof. Using the equality

$$c_{a,b} = \int_r^t t^b(t-\theta)^{-a} r^a(\theta - r)^{b-1} d\theta < \infty,$$

and the Fubini's stochastic theorem, we have

$$\int_0^t u(s)\delta B^{a,b}(s) = c_{a,b}^{-1}\int_0^t t^b(t-r)^{-a}\left(\int_0^r u(s)s^a(r-s)^{b-1}\delta B^{a,b}(s)\right)dr.$$

Chebyshev's inequality results that for some constant $C_{a,b}$

$$\left|\int_0^t u(s)\delta B^{a,b}(s)\right|^2 \leq C_{a,b}\int_0^t \left|\int_0^r u(s)s^a(r-s)^{b-1}\delta B^{a,b}(s)\right|^2 dr,$$

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left|\int_0^t u(s)\delta B^{a,b}(s)\right|^2\right) \leq C_{a,b}\mathbb{E}\int_0^T \left|\int_0^r u(s)s^a(r-s)^{b-1}\delta B^{a,b}(s)\right|^2 dr.$$

Using now inequality (2.3) and then applying (3.1) for $p = a+b+1$, $\alpha_0 = \frac{2a}{a+b+1}$, $\beta - 1 = \frac{2(b-1)}{a+b+1}$ and $f(s) = \Big(\mathbb{E}(|u(s)|)\Big)^{\frac{2}{a+b+1}}$, which satisfies desired condition according to our assumption in the theorem, we obtain

$$\begin{aligned}\mathbb{E}\left(\sup_{t\in[0,T]}\left|\int_0^t u(s)\delta B^{a,b}(s)\right|^2\right) &\leq C_0 C_{a,b}\Big\{\int_0^T\left(\int_0^r s^{\frac{2a}{(a+b+1)}}(r-s)^{\frac{2(b-1)}{(a+b+1)}}|\mathbb{E}u(s)|^{\frac{2}{(a+b+1)}}ds\right)^{(a+b+1)} dr\\ &\leq TC_0'C_0C_{a,b}\left(\int_0^T |\mathbb{E}u(r)|^{\frac{2\alpha}{a+b+1}}dr\right)^{\frac{a+b+1}{\alpha}}\\ &\leq C\int_0^T |\mathbb{E}u(r)|^2 dr.\end{aligned}$$

which completes the proof. □

4. Convergence Rate of Numerical Exponential Euler Scheme

In this section we first show that the SDE (1.1) has a unique solution as the form (1.3). Then we will study the convergence rate of the approximation solution to this exact solution of Equ. (1.1).

Theorem 4.1. *Under Assumption 1.1, the SDE (1.1) has a unique solution as the form (1.3).*

Proof. The proof is motivated from [6]. We first start to prove the uniquness of solutions. Assume that X, Y are two solutions of (1.1). Then from Assumption 1.1

$$\sup_{0\le t\le T} \mathbb{E}(|X(t)-Y(t)|^2) \le TL^2 \int_0^T |e^{2AT}| \sup_{0\le u\le s} \mathbb{E}(|X(u)-Y(u)|^2) ds.$$

Applying Gronwall's inequality results the uniquness of solutions. Now, to prove the existence of the solution, Let $X^0 = 0$ and define a sequence $\{X^n\}$ of processes as

$$\begin{cases} X^n(t) = e^{At}\xi_0 + \int_0^t e^{A(t-s)} f\left(X^{n-1}(s)\right) ds + \int_0^t e^{A(t-s)}\sigma(s) dB^{a,b}(s) \\ X^n(t) = \xi_0 \qquad t\in[-\tau,0] \end{cases}$$

Let $Y^n(t) := \sup_{0\le s\le t} \mathbb{E}(|X^{n+1}(s)-X^n(s)|^2)$ and employ Assumption 1.1 to obtain

$$\mathbb{E}(|X^{n+1}(t)-X^n(t)|^2) \le tL^2 \int_0^t |e^{2AT}| \sup_{0\le u\le s} \mathbb{E}(|X^n(u)-X^{n-1}(u)|^2) ds.$$

