Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 10, No. 2, 2022, pp. 549-566
DOI:10.22034/cmde.2021.40523.1766

Dynamics of combined soliton solutions of unstable nonlinear fractional-order Schrödinger equation by beta-fractional derivative

Majid Bagheri and Ali Khani*
Faculty of Science, Department of Applied Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran.

Abstract

In this article, a new version of the trial equation method is suggested. This method allows new exact solutions of the nonlinear partial differential equations. The developed method is applied to unstable nonlinear fractional-order Schrödinger equation in fractional time derivative form of order α. Some exact solutions of the fractional-order fractional PDE are attained by employing the new powerful expansion approach using by beta-fractional derivatives which are used to get many solitary wave solutions by changing various parameters. New exact solutions are expressed with rational hyperbolic function solutions, rational trigonometric function solutions, 1-soliton solutions, dark soliton solitons, and rational function solutions. We can say that unstable nonlinear Schrödinger equation exist different dynamical behaviors. In addition, the physical behaviors of these new exact solutions are given with two and three dimensional graphs.

Keywords. Unstable nonlinear fractional-order Schrödinger equation, Beta-fractional derivative, New powerful expansion approach, Nonlinear partial differential equations.

2010 Mathematics Subject Classification. 02.60.Lj, 02.70.Wz, 02.90.+p, 04.30.Nk.

1. Introduction

Solitary wave theory plays an important role in explaining the solutions of nonlinear partial differential equations. Acquiring solutions of the partial differential equations helps us to understand many physical phenomena that appears in diversified scientific fields like in optical fibers, chemical kinematics, fluid mechanics, plasma physics, chemical physics, solid-state physics and so on. These physical phenomena is examined by obtaining the exact solutions of nonlinear ordinary and partial differential equations. Investigation of the exact solutions and dynamics of partial differential equations has been done by many researchers in several areas of sciences.

Some methods have suggested for solutions of the partial differential equations such as tanh function method, Hirota bilinear method, exp- function method, G'/G-expansion method, trial equation method, improved G'/G-expansion method, extended trial equation method, multiple extended trial equation method, Weierstrass elliptical function expansion method, Jacobian elliptical function method, first integral method, modified Kudryashov method, generalized Kudrayshov method and F-expansion method [1–3, 15–17, 20–25, 30, 33, 34, 39–50, 52, 53, 55, 57, 58, 60].

Ma and Fuchssteiner [38] proposed a powerful method to find the exact solutions of partial differential equations. Their main purpose was to extend the solution functions of a solvable differential equation to polynomial or rational polynomial functions. In the recent years, this proposed method was improved by many researchers. Liu developed this approach called as a trial equation method and implemented to some evolution differential equations [32–34]. Recently this approach was further developed and called as extended trial equation method by Gurefe et al [21] and Pandir et al [46]. Pandir and Gurefe implemented a more general form of the extended trial equation method to partial differential equations and attained new and different exact solutions.

Received: 27 June 2020 ; Accepted: 20 March 2021.
∗ Corresponding author. Email: khani@azaruniv.ac.ir.

In this article, we present a new version of the trial equation method and then implement this method to the unstable nonlinear Schrödinger equation. A new version of the trial equation method will give us to find combined soliton functions and combined jacobi elliptic functions together in the solution function. With the improved new method has been tried to be attained new and different solutions not found in the literature. We will investigate the solutions of the following unstable nonlinear Schrödinger equation

$$iD_t^\alpha u + u_{xx} + 2\lambda|u|^2u - 2\gamma u = 0, \quad 0 \le \alpha < 1, \quad \lambda, \gamma \in R, \tag{1.1}$$

where λ and γ are arbitrary real constants that explain marginally the development of deteriorations in stable or unstable environments over time[11, 31]. Here, u is a complex function based on the x and t arguments that provide the Eq. (1.1). The unstable nonlinear Schrodinger equation is a type of nonlinear Schrodinger equation that occurs with the displacement of space and time variables. The behavior of this type refers to two-layer baroque clinical instability and lossless symmetrical two-flow plasma instability. Tebue et al. achieved exact solutions using the new Jacobi elliptic function rational expansion method and exponential rational function method [56]. In [29], Ismael et al investigated the dynamic characteristics of nonlinear models that appear in ocean science. Silambarasan and co-workers [54] employed the F expansion method to the far-field equation and studied on the properties of longitudinal strain waves travelling in the cylindrical rod. Authors of [27] investigated soliton solutions for the conformable nonlinear differential equation governing wave-propagation in low-pass electrical transmission lines. In the valuable works some of researchers studied on solve the nonlinear partial differential equations [14, 18, 19, 26–28, 59]. Also, some researchers worked the vigorous study on fractional partial differential equations or nonlinear PDEs in which the interested readers can see herein [4–10].
The rest of the article is regulated as follows: in section 2, a new version of the trial equation method for partial differential equations is acquainted. Then we implement improved method to the unstable nonlinear Schrödinger equation in section 3. Finally, the conclusions are given in section 4.

2. Initial Definitions

Definition 2.1. Definition of β-derivative: Suppose $\chi : [0;1) \to \mathbf{R}$, then the β derivative of χ of order α is defined as

$$D_t^\alpha(\chi)(t) = \lim_{\epsilon \to 0} \frac{\chi\left[t + \epsilon\left(t + \frac{1}{\Gamma(\alpha)}\right)^{1-\alpha}\right] - \chi(t)}{\epsilon}, \quad \alpha \in (0,1], \quad t > 0. \tag{2.1}$$

The properties and new theorems will used as follow:

Theorem 2.2. *Suppose $\alpha \in (0,1]$; χ, ψ be α-differentiable at a point t, therefore we will*

1. $D_t^\alpha(a\chi(t) + b\psi(t)) = aD_t^\alpha(\chi(t)) + bD_t^\alpha(\psi(t)), \quad for \quad a, b \in \mathbf{R}.$
2. $D_t^\alpha(c) = 0, \quad for \quad c \in \mathbf{R}.$
3. $D_t^\alpha(\chi(t)\psi(t)) = \chi(t)D_t^\alpha(\psi(t)) + \psi(t)D_t^\alpha(\chi(t)).$
4. $D_t^\alpha(\frac{\chi(t)}{\psi(t)}) = \frac{\chi(t)D_t^\alpha(\psi(t)) - \psi(t)D_t^\alpha(\chi(t))}{\psi^2(t)}.$
5. $D_t^\alpha\chi(t) = \left(t + \frac{1}{\Gamma(\alpha)}\right)^{1-\alpha} \frac{d\chi(t)}{dt}.$

Theorem 2.3. *Suppose $\chi : [0;1) \to \mathbf{R}$; be a function such that χ is differentiable and also α-differentiable. Assume ψ be a differentiable function defined in the range of χ. Therefore, we have*

$$D_t^\alpha(\chi o \psi)(t) = \left(t + \frac{1}{\Gamma(\alpha)}\right)^{1-\alpha} \psi'(t)\chi'(\psi(t)), \tag{2.2}$$

where prime denotes the classical derivatives with respect to t.

The proofs of the above β-derivative properties were obviously presented in [12]. Also, improvement of fractional derivative have been made in works of Atangana and Baleanu in Refs. [13, 51].

3. Methodology

In this section, we give a description for the direct truncation method and introduce it for partial differential equation.
For a given partial differential equation

$$P(u, u_x, u_{xx}, ..., D_t^{\alpha}u, D_x^{\alpha}u, D_{xx}^{\alpha}u, ...) = 0, \quad 0 < \alpha \leq 1. \tag{3.1}$$

Using a transformation

$$u(x,t) = u(\phi), \qquad \phi = kx + \frac{l}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \tag{3.2}$$

where k and l are constants to be determined later, we can rewrite equation Eq. (3.1) in the following nonlinear ODE

$$Q(u, ku', k^2u'', ..., lu', ...) = 0, \tag{3.3}$$

where the prime denotes derivative with respect to ϕ. If possible, integrate Eq. (3.3) term by term one or more times. This yields constants of integration. For simplicity, the integration constants can be set to zero. Suppose g has the following truncation form

$$g(\phi) = \frac{\sum_{j=0}^{\tau} a_j \xi(\phi)^j}{\zeta(\phi)^{\tau}}, \tag{3.4}$$

in which $\xi(\phi)$ and $\zeta(\phi)$ are introduced as below form

$$\xi(\phi) = p_1F(\chi(\phi)) + q_1G(\chi(\phi)) + r_1,$$
$$\zeta(\phi) = p_2F(\chi(\phi)) + q_2G(\chi(\phi)) + r_2,$$

$$u(\xi) = g(\phi) = \frac{\sum_{j=0}^{\tau} a_j \left(p_1F(\chi(\phi)) + q_1G(\chi(\phi)) + r_1\right)^j}{\left(p_2F(\chi(\phi)) + q_2G(\chi(\phi)) + r_2\right)^{\tau}}, \tag{3.5}$$

where $a_j, p_1, q_1, r_1, p_2, q_2, r_2$ are constants to be determined, $\chi(\phi)$ is given and F, G are functions determined by an ordinary differential system, or F, G are functions given by direct ansatz such that their derivations are combinations of F and G, and $\chi(\phi)$ is determined by an ordinary differential equation

$$\frac{d\chi(\phi)}{d\phi} = H(\chi(\phi)) = LF(\chi(\phi)) + MG(\chi(\phi)) + N, \tag{3.6}$$

in which the function H is also given by a direct ansatz according to the context, the exponent τ is determined by utilizing homogeneous balance method in Eq. (3.1). The value τ is determined by equalizing the maximum order nonlinear term and the maximum order partial derivative term appearing in (3.3). If τ is the rational, then the appropriate transformations can be applied to conquer these hurdles. Substituting (3.5), (3.6) into (3.4) leads to a polynomial in $F(\phi)$ and $G(\phi)$, then set the coefficients of $F^i(\phi)G^j(\phi)$ and the constant term to be zero to get a system of algebraic equations on the unknown parameters in H together with the unknown numbers $a_j, p_1, q_1, r_1, p_2, q_2, r_2$ for $j = 0, 1, ..., \tau$, by solving the system one can get $a_j, p_1, q_1, r_1, p_2, q_2, r_2$ and the unknown parameters in H, then solving Eq. (3.6) to get $\chi(\phi)$ and the solutions of Eq. (3.1) can be obtained.