Consequently, by iteration we result

$$Y^n(t) \le \frac{(tL^2)^{n-1}T^{n-1}}{(n-1)!} Y^1(T),$$

from which the cauchy property of $\{X^n\}$ implies. Now, it is straightforward to show that $X(t)$, as the limit of the sequence, is the solution of the SDE (1.1). Indeed, when $n\to\infty$

$$\mathbb{E}\Big(\Big|\int_0^t e^{A(t-s)}\Big(f(X^{n-1}) - f(X(s))\Big)ds\Big|^2\Big) \le tL^2 \mathbb{E}\left(\int_0^t \left|X^{n-1}(s)-X(s)\right|^2 ds\right) \to 0.$$

□

Lemma 4.2. *Under Assumption 1.1, there exists variable C_1 independent of h such that*

$$(i)\ \mathbb{E}\left(\sup_{0\le t\le T} |y(t)|^2\right) \le C_1, \qquad (ii)\ \mathbb{E}\left(\sup_{0\le t\le T} |x(t)|^2\right) \le C_1. \tag{4.1}$$

Proof. Due to the fact $(a+b+c)^2 \le 3(a^2+b^2+c^2)$, we put forward our argumentation for every $t\ge 0$,

$$|y(t)|^2 \le 3\left[|e^{At}\xi_0|^2 + |\int_0^t e^{A(t-\bar{s})} f(z(s))ds|^2 + |\int_0^t e^{A(t-\bar{s})}\sigma(\bar{s})dB^{a,b}(s)|^2\right]. \tag{4.2}$$

Taking the expectation of both sides and using Hölder inequality, the result obtain as follow

$$\mathbb{E}\left(\sup_{0\le t\le T} |y(t)|^2\right) \le 3|e^{AT}|^2\mathbb{E}|\xi_0|^2 + 3T\mathbb{E}\int_0^T \left|e^{A(t-\bar{s})}\right|^2 |f(z(s))|^2 ds + 3\mathbb{E}\left(\sup_{0\le t\le T}\left|\int_0^t e^{A(t-\bar{s})}\sigma(\bar{s})dB^{a,b}(s)\right|^2\right). \tag{4.3}$$

If we let $M = \max\{1, |e^{At}|^2\}$ then we apply Assumption 1.1 to result the following

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le t\le T}|y(t)|^2\right) &\le 3\left\{M\mathbb{E}|\xi_0|^2 + TM\int_0^T \mathbb{E}|f(z(s))|^2 ds + MC\mathbb{E}\left(\int_0^T |\sigma(\bar{u})|^2 du\right)\right\}\\ &\le 3M\left\{\mathbb{E}|\xi_0|^2 + TK_1\int_0^T (1+\mathbb{E}|z(s)|^2)ds + TCK_2^2\right\}\\ &\le 3M\left\{\mathbb{E}|\xi_0|^2 + T^2K_1 + TCK_2^2\right\} + 3MK_1T\int_0^T \mathbb{E}\left(\sup_{0\le r\le s}|y_r|^2\right)ds.\end{aligned} \tag{4.4}$$

Using Gronwall inequality we derive

$$\mathbb{E}\left(\sup_{0\le t\le T}|y(t)|^2\right) \le C_1,$$

Also, from Eq.(1.3) and by arguments a like, the result (ii) is produced. □

Lemma 4.3. *Under Assumption 1.1, there exists some constant C_2 such that*

$$\mathbb{E}\left(\sup_{0\le t\le T}|y(t)-z(t)|^2\right) \le C_2 h^{\frac{a+b+1}{\alpha}}.$$

Proof. Through defining $z(t)$ and Eq.(1.5) for $t \in [t_k, t_{k+1})$, the following equality yields.

$$y(t)-z(t) = e^{A(t-t_k)}y_k + \int_{t_k}^t e^{A(s-t_k)}f(y_k)ds + \int_{t_k}^t e^{A(s-t_k)}\sigma(t_k)dB^{a,b}(s) - y_k.$$

Then Hölder's inequality and Assumption 1.1 imply

$$|y(t)-z(t)|^2 \le 3\left|e^{A(t-t_k)} - I_n\right|^2 y_k^2 + 3hK_1^2\int_{t_k}^t \left|e^{A(s-t_k)}\right|^2 \left(1+|y_k|^2\right)ds + 3\left|\int_{t_k}^t e^{A(s-t_k)}\sigma(t_k)dB^{a,b}(s)\right|^2 , \tag{4.5}$$

where I_n is an identity matrix. Take the expectation on the supremum with respect to t of the both sides of (4.5). Applying Theorem (3.1) and then Lemma 4.2 we derive