4. Application to unstable nonlinear fractional order form Schrödinger equation

The given section deals with application of new powerful expansion technique by determining the traveling wave form solutions of fractional order Schrödinger equation. In order to apply the new version trial equation method to the Eq. (1.1), first let's take traveling wave transformation

$$u(x,t) = e^{i\phi}\Psi(\xi), \quad \xi = x + \frac{\nu}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \tag{4.1}$$

applying aforementioned method. By using the fractional beta complex transform

$$\phi = \beta x + \frac{\mu}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha},$$

Eq. (1.1) is reduced to an ODE as

$$\Psi'' - (\beta^2 + \mu + 2\gamma)\Psi + 2\lambda\Psi^3 = 0. \tag{4.2}$$

where β, μ and $\nu = -2\beta$ are constants. Balancing the Ψ'' and Ψ^3 by employing the homogenous principle, we get

$$M + 2 = 3M, \qquad \Rightarrow M = 1. \tag{4.3}$$

4.1. **Case I:.** Then the exact solution will be as

$$\Psi(\xi) = \frac{\mathrm{e}^{2\,\chi(\xi)}a_1p_1 + \mathrm{e}^{\chi(\xi)}a_1q_1 + a_1r_1 + a_0}{p_2\mathrm{e}^{2\,\chi(\xi)} + q_2\mathrm{e}^{\chi(\xi)} + r_2}. \tag{4.4}$$

Inserting (4.4) in to Eq. (4.2), we obtain

$$\left(\left(\mathrm{p}_2\mathrm{e}^{2\,\chi(\xi)} + \mathrm{q}_2\mathrm{e}^{\chi(\xi)} + \mathrm{r}_2\right)^3\right)^{-1}\sum_{\mathrm{n}=0}^{18}\mathrm{C}_\mathrm{n}\exp(\mathrm{n}\chi(\xi)) = 0, \tag{4.5}$$

where $\mathrm{C_n}(0 \le \mathrm{n} \le 18)$ are polynomial statements in terms of $a_0, a_1, p_1, p_2, q_1, q_2, r_1$ and r_2. Hence, solving the resulting system $\mathrm{C_n} = 0(0 \le \mathrm{n} \le 18)$ simultaneously, we acquire the below set of parameters of solutions

Set I:

$$L = 0, \quad N = 0, \quad M = \frac{a_1p_1\sqrt{-\lambda}}{q_2}, \quad \beta = \beta, \quad a_0 = -a_1r_1, \quad q_1 = r_2 = 0, \tag{4.6}$$

$$\mu = -\beta^2 - 2\gamma, \quad a_1 = a_1, \quad p_1 = p_1, \quad p_2 = p_2, \quad q_2 = q_2, \quad r_1 = r_1.$$

We, therefore, gained the following generalized solitary solution

$$u_1(\phi, \xi) = \left[\frac{a_1\mathrm{e}^{\chi(\xi)}p_1}{\mathrm{e}^{\chi(\xi)}p_2 + q_2}\right]e^{i\phi}, \quad \chi(\xi) = \ln\left(-\frac{1}{M(\xi)}\right), \tag{4.7}$$

in which

$$\phi = \beta x - \frac{\beta^2 + 2\gamma}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \quad \xi = x - \frac{2\beta}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{4.8}$$

Set II:

$$L = 0, \quad M = M, \quad N = N, \quad \beta = \beta, \quad a_0 = -a_1r_1, \quad a_1 = \frac{q_2N}{2q_1\sqrt{-\lambda}}, \quad r_2 = 0, \tag{4.9}$$

$$\mu = -\frac{1}{2}N^2 - \beta^2 - 2\gamma, \quad a_1 = a_1, \quad p_1 = \frac{(2\,Mq_2 - Np_2)\,q_1}{Nq_2}, \quad p_2 = p_2, \quad q_1 = q_1, \quad q_2 = q_2, \quad r_1 = r_1.$$

We, therefore, gained the following generalized solitary solution

$$u_2(\phi, \xi) = \left[\frac{1}{2}\frac{2\,M\mathrm{e}^{\chi(\xi)}q_2 - N\mathrm{e}^{\chi(\xi)}p_2 + Nq_2}{\sqrt{-\lambda}\left(\mathrm{e}^{\chi(\xi)}p_2 + q_2\right)}\right]e^{i\phi}, \quad \chi(\xi) = N(\xi + C) + \ln\left(-\frac{N}{-1 + M\mathrm{e}^{N(\xi+C)}}\right), \tag{4.10}$$

in which

$$\phi = \beta x - \frac{\frac{1}{2}N^2 + \beta^2 + 2\gamma}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \quad \xi = x - \frac{2\beta}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{4.11}$$

Set III:

$$L = 0, \quad N = 0, \quad M = \frac{a_1p_1\sqrt{-\lambda}}{q_2}, \quad \beta = \beta, \quad a_0 = -a_1r_1, \quad q_1 = q_1, \quad r_2 = r_2, \tag{4.12}$$

$$\mu = -\beta^2 - 2\gamma, \quad a_1 = a_1, \quad p_1 = p_1, \quad p_2 = -\frac{(p_1r_2 - q_1q_2)\,p_1}{{q_1}^2}, \quad q_2 = q_2, \quad r_1 = r_1.$$

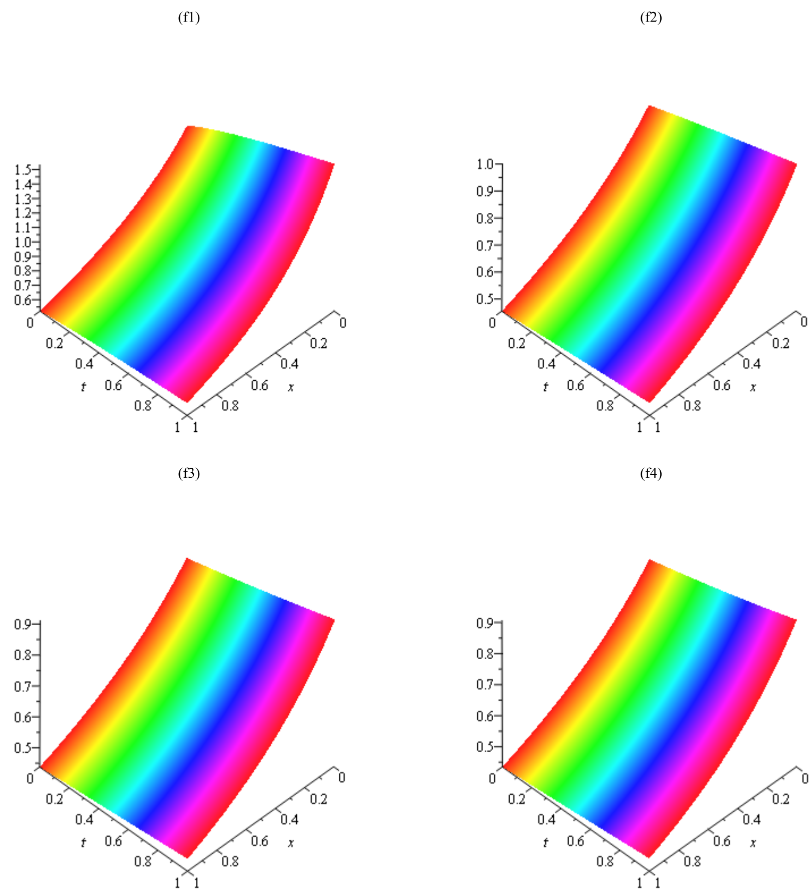

FIGURE 1. The 3D plot of (4.7) at: $\beta = 0.1, \gamma = -1, a_1 = -1, p_1 = 1,\ p_2 = 1.5\ q_2 = 1, M = -1, \lambda = -1$ when (f1): $\alpha = 0.25$, (f2): $\alpha = 0.5$, (f3): $\alpha = 0.85$, and (f4): $\alpha = 0.99$.

We, therefore, gained the following generalized solitary solution

$$u_3(\phi,\xi) = \left[-\frac{\mathrm{e}^{\chi(\xi)} a_1 {q_1}^2}{\mathrm{e}^{\chi(\xi)} p_1 r_2 - \mathrm{e}^{\chi(\xi)} q_1 q_2 - q_1 r_2}\right] e^{i\phi}, \quad \chi(\xi) = \ln\left(-\frac{1}{M\xi}\right), \tag{4.13}$$

in which

$$\phi = \beta x - \frac{\beta^2 + 2\gamma}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \quad \xi = x - \frac{2\beta}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{4.14}$$

Set IV:

$$L = 0, \quad M = \frac{N p_1}{q_1}, \quad N = N, \quad \beta = \beta, \quad a_0 = -a_1 r_1, \quad a_1 = \frac{q_2 N}{2 q_1 \sqrt{-\lambda}}, \quad r_2 = 0, \tag{4.15}$$

$$\mu = N^2 - \beta^2 - 2\gamma, \quad a_1 = a_1,$$

$$p_1 = \frac{(2\,M q_2 - N p_2)\, q_1}{N q_2}, \quad p_2 = \frac{1}{4}\,\frac{\lambda\, {a_1}^2 {q_1}^4 + 4\, N^2 {p_1}^2 {r_2}^2}{{q_1}^2 N^2 r_2},$$

$$q_1 = q_1, \quad q_2 = 2\,\frac{p_1 r_2}{q_1}, \quad r_1 = r_1.$$

We, therefore, gained the following generalized solitary solution

$$u_4(\phi,\xi) = \left[4\,\frac{N^2 e^{\chi(\xi)} a_1 {q_1}^2 r_2 \left(e^{\chi(\xi)} p_1 + q_1\right)}{e^{2\,\chi(\xi)} \lambda\, {a_1}^2 {q_1}^4 + 4\,N^2 e^{2\,\chi(\xi)} {p_1}^2 {r_2}^2 + 8\,e^{\chi(\xi)} p_1 {r_2}^2 q_1 N^2 + 4\,{q_1}^2 N^2 {r_2}^2}\right] e^{i\phi}, \tag{4.16}$$

$$\chi(\xi) = N(\xi + C) + \ln\left(-\frac{N}{-1 + \frac{N p_1}{q_1} e^{N(\xi+C)}}\right),$$

in which

$$\phi = \beta x - \frac{-N^2 + \beta^2 + 2\gamma}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \quad \xi = x - \frac{2\beta}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{4.17}$$

Set V:

$$L = 0, \quad M = 0, \quad N = \frac{\sqrt{-\lambda}\,(a_1 r_1 + a_0)}{r_2}, \quad \beta = \beta, \quad \mu = -\frac{-2\,\lambda\,{a_1}^2 {r_1}^2 + \beta^2 {r_2}^2 - 4\,\lambda\, a_0 a_1 r_1 + 2\,\gamma\, {r_2}^2 - 2\,\lambda\, {a_0}^2}{{r_2}^2}, \tag{4.18}$$

$$a_0 = a_0, \quad a_1 = a_1, \quad p_1 = -\frac{p_2\,(a_1 r_1 + a_0)}{a_1 r_2}, \quad p_2 = p_2, \quad q_1 = q_2 = 0, \quad r_1 = r_1, \quad r_2 = r_2.$$

We, therefore, gained the following generalized solitary solution

$$u_5(\phi,\xi) = \left[-\frac{e^{2\,\chi(\xi)} a_1 p_2 r_1 + e^{2\,\chi(\xi)} a_0 p_2 - a_1 r_1 r_2 - a_0 r_2}{\left(p_2 e^{2\,\chi(\xi)} + r_2\right) r_2}\right] e^{i\phi}, \quad \chi(\xi) = \frac{\sqrt{-\lambda}\,(a_1 r_1 + a_0)}{r_2}(\xi + C), \tag{4.19}$$

in which

$$\phi = \beta x - \frac{-2\,\lambda\,{a_1}^2 {r_1}^2 + \beta^2 {r_2}^2 - 4\,\lambda\, a_0 a_1 r_1 + 2\,\gamma\, {r_2}^2 - 2\,\lambda\, {a_0}^2}{\alpha {r_2}^2}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \quad \xi = x - \frac{2\beta}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{4.20}$$

Set VI:

$$L = 0, \quad M = 0, \quad N = \frac{2\sqrt{-\lambda}\,(a_1 r_1 + a_0)}{r_2}, \quad \beta = \beta, \quad \mu = -\frac{-2\,\lambda\,{a_1}^2 {r_1}^2 + \beta^2 {r_2}^2 - 4\,\lambda\, a_0 a_1 r_1 + 2\,\gamma\, {r_2}^2 - 2\,\lambda\, {a_0}^2}{{r_2}^2}, \tag{4.21}$$

$$a_0 = a_0, \quad a_1 = a_1, \quad p_1 = -\frac{1}{4}\,\frac{-{a_1}^2 {q_1}^2 {r_2}^2 + {a_1}^2 {q_2}^2 {r_1}^2 + 2\,a_0 a_1 {q_2}^2 r_1 + {a_0}^2 {q_2}^2}{a_1\,(a_1 r_1 + a_0)\, {r_2}^2},$$

$$p_2 = \frac{1}{4}\,\frac{-{a_1}^2 {q_1}^2 {r_2}^2 + {a_1}^2 {q_2}^2 {r_1}^2 + 2\,a_0 a_1 {q_2}^2 r_1 + {a_0}^2 {q_2}^2}{r_2\,({a_1}^2 {r_1}^2 + 2\,a_0 a_1 r_1 + {a_0}^2)}, \quad q_1 = q_2 = 0, \; r_1 = r_1, \; r_2 = r_2.$$

We, therefore, gained the following generalized solitary solution

$$u_6(\phi,\xi) = \left[-\frac{(a_1 r_1 + a_0)\left(-a_1 e^{\chi(\xi)} q_1 r_2 + a_1 e^{\chi(\xi)} q_2 r_1 + e^{\chi(\xi)} a_0 q_2 - 2\,a_1 r_1 r_2 - 2\,a_0 r_2\right)}{r_2\left(-a_1 e^{\chi(\xi)} q_1 r_2 + a_1 e^{\chi(\xi)} q_2 r_1 + e^{\chi(\xi)} a_0 q_2 + 2\,a_1 r_1 r_2 + 2\,a_0 r_2\right)}\right] e^{i\phi}, \tag{4.22}$$

$$\chi(\xi) = \frac{2\sqrt{-\lambda}\,(a_1 r_1 + a_0)}{r_2}(\xi + C),$$

in which

$$\phi = \beta x - \frac{-2\,\lambda\,{a_1}^2 {r_1}^2 + \beta^2 {r_2}^2 - 4\,\lambda\, a_0 a_1 r_1 + 2\,\gamma\, {r_2}^2 - 2\,\lambda\, {a_0}^2}{\alpha {r_2}^2}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \quad \xi = x - \frac{2\beta}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{4.23}$$

Set VII:

$$L = 0, \quad M = \frac{\sqrt{-\lambda}\, a_0 q_2}{{r_2}^2}, \quad N = 2\,\frac{\sqrt{-\lambda}\, a_0}{r_2}, \quad \beta = \beta, \quad a_0 = a_0, \quad a_1 = 0, \quad p_2 = 0, \tag{4.24}$$

$$\mu=-\frac{\beta^2{r_2}^2+2\,\gamma\,{r_2}^2-2\,\lambda\,{a_0}^2}{{r_2}^2},\quad p_1=p_1,\quad r_2=r_2,\quad q_1=q_1,\quad q_2=q_2,\quad r_1=r_1.$$

We, therefore, gained the following generalized solitary solution

$$u_7(\phi,\xi)=\left[\frac{a_0}{q_2 \mathrm{e}^{\chi(\xi)}+r_2}\right]e^{i\phi},\quad \chi\left(\xi\right)=2\,\frac{\sqrt{-\lambda}a_0}{r_2}\left(\xi+C\right)+\ln\left(-\frac{2\,\frac{\sqrt{-\lambda}a_0}{r_2}}{-1+\frac{\sqrt{-\lambda}a_0 q_2}{{r_2}^2}\mathrm{e}^{2\,\frac{\sqrt{-\lambda}a_0}{r_2}(\xi+C)}}\right), \tag{4.25}$$

in which

$$\phi=\beta x-\frac{\beta^2{r_2}^2+2\,\gamma\,{r_2}^2-2\,\lambda\,{a_0}^2}{\alpha{r_2}^2}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha},\quad \xi=x-\frac{2\beta}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{4.26}$$

4.2. **Case II:.** Then the exact solution will be as

$$\Psi(\xi)=\frac{\sinh\left(\chi\left(\xi\right)\right)a_1p_1+\cosh\left(\chi\left(\xi\right)\right)a_1q_1+a_1r_1+a_0}{p_2\sinh\left(\chi\left(\xi\right)\right)+q_2\cosh\left(\chi\left(\xi\right)\right)+r_2}. \tag{4.27}$$

Inserting (4.27) in to Eq. (4.2), we obtain

$$\left((\mathrm{p}_2\sinh\left(\chi\left(\xi\right)\right)+\mathrm{q}_2\cosh\left(\chi\left(\xi\right)\right)+\mathrm{r}_2)^3\right)^{-1}\sum_{\mathrm{i+j=5}}\mathrm{C_{ij}}\sinh^{\mathrm{i}}(\chi\left(\xi\right))\cosh^{\mathrm{j}}(\chi\left(\xi\right))=0, \tag{4.28}$$

where $\mathrm{C_{ij}}(\mathrm{i}+\mathrm{j}=5,0\le\mathrm{i},\mathrm{j}\le5)$ are polynomial statements in terms of $a_0,a_1,p_1,p_2,q_1,q_2,r_1$ and r_2. Hence, solving the resulting system $\mathrm{C_{ij}}=0(\mathrm{i}+\mathrm{j}=5,0\le\mathrm{i},\mathrm{j}\le5)$ simultaneously, we acquire the below set of parameters of solutions
Set I:

$$L=0,\quad M=0,\quad N=\frac{\sqrt{-\lambda}p_1a_1}{q_2},\quad \beta=\beta,\quad \mu=-\frac{-2\,\lambda\,{a_1}^2{p_1}^2+\beta^2{q_2}^2+2\,\gamma\,{q_2}^2}{{q_2}^2}, \tag{4.29}$$

$$a_0=-a_1r_1,\quad a_1=0,\quad p_1=p_1,\quad p_2=0,\quad q_1=0,\quad q_2=q_2,\quad r_1=r_1,\quad r_2=0.$$

We, therefore, gained the following generalized solitary solution

$$u_1(\phi,\xi)=\left(\frac{a_1p_1\sinh\left(\chi\left(\xi\right)\right)}{q_2\cosh\left(\chi\left(\xi\right)\right)}\right)e^{i\phi},\quad \chi\left(\xi\right)=\frac{\sqrt{-\lambda}p_1a_1}{q_2}(\xi+C), \tag{4.30}$$

in which

$$\phi=\beta x-\frac{-2\,\lambda\,{a_1}^2{p_1}^2+\beta^2{q_2}^2+2\,\gamma\,{q_2}^2}{\alpha{q_2}^2}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha},\quad \xi=x-\frac{2\beta}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{4.31}$$

Set II:

$$L=0,\quad M=0,\quad N=\frac{2\sqrt{-\lambda}q_1a_1}{q_2},\quad \beta=\beta,\quad \mu=-\frac{-2\,\lambda\,{a_1}^2{q_1}^2+\beta^2{q_2}^2+2\,\gamma\,{q_2}^2}{{q_2}^2}, \tag{4.32}$$

$$a_0=-\frac{a_1\left(q_1r_2+q_2r_1\right)}{q_2},\quad a_1=a_1,\quad p_1=q_1,\quad p_2=q_2,\quad q_1=q_1,\quad q_2=q_2,\quad r_1=r_1,\quad r_2=r_2.$$