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le k\le n}\sup_{t_k\le t<t_{k+1}}|y(t)-z(t)|^2\right) &\le 3|e^{A(t-t_k)} - I_n|^2\mathbb{E}\left(\sup_k |y_k|^2\right) + 3MhK_1^2\mathbb{E}\left(\int_{t_{k_1}}^{t_{k_1+1}}(1+|y_k|^2)ds\right)\\ &\quad + 3MC\mathbb{E}\left(\int_{t_{k_2}}^{t_{k_2+1}}|\sigma(t_k)|^{\frac{2\alpha}{a+b+1}}ds\right)^{\frac{a+b+1}{\alpha}}\\ &\le 3\left\{|e^{A(t_{k+1}-t_k)} - I_n|^2C_1 + h^2MK_1^2(1+C_1)\right\} + 3MCK_2^{\frac{2\alpha}{a+b+1}}h^{\frac{a+b+1}{\alpha}}.\end{aligned}$$

in which two last supremum is taken for some $0 \le k_1, k_2 \le n$ for the first inequality.
For every $t_k \le t \le t_{k+1}$, we know $|e^{A(t-t_k)} - I_n| \le e^{|A|h} - 1 \le |A|hM$, therefore for some constant C_2, independent of h,

$$\mathbb{E}\left(\sup_{0\le t\le T}|y(t)-z(t)|^2\right) \le C_2 h^{\frac{a+b+1}{\alpha}}.$$

□

Theorem 4.4. *Under Assumptions 1.1 and conditions in Theorem 3.1, the numerical approximated solution $y(t)$ converges to the exact solution of Eq.(1.1) i.e.,*

$$\lim_{h\to 0}\mathbb{E}\left(\sup_{0\le t\le T} x(t)-y(t)|^2\right) = 0.$$

In fact, we prove the more efficient inequality and show that for some constant C_6

$$\mathbb{E}\left(\sup_{0\le t\le T}|x(t)-y(t)|^2\right)\le C_6 h^{\frac{a+b+1}{\alpha}}. \tag{4.6}$$

Proof. From Eq.(1.3) and Eq.(1.5) we know

$$\begin{aligned}
(x(t)-y(t))^2 &\le 2\Big[\int_0^t \left|e^{A(t-s)}f(x(s))-e^{A(t-\bar{s})}f(z(\bar{s}))\right|ds\Big]^2+2\Big[\int_0^t \left|e^{A(t-s)}\sigma(s)-e^{A(t-\bar{s})}\sigma(\bar{s})\right|dB^{a,b}(s)\Big]^2\\
&\le 4\Big[\int_0^t \left|e^{A(t-s)}-e^{A(t-\bar{s})}\right|f(x(s))ds\Big]^2+4\Big[\int_0^t [f(x(s))-f(z(\bar{s}))|\,e^{A(t-\bar{s})}ds\Big]^2\\
&\quad+4\Big[\int_0^t \left|e^{A(t-s)}-e^{A(t-\bar{s})}\right|\sigma(s)dB^{a,b}(s)\Big]^2+4\Big[\int_0^t |\sigma(s)-\sigma(\bar{s})|\,e^{A(t-\bar{s})}dB^{a,b}(s)\Big]^2\\
&=: I_1(t)+I_2(t)+I_3(t)+I_4(t).
\end{aligned} \tag{4.7}$$

In sequence, we will find some upper bounds to $\mathbb{E}\left(\sup_{0\le t\le T}|I_i(t)|^2\right)$ for every $1\le i\le 4$ with respect to $\mathbb{E}\left(\sup_{0\le t\le T}|x(t)-y(t)|^2\right)$ and h. To do this. for $i=1$ from Lemma 4.2, assumption 1.1 part a and Lemma 4.3 we result