We, therefore, gained the following generalized solitary solution

$$u_2(\phi,\xi)=\left[\frac{a_1q_1\left(q_2\cosh\left(\chi\left(\xi\right)\right)+q_2\sinh\left(\chi\left(\xi\right)\right)-r_2\right)}{q_2\left(q_2\cosh\left(\chi\left(\xi\right)\right)+q_2\sinh\left(\chi\left(\xi\right)\right)+r_2\right)}\right]e^{i\phi},\quad \chi\left(\xi\right)=\frac{2\sqrt{-\lambda}q_1a_1}{q_2}(\xi+C), \tag{4.33}$$

in which

$$\phi=\beta x-\frac{-2\,\lambda\,{a_1}^2{q_1}^2+\beta^2{q_2}^2+2\,\gamma\,{q_2}^2}{\alpha{q_2}^2}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha},\quad \xi=x-\frac{2\beta}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{4.34}$$

Set III:

$$L=0,\quad M=0,\quad N=\frac{2\sqrt{-\lambda}q_1a_1}{q_2},\quad \beta=\beta,\quad \mu=-\frac{-2\,\lambda\,{a_1}^2{q_1}^2+\beta^2{q_2}^2+2\,\gamma\,{q_2}^2}{{q_2}^2}, \tag{4.35}$$

$$a_0 = -\frac{a_1\left(q_1 r_2 + q_2 r_1\right)}{q_2}, \quad a_1 = a_1, \quad p_1 = -q_1, \quad p_2 = -q_2, \quad q_1 = q_1, \quad q_2 = q_2, \quad r_1 = r_1, \quad r_2 = r_2.$$

We, therefore, gained the following generalized solitary solution

$$u_3(\phi, \xi) = \left[\frac{a_1 q_1\left(q_2 \cosh\left(\chi\left(\xi\right)\right) - q_2 \sinh\left(\chi\left(\xi\right)\right) - r_2\right)}{q_2\left(-q_2 \cosh\left(\chi\left(\xi\right)\right) + q_2 \sinh\left(\chi\left(\xi\right)\right) + r_2\right)}\right] e^{i\phi}, \quad \chi\left(\xi\right) = \frac{2\sqrt{-\lambda} q_1 a_1}{q_2}(\xi + C), \tag{4.36}$$

in which

$$\phi = \beta x - \frac{-2\,\lambda\, a_1{}^2 q_1{}^2 + \beta^2 q_2{}^2 + 2\,\gamma\, q_2{}^2}{\alpha q_2{}^2}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \quad \xi = x - \frac{2\beta}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{4.37}$$

Set IV:

$$L = 0, \quad M = 0, \quad N = \frac{\sqrt{\lambda} q_1 a_1}{q_2}, \quad \beta = \beta, \quad \mu = -\frac{-2\,\lambda\, a_1{}^2 q_1{}^2 + \beta^2 q_2{}^2 + 2\,\gamma\, q_2{}^2}{q_2{}^2}, \tag{4.38}$$

$$a_0 = -a_1 r_1, \quad a_1 = a_1, \quad p_1 = -i q_1, \quad p_2 = i q_2, \quad q_1 = q_1, \quad q_2 = q_2, \quad r_1 = r_1, \quad r_2 = 0.$$

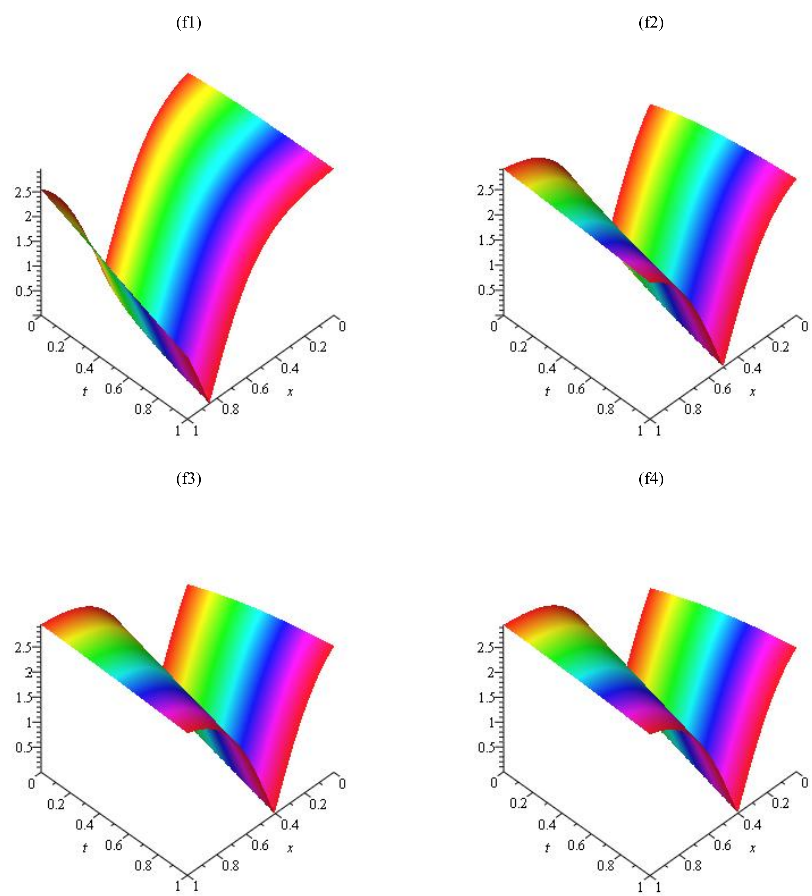

FIGURE 2. The 3D plot of (4.30) at: $\beta = 0.1, \gamma = -1, a_1 = 1.5, p_1 = 2, q_2 = 1, \lambda = -1$ when (f1): $\alpha = 0.25$, (f2): $\alpha = 0.5$, (f3): $\alpha = 0.85$, and (f4): $\alpha = 0.99$.

We, therefore, gained the following generalized solitary solution

$$u_4(\phi,\xi)=\left[-\frac{a_1q_1\left(2\,\sinh\left(\chi\left(\xi\right)\right)i\cosh\left(\chi\left(\xi\right)\right)-1\right)}{q_2\left(2\,\left(\cosh\left(\chi\left(\xi\right)\right)\right)^2-1\right)}\right]e^{i\phi},\quad \chi\left(\xi\right)=\frac{\sqrt{\lambda}q_1a_1}{q_2}(\xi+C), \tag{4.39}$$

in which

$$\phi=\beta x-\frac{-2\,\lambda\,{a_1}^2{q_1}^2+\beta^2{q_2}^2+2\,\gamma\,{q_2}^2}{\alpha{q_2}^2}\left(t+\frac{1}{\Gamma(\alpha)}\right)^\alpha,\quad \xi=x-\frac{2\beta}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^\alpha. \tag{4.40}$$

Set V:

$$L=0,\quad M=0,\quad N=\frac{\sqrt{-\lambda}q_1a_1}{p_2},\quad \beta=\beta,\quad \mu=-\frac{-2\,\lambda\,{a_1}^2{q_1}^2+\beta^2{p_2}^2+2\,\gamma\,{p_2}^2}{{p_2}^2}, \tag{4.41}$$

$$a_0=-a_1r_1,\quad a_1=a_1,\quad p_1=\frac{q_1q_2}{p_2},\quad p_2=p_2,\quad q_1=q_1,\quad q_2=q_2,\quad r_1=r_1,\quad r_2=0.$$

We, therefore, gained the following generalized solitary solution

$$u_5(\phi,\xi)=\left[\frac{q_1a_1\left(\cosh\left(\chi\left(\xi\right)\right)p_2+q_2\sinh\left(\chi\left(\xi\right)\right)\right)}{\left(p_2\sinh\left(\chi\left(\xi\right)\right)+q_2\cosh\left(\chi\left(\xi\right)\right)\right)p_2}\right]e^{i\phi},\quad \chi\left(\xi\right)=\frac{\sqrt{-\lambda}q_1a_1}{p_2}(\xi+C), \tag{4.42}$$

in which

$$\phi=\beta x-\frac{-2\,\lambda\,{a_1}^2{q_1}^2+\beta^2{p_2}^2+2\,\gamma\,{p_2}^2}{\alpha{p_2}^2}\left(t+\frac{1}{\Gamma(\alpha)}\right)^\alpha,\quad \xi=x-\frac{2\beta}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^\alpha. \tag{4.43}$$

Set VI:

$$L=0,\quad M=0,\quad N=\frac{2\sqrt{-\lambda}q_1a_1}{q_2},\quad \beta=\beta,\quad \mu=-\frac{-8\,\lambda\,{a_1}^2{q_1}^2+\beta^2{q_2}^2+2\,\gamma\,{q_2}^2}{{q_2}^2}, \tag{4.44}$$

$$a_0=-a_1r_1,\quad a_1=a_1,\quad p_1=-2q_1,\quad p_2=-\frac{1}{2}q_2,\quad q_1=q_1,\quad q_2=q_2,\quad r_1=r_1,\quad r_2=0.$$

We, therefore, gained the following generalized solitary solution

$$u_6(\phi,\xi)=\left[2\,\frac{q_1a_1\left(\cosh\left(\chi\left(\xi\right)\right)-2\,\sinh\left(\chi\left(\xi\right)\right)\right)}{q_2\left(-\sinh\left(\chi\left(\xi\right)\right)+2\,\cosh\left(\chi\left(\xi\right)\right)\right)}\right]e^{i\phi},\quad \chi\left(\xi\right)=\frac{2\sqrt{-\lambda}q_1a_1}{q_2}(\xi+C), \tag{4.45}$$

in which

$$\phi=\beta x-\frac{-8\,\lambda\,{a_1}^2{q_1}^2+\beta^2{q_2}^2+2\,\gamma\,{q_2}^2}{\alpha{q_2}^2}\left(t+\frac{1}{\Gamma(\alpha)}\right)^\alpha,\quad \xi=x-\frac{2\beta}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^\alpha. \tag{4.46}$$