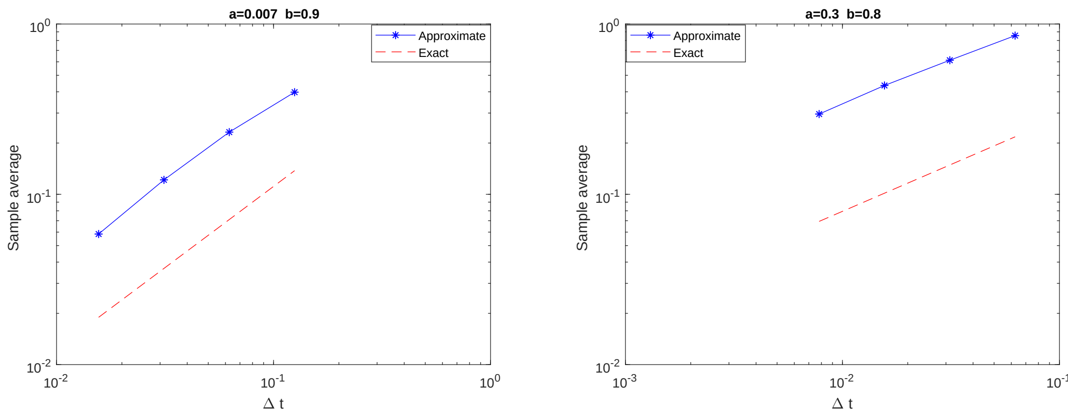
$$\begin{aligned}
\mathbb{E}\left(\sup_{0\le t\le T}|I_1(t)|^2+|I_2(t)|^2\right) &= \mathbb{E}\left(\sup_{0\le t\le T}\int_0^t e^{A(t-\bar{s})}\left|e^{A(\bar{s}-s)}-I_n\right|f(x(s))ds\right)^2\\
&\quad+TL^2\mathbb{E}\left(\sup_{0\le t\le T}\int_0^t |e^{A(t-\bar{s})}|^2|x(s)-z(\bar{s})|^2ds\right)\\
&\le TK_1M\int_0^T\left|e^{A(\bar{s}-s)}-I_n\right|^2\left(1+\mathbb{E}\Big(\sup_{0\le u\le s}|x(u)|^2\Big)\right)ds\\
&\quad+2L^2TM\int_0^T\mathbb{E}\left(\sup_{0\le u\le s}|x(u)-y(u)|^2\right)ds+2L^2T^2MC_2h^{\frac{a+b+1}{\alpha}}\\
&\le T^2K_1M^2|F|^2h^2(1+C_1)+2L^2TM\int_0^T\mathbb{E}\left(\sup_{0\le u\le s}|x(u)-y(u)|^2\right)ds\\
&\quad+2L^2T^2MC_2h^{\frac{a+b+1}{\alpha}}.
\end{aligned} \tag{4.8}$$

To bound $\mathbb{E}\left(\sup_{0\le t\le T}|I_3(t)|^2\right)$ and $\mathbb{E}\left(\sup_{0\le t\le T}|I_4(t)|^2\right)$ we use Theorem 3.1. So, as we done in 4.8, we deduce

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\le t\le T}|I_3(t)|^2+|I_4(t)|^2\right) &\le C\left(\int_0^T\left[e^{A(t-s)}-e^{A(t-\bar{s})}\right]^{\frac{2\alpha}{a+b+1}}|\sigma(s)|^{\frac{2\alpha}{a+b+1}}ds\right)^{\frac{a+b+1}{\alpha}}\\
&\quad+M\left(\int_0^T|\sigma(s)-\sigma(\bar{s})|^{\frac{2\alpha}{a+b+1}}ds\right)^{\frac{a+b+1}{\alpha}}\\
&\le C(MK_2|A|)^{\frac{2\alpha}{a+b+1}}T^{a+b+1}h^{\frac{a+b+1}{\alpha}}+L^2h^2MT^{\frac{a+b+1}{\alpha}}.
\end{aligned} \tag{4.9}$$

Now, substituting Eq.(4.8) and (4.9) into (4.7) can conclude that

$$\begin{aligned}
\mathbb{E}(\sup_{0\le t\le T}|x(t)-y(t)|^2) &\le T^2K_1M^2|A|^2h^2(1+C_1)+2L^2TM\int_0^T\mathbb{E}\left(\sup_{0\le u\le s}|x(u)-y(u)|^2\right)ds\\
&\quad+2L^2T^2MC_2h^{\frac{a+b+1}{\alpha}}+C(MK_2|A|)^{\frac{2\alpha}{a+b+1}}T^{a+b+1}h^{\frac{a+b+1}{\alpha}}+L^2h^2MT^{\frac{a+b+1}{\alpha}}\\
&\le C_3h^2+C_4h^{\frac{a+b+1}{\alpha}}+C_5\int_0^T\mathbb{E}\left(\sup_{0\le u\le s}|x(u)-y(u)|^2\right)ds,
\end{aligned}$$



(a) (b)

FIGURE 1. (a) Log-Log scale of method for example 1, a=0.2, b=0.9, (b) Log-Log scale for exponential Euler method for example 1, T=1, a=0.3, b=0.8.