Set VII:

$$L=\frac{1}{4a_0}\,\frac{N^2q_2}{\sqrt{-\lambda}},\quad M=\frac{1}{4a_0}\,\frac{N^2q_2}{\sqrt{-\lambda}},\quad N=N,\quad \beta=\beta,\quad \mu=-\frac{1}{2}\,N^2-\beta^2-2\,\gamma, \tag{4.47}$$

$$a_0=a_0,\quad a_1=0,\quad p_1=p_1,\quad p_2=q_2,\quad q_1=q_1,\quad q_2=q_2,\quad r_1=r_1,\quad r_2=2\,\frac{a_0\sqrt{-\lambda}}{N}.$$

We, therefore, gained the following generalized solitary solution

$$u_7(\phi,\xi)=\left[\frac{a_0N\left(N\cosh\left(\chi\left(\xi\right)\right)q_2+N\sinh\left(\chi\left(\xi\right)\right)q_2+2\,\frac{a_0\lambda}{\sqrt{-\lambda}}\right)}{2\,N^2\left(\cosh\left(\chi\left(\xi\right)\right)\right)^2{q_2}^2+2\,N^2\cosh\left(\chi\left(\xi\right)\right)\sinh\left(\chi\left(\xi\right)\right){q_2}^2-N^2{q_2}^2+4\,\lambda\,{a_0}^2}\right]e^{i\phi}, \tag{4.48}$$

$$\chi\left(\xi\right)=2arc\tanh\left(8\,\sqrt{-\frac{\lambda\,{a_0}^2}{\left(-1+Nq_2\mathrm{e}^{N(\phi+C)}-4\,\sqrt{-\lambda}a_0\mathrm{e}^{N(\phi+C)}\right)^2}}\mathrm{e}^{N(\phi+C)}-1\right),$$

in which

$$\phi=\beta x+\frac{-\frac{1}{2}\,N^2-\beta^2-2\,\gamma}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^\alpha,\quad \xi=x-\frac{2\beta}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^\alpha. \tag{4.49}$$

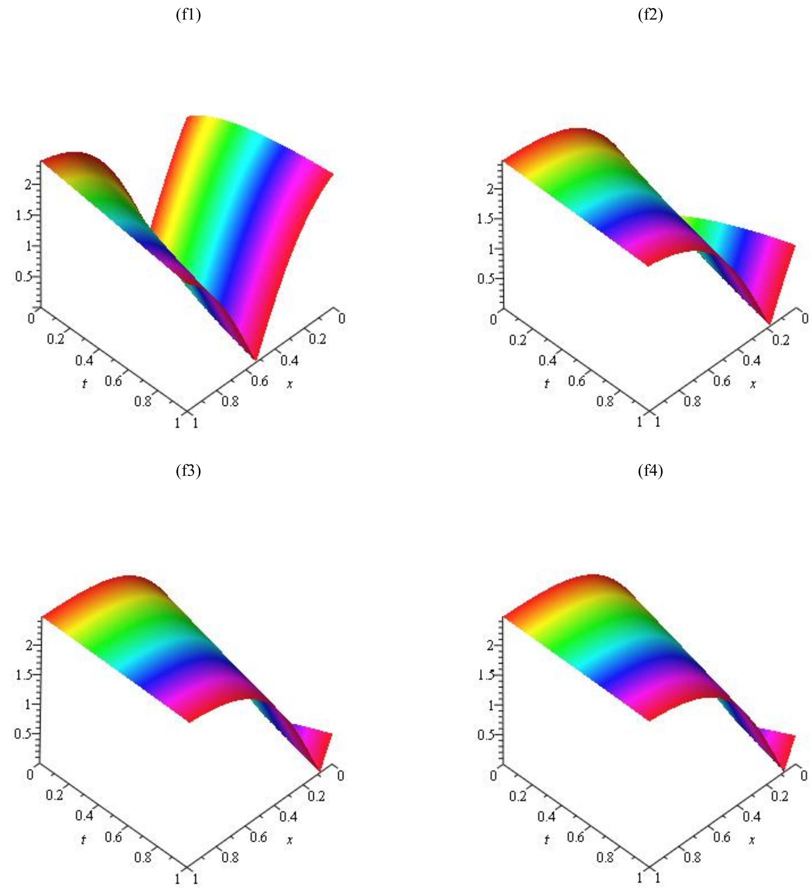

FIGURE 3. The 3D plot of (4.48) at: $\beta = 0.1, \gamma = -1, a_0 = 1.5, N = 5, q_2 = 1, \lambda = -1$ when (f1): $\alpha = 0.25$, (f2): $\alpha = 0.5$, (f3): $\alpha = 0.85$, and (f4): $\alpha = 0.99$.

Set VIII:

$$L = \frac{1}{4a_0}\frac{N^2 q_2}{\sqrt{-\lambda}}, \quad M = \frac{1}{4a_0}\frac{N^2 q_2}{\sqrt{-\lambda}}, \quad N = N, \quad \beta = \beta, \quad \mu = -\frac{1}{2}N^2 - \beta^2 - 2\gamma, \tag{4.50}$$

$$a_0 = a_0, \quad a_1 = 0, \quad p_1 = p_1, \quad p_2 = -q_2, \quad q_1 = q_1, \quad q_2 = q_2, \quad r_1 = r_1, \quad r_2 = 2\,\frac{a_0\sqrt{-\lambda}}{N}.$$

We, therefore, gained the following generalized solitary solution

$$u_8(\phi,\xi) = \left[\frac{a_0 N\left(N\cosh(\chi(\xi))\,q_2 - N\sinh(\chi(\xi))\,q_2 + 2\,\frac{a_0\lambda}{\sqrt{-\lambda}}\right)}{2\,N^2(\cosh(\chi(\xi)))^2\,{q_2}^2 - 2\,N^2\cosh(\chi(\xi))\sinh(\chi(\xi))\,{q_2}^2 - N^2{q_2}^2 + 4\,\lambda\,{a_0}^2}\right]e^{i\phi}, \tag{4.51}$$

$$\chi(\xi) = 2arc\tanh\left(8\sqrt{-\frac{\lambda\,{a_0}^2}{\left(-1 + Nq_2\mathrm{e}^{N(\phi+C)} - 4\sqrt{-\lambda}a_0\mathrm{e}^{N(\phi+C)}\right)^2}}\mathrm{e}^{N(\phi+C)} - 1\right),$$

in which

$$\phi = \beta x + \frac{-\frac{1}{2}N^2 - \beta^2 - 2\gamma}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \quad \xi = x - \frac{2\beta}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{4.52}$$

Set IX:

$$L=L,\quad M=M,\quad N=N,\quad \beta=\beta,\quad \mu=\mu, \tag{4.53}$$

$$a_0=-a_1r_1,\quad a_1=a_1,\quad p_1=0,\quad p_2=p_2,\quad q_1=0,\quad q_2=q_2,\quad r_1=r_1,\quad r_2=r_2.$$

We, therefore, gained the following generalized solitary solution

$$u_9(\phi,\xi)=0. \tag{4.54}$$

Set X:

$$L=\frac{\sqrt{-\lambda}\,(a_1r_1+a_0)\,q_2}{{r_2}^2},\quad M=\frac{\sqrt{-\lambda}\,(a_1r_1+a_0)\,q_2}{{r_2}^2},\quad N=\frac{2\sqrt{-\lambda}\,(a_1r_1+a_0)}{r_2},\quad \beta=\beta,\quad p_2=q_2,\quad q_1=0, \tag{4.55}$$

$$\mu=-\frac{-2\,\lambda\,{a_1}^2{r_1}^2+\beta^2{r_2}^2-4\,\lambda\,a_0a_1r_1+2\,\gamma\,{r_2}^2-2\,\lambda\,{a_0}^2}{{r_2}^2},$$

$$a_0=a_0,\quad a_1=a_1,\quad p_1=0,\quad q_2=q_2,\quad r_1=r_1,\quad r_2=r_2.$$

We, therefore, gained the following generalized solitary solution

$$u_{10}(\phi,\xi)=\left(\frac{a_1r_1+a_0}{q_2\sinh(\chi(\xi))+q_2\cosh(\chi(\xi))+r_2}\right)e^{i\phi}, \tag{4.56}$$

$$\chi\left(\xi\right)=2arc\tanh\left(1-4\,r_2\mathrm{e}^{2\,\frac{\sqrt{-\lambda}(a_1r_1+a_0)(\xi+C)}{r_2}}\left(1-\mathrm{e}^{2\,\frac{\sqrt{-\lambda}(a_1r_1+a_0)(\xi+C)}{r_2}}\left(q_2-2\,r_2\right)\right)^{-1}\right),$$

in which

$$\phi=\beta x-\frac{-2\,\lambda\,{a_1}^2{r_1}^2+\beta^2{r_2}^2-4\,\lambda\,a_0a_1r_1+2\,\gamma\,{r_2}^2-2\,\lambda\,{a_0}^2}{\alpha{r_2}^2}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha},\quad \xi=x-\frac{2\beta}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{4.57}$$

Set XI:

$$L=\frac{\sqrt{-\lambda}\,(a_1r_1+a_0)\,q_2}{{r_2}^2},\quad M=\frac{\sqrt{-\lambda}\,(a_1r_1+a_0)\,q_2}{{r_2}^2},\quad N=\frac{2\sqrt{-\lambda}\,(a_1r_1+a_0)}{r_2},\quad \beta=\beta,\quad p_2=-q_2,\quad q_1=0, \tag{4.58}$$

$$\mu=-\frac{-2\,\lambda\,{a_1}^2{r_1}^2+\beta^2{r_2}^2-4\,\lambda\,a_0a_1r_1+2\,\gamma\,{r_2}^2-2\,\lambda\,{a_0}^2}{{r_2}^2},$$

$$a_0=a_0,\ a_1=a_1,\ p_1=0,\ q_2=q_2,\ r_1=r_1,\ r_2=r_2.$$

We, therefore, gained the following generalized solitary solution

$$u_{11}(\phi,\xi)=\left(\frac{a_1r_1+a_0}{-q_2\sinh(\chi(\xi))+q_2\cosh(\chi(\xi))+r_2}\right)e^{i\phi}, \tag{4.59}$$

$$\chi\left(\xi\right)=2arc\tanh\left(1-4\,r_2\mathrm{e}^{2\,\frac{\sqrt{-\lambda}(a_1r_1+a_0)(\xi+C)}{r_2}}\left(1-\mathrm{e}^{2\,\frac{\sqrt{-\lambda}(a_1r_1+a_0)(\xi+C)}{r_2}}\left(q_2-2\,r_2\right)\right)^{-1}\right),$$

in which

$$\phi=\beta x-\frac{-2\,\lambda\,{a_1}^2{r_1}^2+\beta^2{r_2}^2-4\,\lambda\,a_0a_1r_1+2\,\gamma\,{r_2}^2-2\,\lambda\,{a_0}^2}{\alpha{r_2}^2}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha},\quad \xi=x-\frac{2\beta}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{4.60}$$