TABLE 1. The mean, standard deviation of error.

	$a=0.2$ $b=0.9$		$a=0.3$ $b=0.8$	
$n-dimensional$	$\bar{x}_E$	S_E	$\bar{x}_E$	S_E
$1-dimensional$	0.058078	0.042101	0.063612	0.050348
$2-dimensional$	0.047122	0.042627	0.045524	0.048724
$3-dimensional$	0.052205	0.050952	0.062094	0.045738

for some positive constants C_3, C_4 and C_5.
Hence, Gronwall's inequality results the assertion (4.6) for some constant C_6. □

5. Numerical Results

In this section, we show our proposed method for numerical results to demonstrate the results of the convergence of the numerical solution for stochastic differential equation.

Example 5.1. Consider the stochastic differential equation:

$$dx(t) = (Ax(t) + 1 - x(t))\,dt + dB^{a,b}(t),\ t \in [0,1],$$
$$x(t) = 0, \quad A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}. \tag{5.1}$$

Exponential Euler method with step sizes $h = 2^{p-1}2^{-8}$ for $1 < p \leq 5$ is used for discretization. The numerical results obtained from exponential Euler method show the order of convergence $\frac{1}{2}$ as $\alpha = a + b + 1$, displayed in Figures 1 and 2. We used the parameters $a = 0.2,\ b = 0.9$ produces the slop=0.5009, residual=0.0609 and $a = 0.3,\ b = 0.8$ produces the slop=0.5083, residual=0.0287 displayed in Figures 1. In table 1, mean of error $\bar{x}_E$, and standard deviation of error S_E for 2000 iterations of exponential Euler method is presented.

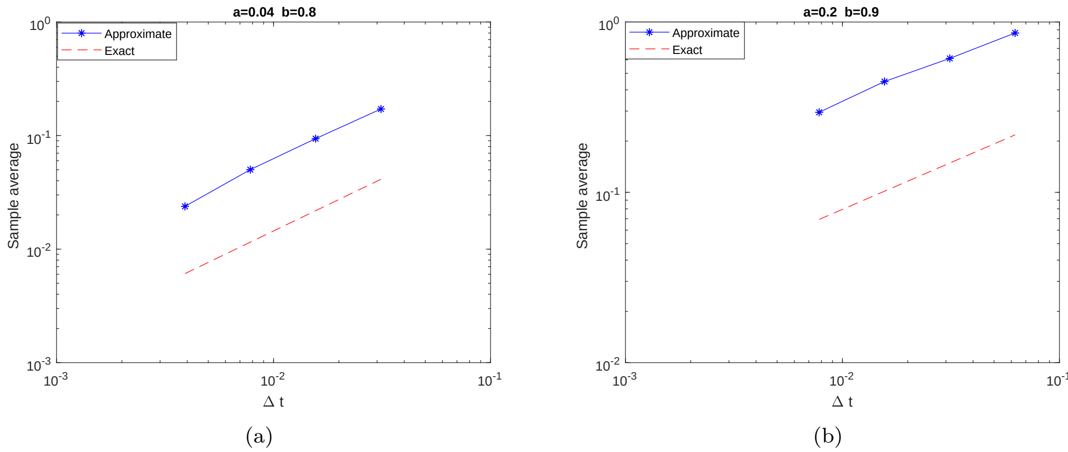


FIGURE 2. (a) Log-Log scale for exponential Euler method for example 2, T=1, a=0.4, b=0.7, (b) Log-Log scale for exponential Euler method for example 2, T=1, a=0.2, b=0.9.

TABLE 2. The mean, standard deviation of error.