Set XII:

$$L=L,\quad M=M,\quad N=N,\quad \beta=\beta,\quad \mu=-\frac{-2\,\lambda\,{a_1}^2{q_1}^2+\beta^2{q_2}^2+2\,\gamma\,{q_2}^2}{{q_2}^2}, \tag{4.61}$$

$$a_0=\frac{a_1\,(q_1r_2-q_2r_1)}{q_2},\quad a_1=a_1,\quad p_1=\frac{p_2q_1}{q_2},\quad p_2=p_2,\quad q_1=q_1,\quad q_2=q_2,\quad r_1=r_1,\quad r_2=r_2.$$

We, therefore, gained the following generalized solitary solution

$$u_{12}(\phi,\xi) = \left(\frac{a_1 q_1}{q_2}\right) e^{i\phi}, \tag{4.62}$$

$$\chi(\xi) = 2arc\tanh\left(\frac{-\tan\left(\frac{1}{2}\,(\xi+C)\sqrt{-L^2+M^2-N^2}\right)\sqrt{-L^2+M^2-N^2}+L}{M-N}\right),$$

in which

$$\phi = \beta x - \frac{-2\,\lambda\, a_1{}^2 q_1{}^2 + \beta^2 q_2{}^2 + 2\,\gamma\, q_2{}^2}{\alpha q_2{}^2}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha},\quad \xi = x - \frac{2\beta}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{4.63}$$

Set XIII:

$$L = L,\quad M = M,\quad N = N,\quad \beta = \beta,\quad \mu = -\frac{1}{2}N^2 - \beta^2 - 2\,\gamma, \tag{4.64}$$

$$a_0 = \frac{1}{2}\,\frac{(2\,M r_1 r_2 - N q_1 r_2 - N q_2 r_1)\sqrt{-\lambda}}{\lambda\, q_1},\quad a_1 = \frac{1}{2}\,\frac{2\,M r_2 - N q_2}{\sqrt{-\lambda} q_1},$$
$$p_1 = q_1,\ p_2 = q_2,\ q_1 = q,\ q_2 = q_2,\ r_1 = r_1,\ r_2 = r_2.$$

We, therefore, gained the following generalized solitary solution

$$u_{13}(\phi,\xi) = \left[\frac{N\left(\frac{1}{2}\,\frac{(2\,M r_1 r_2 - N q_1 r_2 - N q_2 r_1)^2}{\sqrt{-\lambda}} - \frac{1}{2}\,\frac{N q_1 q_2 (\sinh(\chi(\xi)) - \cosh(\chi(\xi)))(2\,M r_1 r_2 - N q_1 r_2 - N q_2 r_1)}{\sqrt{-\lambda}}\right)}{2\,N^2\cosh(\chi(\xi))\, q_1{}^2 q_2{}^2 (\sinh(\chi(\xi)) - \cosh(\chi(\xi))) + G}\right] e^{i\phi}, \tag{4.65}$$

$$G = 4\,M^2 r_1{}^2 r_2{}^2 - 4\,MN q_1 r_1 r_2{}^2 - 4\,MN q_2 r_1{}^2 r_2 + N^2 q_1{}^2 q_2{}^2 + N^2 q_1{}^2 r_2{}^2 + 2\,N^2 q_1 q_2 r_1 r_2 + N^2 q_2{}^2 r_1{}^2,$$

$$\chi(\xi) = 2arc\tanh\left(\frac{-\tan\left(\frac{1}{2}\,(\xi+C)\sqrt{-L^2+M^2-N^2}\right)\sqrt{-L^2+M^2-N^2}+L}{M-N}\right),$$

in which

$$\phi = \beta x + \frac{-\frac{1}{2}N^2 - \beta^2 - 2\,\gamma}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha},\quad \xi = x - \frac{2\beta}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{4.66}$$

Set XIV:

$$L = \frac{\lambda\, a_1 q_1}{\sqrt{-\lambda} r_2},\quad M = \frac{\sqrt{-\lambda} q_1 a_1}{r_2},\quad N = N,\quad \beta = \beta,\quad \mu = -\frac{1}{2}N^2 - \beta^2 - 2\,\gamma, \tag{4.67}$$

$$a_0 = \frac{1}{2}\,\frac{-2\sqrt{-\lambda} a_1 r_1 + N r_2}{\sqrt{-\lambda}},\quad a_1 = a_1,\quad p_1 = p_1,\quad p_2 = 0,\quad q_1 = q_1,\quad q_2 = 0,\quad r_1 = r_1,\quad r_2 = r_2.$$

We, therefore, gained the following generalized solitary solution

$$u_{14}(\phi,\xi) = \left[-\frac{1}{2}\,\frac{-2\,\lambda\, a_1 q_1 \sinh(\chi(\xi)) - 2\,\lambda\, a_1 q_1 \cosh(\chi(\phi)) + N\sqrt{-\lambda} r_2}{\lambda\, r_2}\right] e^{i\phi}, \tag{4.68}$$

$$\chi(\xi) = 2arc\tanh\left(2\sqrt{-\frac{\lambda\, N^2 r_2{}^2}{\left(-1 + \mathrm{e}^{N(\phi+C)}\left(N\sqrt{-\lambda} r_2 + \lambda\, a_1 q_1\right)\right)^2}\mathrm{e}^{N(\phi+C)}} - 1\right),$$

in which

$$\phi = \beta x + \frac{-\frac{1}{2}N^2 - \beta^2 - 2\,\gamma}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha},\quad \xi = x - \frac{2\beta}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{4.69}$$

4.3. **Case III:.** Thereafter we search the exact solution with the below form

$$\Psi(\xi) = \frac{\sin(\chi(\xi))\, a_1 p_1 + \cos(\chi(\xi))\, a_1 q_1 + a_1 r_1 + a_0}{p_2 \sin(\chi(\xi)) + q_2 \cos(\chi(\xi)) + r_2}. \tag{4.70}$$

Inserting (4.70) in to Eq. (4.2), we obtain

$$\left((\mathrm{p_2} \sin(\chi(\xi)) + \mathrm{q_2} \cos(\chi(\xi)) + \mathrm{r_2})^3 \right)^{-1} \sum_{\mathrm{i+j=5}} \mathrm{C_{ij}} \sin^{\mathrm{i}}(\chi(\xi)) \cos^{\mathrm{j}}(\chi(\xi)) = 0, \tag{4.71}$$

where $\mathrm{C_{ij}}(\mathrm{i+j}=5, 0 \le \mathrm{i,j} \le 5)$ are polynomial statements in terms of $a_0, a_1, p_1, p_2, q_1, q_2, r_1$ and r_2. Hence, solving the resulting system $\mathrm{C_{ij}} = 0(\mathrm{i+j}=5, 0 \le \mathrm{i,j} \le 5)$ simultaneously, we acquire the below set of parameters of solutions
Set I:

$$L = 0, \quad M = 0, \quad N = \frac{\sqrt{-\lambda} p_1 a_1}{q_2}, \quad \beta = \beta, \quad \mu = -\frac{2\,\lambda\, {a_1}^2 {p_1}^2 + \beta^2 {q_2}^2 + 2\,\gamma\, {q_2}^2}{{q_2}^2}, \tag{4.72}$$

$$a_0 = -a_1 r_1, \quad a_1 = a_1, \quad p_1 = p_1, \quad p_2 = 0, \quad q_1 = 0, \quad q_2 = q_2, \quad r_1 = r_1, \quad r_2 = 0.$$

We, therefore, gained the following generalized solitary solution

$$u_1(\phi, \xi) = \left(\frac{a_1 p_1 \sin(\chi(\xi))}{q_2 \cos(\chi(\xi))} \right) e^{i\phi}, \quad \chi(\xi) = \frac{\sqrt{-\lambda} p_1 a_1}{q_2}(\xi + C), \tag{4.73}$$

in which

$$\phi = \beta x - \frac{2\,\lambda\, {a_1}^2 {p_1}^2 + \beta^2 {q_2}^2 + 2\,\gamma\, {q_2}^2}{\alpha {q_2}^2} \left(t + \frac{1}{\Gamma(\alpha)} \right)^{\alpha}, \quad \xi = x - \frac{2\beta}{\alpha} \left(t + \frac{1}{\Gamma(\alpha)} \right)^{\alpha}. \tag{4.74}$$

Set II:

$$L = 0, \quad M = 0, \quad N = \frac{2\sqrt{-\lambda} q_1 a_1}{q_2}, \quad \beta = \beta, \quad \mu = -\frac{-2\,\lambda\, {a_1}^2 {q_1}^2 + \beta^2 {q_2}^2 + 2\,\gamma\, {q_2}^2}{{q_2}^2}, \tag{4.75}$$

$$a_0 = -\frac{a_1 (q_1 r_2 + q_2 r_1)}{q_2}, \quad a_1 = a_1, \quad p_1 = i q_1, \quad p_2 = i q_2, \quad q_1 = q_1, \quad q_2 = q_2, \quad r_1 = r_1, \quad r_2 = r_2, \quad i = \sqrt{-1}.$$