$a = 0.4\ b = 0.7$			$a = 0.2\ b = 0.9$	
$n - dimensional$	$\bar{x}_E$	S_E	$\bar{x}_E$	S_E
$1 - dimensional$	0.071499	0.043180	0.068295	0.044134
$2 - dimensional$	0.046365	0.035848	0.057645	0.032884
$3 - dimensional$	0.049738	0.034820	0.041542	0.039798

Example 5.2. Consider the stochastic differential equation:

$$\begin{aligned} &dx(t) = Ax(t)dt + dB^{a,b}(t),\ t \in [0, 1], \\ &x(t) = 0, \quad A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}. \end{aligned} \tag{5.2}$$

Exponential Euler method with step sizes $h = 2^{p-1}2^{-10}$ for $1 < p \leq 5$ is used for discretization. The numerical results obtained from exponential Euler method show the order of convergence $\frac{1}{2}$ as $\alpha = a + b + 1$, displayed in Figures 5.2 and 5.2. The parameters $a = 0.4,\ b = 0.7$ produce the slop=0.5008, residual=0.0438 and $a = 0.2,\ b = 0.9$ produce the slop=0.5048, residual=0.0570.

Example 5.3. Consider the stochastic differential equation:

$$\begin{aligned} &dx(t) = \left(Ax(t) + 1 - \sqrt{x(t)}\right) dt + dB^{a,b}(t),\ t \in [0, 1], \\ &x(t) = 0. \end{aligned} \tag{5.3}$$

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

Exponential Euler method with step sizes $h = 2^{p-1}2^{-9}$ for $1 < p \leq 5$ is used for discretization. The numerical results obtained from exponential Euler method shows that the order of convergence is of $(a+b+1)/2$ as $\alpha = 1$. We used the parameters $a = 0.05,\ b = 0.4$ which produce the slop=0.7294, residual=0.1060 and $a = 0.04,\ b = 0.7$ which produce

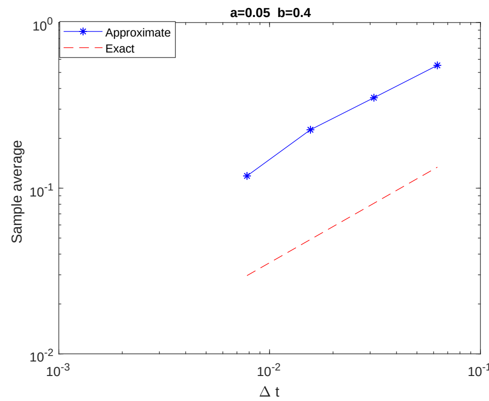


FIGURE 3. Log-Log scale for exponential Euler method for example 3, T=1, a=0.05, b=0.4.

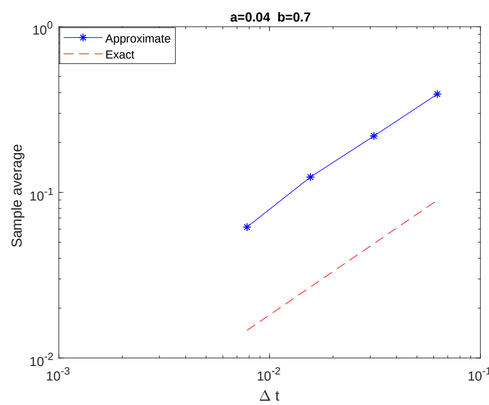


FIGURE 4. The numerical solutions for exponential Euler method for example 3, T=1, a=0.04, b=0.7.

TABLE 3. The mean, standard deviation of error.

$a = 0.04$ $b = 0.7$			$a = 0.05$ $b = 0.4$	
$n - dimensional$	$\bar{x}_E$	S_E	$\bar{x}_E$	S_E
$1 - dimensional$	0.009902	0.007343	0.055054	0.022718
$2 - dimensional$	0.010415	0.01058	0.057804	0.028336
$3 - dimensional$	0.022413	0.012454	0.038389	0.030996

the slop=0.8827, residual=0.0642. In table 3, exponential Euler method is the number of iterations, $\bar{x}_E$ is mean of error, and S_E is standard deviation of error.

6. Conclusions

In this paper, we study convergence of stochastic integral of the numerical solution for the exponential Euler method to stochastic functional differential equation driven by weighted fractional Brownian motion in additrive case. We also establish the convergence rate $\frac{a+b+1}{2\alpha}$ for every $1 \leq \alpha \leq a + b + 1$. A numerical example is given to verify the exponential Euler method, the conclusion is correct.

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