We, therefore, gained the following generalized solitary solution

$$u_2(\phi, \xi) = \left[\frac{q_1 a_1 \left({q_2}^2 + 2\, i q_2 \sin(\chi(\xi))\, r_2 - {r_2}^2 \right)}{q_2 \left({q_2}^2 + 2\, \cos(\chi(\xi))\, q_2 r_2 + {r_2}^2 \right)} \right] e^{i\phi}, \quad \chi(\xi) = \frac{2\sqrt{-\lambda} q_1 a_1}{q_2}(\xi + C), \tag{4.76}$$

in which

$$\phi = \beta x - \frac{-2\,\lambda\, {a_1}^2 {q_1}^2 + \beta^2 {q_2}^2 + 2\,\gamma\, {q_2}^2}{\alpha {q_2}^2} \left(t + \frac{1}{\Gamma(\alpha)} \right)^{\alpha}, \quad \xi = x - \frac{2\beta}{\alpha} \left(t + \frac{1}{\Gamma(\alpha)} \right)^{\alpha}. \tag{4.77}$$

Set III:

$$L = 0, \quad M = 0, \quad N = \frac{\sqrt{\lambda} q_1 a_1}{q_2}, \quad \beta = \beta, \quad \mu = -\frac{-2\,\lambda\, {a_1}^2 {q_1}^2 + \beta^2 {q_2}^2 + 2\,\gamma\, {q_2}^2}{{q_2}^2}, \tag{4.78}$$

$$a_0 = -a_1 r_1, \quad a_1 = a_1, \quad p_1 = q_1, \quad p_2 = -q_2, \quad q_1 = q_1, \quad q_2 = q_2, \quad r_1 = r_1, \quad r_2 = 0.$$

We, therefore, gained the following generalized solitary solution

$$u_3(\phi, \xi) = \left[\frac{q_1 a_1 (\cos(\chi(\xi)) + \sin(\chi(\xi)))}{q_2 (\cos(\chi(\xi)) - \sin(\chi(\xi)))} \right] e^{i\phi}, \quad \chi(\xi) = \frac{\sqrt{\lambda} q_1 a_1}{q_2}(\xi + C), \tag{4.79}$$

in which

$$\phi = \beta x - \frac{-2\,\lambda\, {a_1}^2 {q_1}^2 + \beta^2 {q_2}^2 + 2\,\gamma\, {q_2}^2}{\alpha {q_2}^2} \left(t + \frac{1}{\Gamma(\alpha)} \right)^{\alpha}, \quad \xi = x - \frac{2\beta}{\alpha} \left(t + \frac{1}{\Gamma(\alpha)} \right)^{\alpha}. \tag{4.80}$$

Set IV:

$$L = 0, \quad M = 0, \quad N = \frac{\sqrt{-\lambda} q_1 a_1}{p_2}, \quad \beta = \beta, \quad \mu = -\frac{2\,\lambda\, {a_1}^2 {q_1}^2 + \beta^2 {p_2}^2 + 2\,\gamma\, {p_2}^2}{{p_2}^2}, \tag{4.81}$$

$$a_0 = -a_1 r_1, \quad a_1 = a_1, \quad p_1 = -\frac{q_1 q_2}{p_2}, \quad p_2 = p_2, \quad q_1 = q_1, \quad q_2 = q_2, \quad r_1 = r_1, \quad r_2 = 0.$$

We, therefore, gained the following generalized solitary solution

$$u_4(\phi, \xi) = \left[\frac{q_1 a_1 \left(\cos\left(\chi\left(\xi\right)\right) p_2 - \sin\left(\chi\left(\xi\right)\right) q_2\right)}{\left(p_2 \sin\left(\chi\left(\xi\right)\right) + \cos\left(\chi\left(\xi\right)\right) q_2\right) p_2}\right] e^{i\phi}, \quad \chi\left(\xi\right) = \frac{\sqrt{-\lambda} q_1 a_1}{p_2}(\xi + C), \tag{4.82}$$

in which

$$\phi = \beta x - \frac{2\,\lambda\, {a_1}^2 {q_1}^2 + \beta^2 {p_2}^2 + 2\,\gamma\, {p_2}^2}{\alpha {p_2}^2} \left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \quad \xi = x - \frac{2\beta}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{4.83}$$

Set V:

$$L = 0, \quad M = 0, \quad N = \frac{2\sqrt{-\lambda} q_1 a_1}{q_2}, \quad \beta = \beta, \quad \mu = -\frac{-8\,\lambda\, {a_1}^2 {q_1}^2 + \beta^2 {q_2}^2 + 2\,\gamma\, {q_2}^2}{{q_2}^2}, \tag{4.84}$$

$$a_0 = -a_1 r_1, \quad a_1 = a_1, \quad p_1 = 2 i q_1, \quad p_2 = \frac{1}{2} i q_2, \quad q_1 = q_1, \quad q_2 = q_2, \quad r_1 = r_1, \quad r_2 = 0.$$

We, therefore, gained the following generalized solitary solution

$$u_5(\phi, \xi) = \left[2\,\frac{q_1 a_1 \left(3\, i \sin\left(\chi\left(\xi\right)\right) \cos\left(\chi\left(\xi\right)\right) + 2\right)}{q_2 \left(3\,\left(\cos\left(\chi\left(\xi\right)\right)\right)^2 + 1\right)}\right] e^{i\phi}, \quad \chi\left(\xi\right) = \frac{2\sqrt{-\lambda} q_1 a_1}{q_2}(\xi + C), \tag{4.85}$$

in which

$$\phi = \beta x - \frac{-8\,\lambda\, {a_1}^2 {q_1}^2 + \beta^2 {q_2}^2 + 2\,\gamma\, {q_2}^2}{\alpha {q_2}^2} \left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \quad \xi = x - \frac{2\beta}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \tag{4.86}$$

5. Graphical Representation

All of obtained exact solutions for the unstable nonlinear Schrodinger equation are examined, the exact solution (4.30) is similar to the solution of Arbabi [11] $u_{32}(x, t)$, the solution of Lu [31] (4.54) and the solution of Lu [36] (4.65) in the literature. In addition, the exact solution (49) is similar to Lu's [36] (4.39) solution. The other obtained exact solutions that are not included in the literature, and it can be said that they are new exact solutions obtained by the new version trial equation method. Also, two and three dimensional graphics of the obtained solution functions are illustrated in Figures:(1)-(4) which demonstrate with suitable parametric choices.

6. Conclusion

In this paper, the new version of the direct truncation method has been successfully applied to achieve the new exact solutions of unstable nonlinear time-order fractional Schrodinger equation. This method make it possible to attain combined new function solutions. These solutions include hyperbolic function solutions, 1-soliton solutions, dark soliton solitons, rational function solutions, combined dark-bright function solutions and combined soliton solutions. Two and three dimensional graphics of the obtained solution functions were illustrated in Figures:(1)-(4) which demonstrate with suitable parametric choices. We conclude that new solutions of other nonlinear fractional partial differential equations can be obtained with this method. So,this gives the efficient applications of new analytical expansion for fractional PDEs. Moreover, the method applied in this paper provides an effective tool to obtain exact solutions of nonlinear system and can in common use for other NLDEs.

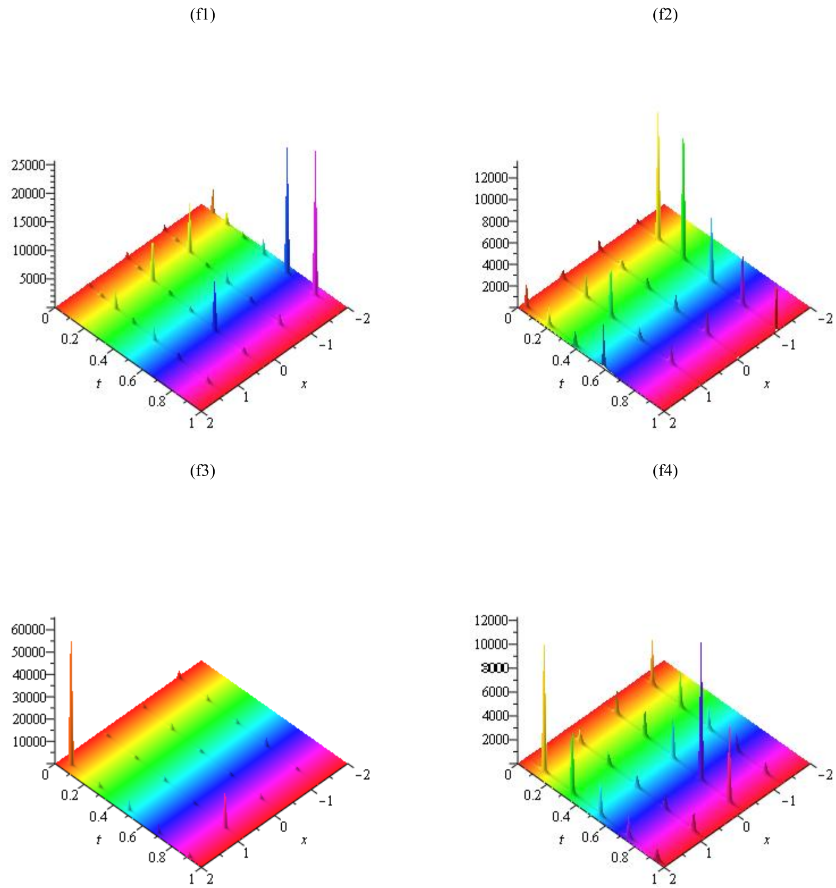

FIGURE 4. The 3D plot of (4.73) at: $\beta = 0.1, \gamma = -1, a_1 = 1.5, p_1 = 2, q_2 = 1, \lambda = -1$ when (f1): $\alpha = 0.25$, (f2): $\alpha = 0.5$, (f3): $\alpha = 0.85$, and (f4): $\alpha = 0.99$.

REFERENCES

[1] S. Abbasbandy and A. Shirzadi, *The first integral method for modified Benjamin-Bona-Mahony equation*, Commun. Nonlinear Sci. Numer. Simul., *15* (2010), 1759-1764.

[2] M. A. Abdou, *Further improved F-expansion and new exact solutions for nonlinear evolution equations*, Nonlinear Dyn., *52* (2008), 277-288.

[3] M. A. Akbar, N. H. M. Ali, and S. T. Mohyud-Din, *The modified alternative G'/G-expansion method to nonlinear evolution equation: application to the (1+1)-dimensional Drinfel'd-Sokolov-Wilson equation*, Springer Plus, *327* (2013), 2-16.

[4] M. N. Alam, *Exact solutions to the foam drainage equation by using the new generalized G'/G-expansion method*, Results Phys., *5* (2015), 168-177.

[5] M. N. Alam and M. A. Akbar, *Traveling wave solutions for the mKdV equation and the Gardner equations by new approach of the generalized (G'/G)-expansion method*, J. Egyptian Math. Soc., *22* (2014), 402-406.

[6] M. N. Alam, M. A. Akbar, and M. F. Houque, *Exact travelling wave solutions of the (3+1)-dimensional mKdV-ZK equation and the (1+1)-dimensional compound KdVB equation using the new approach of generalized (G'/G)-expansion method*, Pramana, *83* (2014), 317-329.

[7] M. N. Alam and M. M. Alam, *An analytical method for solving exact solutions of a nonlinear evolution equation describing the dynamics of ionic currents along microtubules*, J. Taibah Univ. Sci., *11* (2017), 939-948.

[8] M. N. Alam and X. Li, *Exact traveling wave solutions to higher order nonlinear equations*, J. Ocean Eng. Sci., *4* (2019), 276-288.

[9] M. N. Alam and X. Li, *New soliton solutions to the nonlinear complex fractional Schrodinger equation and the conformable time-fractional Klein-Gordon equation with quadratic and cubic nonlinearity*, Phys. Scr., *95* (2020), 045224.

[10] M. N. Alam and C. Tunç, *The new solitary wave structures for the (2 + 1)-dimensional time-fractional Schrodinger equation and the space-time nonlinear conformable fractional Bogoyavlenskii equations*, Alexandria Eng. J., *59* (2020), 2221-2232.

[11] S. Arbabi and M. Najafi, *Exact solitary wave solutions of the complex nonlinear Schrödinger equations*, Optik., *127* (2016), 4682-4688.

[12] A. Atangana and D. Baleanu, *New fractional derivatives with nonlocal and non-singular kernel, Theory and application to heat transfer model*, Therm Sci., *20* (2016), 763-769.

[13] A. Atangana and I. Koca, *Chaos in a simple nonlinear system with Atangana-Baleanu derivative with fractional order*, Chaos Solitons Frac., *89* (2016), 447-454.

[14] H. M. Baskonus, A. Kumar, A. Kumar, and W. Gao, *Deeper investigations of the (4+1)-dimensional Fokas and (2+1)-dimensional Breaking soliton equations*, Int. J. Modern Phys. B, *34* (2020) 2050152.

[15] S. T. Demiray, Y. Pandir, and H. Bulut, *New solitary wave solutions of Maccari system*, Ocean Eng., *103* (2015), 153-159.

[16] S. T. Demiray, Y. Pandir, and H. Bulut, *New soliton solutions for Sasa-Satsuma equation*, Waves Random Complex Media, 25 (3) (2015), 417-428.

[17] Z. Fu, S. Liu and Q. Zhao, *New Jacobi elliptic function expansion and new periodic solutions of nonlinear wave equations*, Phys. Lett. A., *290* (2001), 72-76.

[18] W. Gao, G. Yel, H. M. Baskonus, and C. Cattani, *Complex Solitons in the Conformable (2+1)-dimensional Ablowitz-Kaup-Newell-Segur Equation*, Aims Math., *5* (2020), 507-521.

[19] J. L. G. Guirao, H. M. Baskonus and A. Kumar, *Regarding New Wave Patterns of the Newly Extended Nonlinear (2+1)-Dimensional Boussinesq Equation with Fourth Order*, Math., *8* (2020), 341.

[20] S. Guo and and Y. Zhou, *The extended G'/G-expansion method and its applications to the Whitham-Broer-Kaup like equations and coupled Hirota-Satsuma KdV equations*, Appl. Math. Comput., *215* (2010), 3214-3221.

[21] Y. Gurefe, E. Misirli, A. Sonmezoglu, and M. Ekici, *Extended trial equation method to generalized nonlinear partial differential equations*, Appl. Math. Comput., *219* (2013), 5253-5260.

[22] Y. Gurefe, A. Sonmezoglu, and E. Misirl, *Application of an irrational trial equation method to high dimensional nonlinear evolution equations*, J. Adv. Math. Stud., *5* (2012), 41-47.

[23] Y. Gurefe, A. Sonmezoglu, and E. Misirl, *Application of trial equation method to the nonlinear partial differential equations arising in mathematical physics*, Pramana-J. Phys., *77* (2011), 1023-1029.

[24] J. H. He and X. H. Wu, *Exp-function method for nonlinear wave equations*, Chaos Solitons Frac., *30* (2006), 700-708.

[25] J. Hietarinta, *Hirota's bilinear method and its generalization*, Int. J. Mod. Phys. A., *12* (1997), 43-51.

[26] K. Hosseini, M. Samavat, M. Mirzazadeh, W. X. Ma, and Z. Hammouch, *A New (3+1)-dimensional Hirota Bilinear Equation: Its Bäcklund Transformation and Rational-type Solutions*, Regular Chaotic Dyn., *25* (2020), 383-391.

[27] A. Houwe, J. Sabi'u, Z. Hammouch, and S. Y. Doka, *Solitary pulses of a conformable nonlinear differential equation governing wave propagation in low-pass electrical transmission line*, Phys. Scr., *5* (2020), 4027-4044.

[28] E. Ilhan and I. O. Kiymaz, *A generalization of truncated M-fractional derivative and applications to fractional differential equations*, Appl. Math. Nonlinear Sci., *5* (2020), 171-188.

[29] H. F. Ismael, H. Bulut, H. M. Baskonus, and W. Gao, *Newly modified method and its application to the coupled Boussinesq equation in ocean engineering with its linear stability analysis*, Commun. Theo. Phys., *72* (2020), 115002 (8pp).

[30] X. J. Laia, J. F. Zhang, and S. H. Mei, *Application of the Weierstrass elliptic expansion method to the long-wave and short-wave resonance interaction system*, Z. Naturforsch., *63a* (2008), 273-279.

[31] D. Lu, A. R. Seadawy and A. Ali, *Structure of traveling wave solutions for some nonlinear models via modified mathematical method*, Open Phys., *16* (2018), 854-860.

[32] C. S. Liu, *A new trial equation method and its applications*, Commun. Theor. Phys., *45* (2006), 395-397.

[33] C. S. Liu, *Applications of complete discrimination system for polynomial for classifications of traveling wave solutions to nonlinear differential equations*, Commun. The. Phys., 181 (2010), 317-324.

[34] C. S. Liu, *Trial equation method and its applications to nonlinear evolution equations*, Phys. Sinica, *54* (2005), 2505-2509.

[35] C. S. Liu, *Using trial equation to solve the exact solutions for two kinds of KdV equations with variable coeffients*, Acta Phys. Sinica, *54* (2005), 4506-4510.

[36] D. Lu, A. R. Seadawy, and M. Arshad, *Bright-dark solitary wave and elliptic function solutions of unstable nonlinear Schrodinger equation and their applications*, Opt. Quant. Elec., *50* (2018), 1-10.

[37] C. S. Liu, *Trial equation method for nonlinear evolution equations with rank inhomogeneous: mathematical discussions and applications*, Commun. The. Phys., 45 (2006), 219-223.

[38] W. X. Ma and B. Fuchssteiner, *Explicit and exact solutions to a Kolmogrov- Petrovski-Piskunov equation*, Int. J. Nonlinear Mech., *31* (1996), 329-338.

[39] W. Malfliet, *The tanh method: a tool for solving certain classes of nonlinear evolution and wave equations*, J. Comput. Appl. Math., *164 -165* (2004), 529-541.

[40] W. Malfliet and W. Hereman, *The tanh method: I exact solutions of nonlinear evolution and wave equations*, Phys. Scr., *54* (1996), 563-568.

[41] Y. Pandir, *A new type of the generalized F-expansion method and its application to Sine-Gordon equation*, Celal Bayar Univ. J. Sci., *13* (2017), 647-650.

[42] Y. Pandir, *Symmetric fibonacci function solutions of some nonlinear partial differential equations*, Appl. Math. Inf. Sci., *8* (2014), 2237-2241.

[43] Y. Pandir, S. T. Demiray, and H. Bulut, *A new approach for some NLDEs with variable coefficients*, Optik., *127* (2016), 11183-11190.

[44] Y. Pandir, Y. Gurefe, U. Kadak, and E. Misirli, *Classifications of exact solutions for some nonlinear partial differential equations with generalized evolution*, Abst. Appl. Anal., 2012 (2012), ID 478531.

[45] Y. Pandir, Y. Gurefe, and E. Misirli, *Classification of exact solutions to the generalized Kadomtsev-Petviashvili equation*, Phys. Scr., *87* (2013), 025003.

[46] Y. Pandir, Y. Gurefe, and E. Misirli, *A multiple extended trial equation method for the fractional Sharma-Tasso-Olver equation*, AIP Conf. Proc., *1558* (2013), 1927.

[47] Y. Pandir, A. Sonmezoglu, H. H. Duzgun, and N.Turhan, *Exact solutions of nonlinear Schrodinger's equation by using generalized Kudryashov method*, AIP Conf. Proc., *1648* (2015), 370004.

[48] Y. Pandir and N. Turhan, *A new version of the generalized F-expansion method and its applications*, AIP Conf. Proc., *1798* (2017), 020122.

[49] O. Pashaev and G. Tanoglu, *Vector shock soliton and the Hirota bilinear method*, Chaos Solitons Frac., *26* (2005) 95-105.

[50] L. K. Rav, S. S. Ray, and S. Sahoo, *New exact solutions of coupled Boussinesq-Burgers equations by exp-function method*, J. Ocean Eng. Sci., *2* (2017), 34-46.

[51] K.M. Saad, A. Atangana, and D. Baleanu, New fractional derivatives with non-singular kernel applied to the burgers equation, Chaos, 28 (2018), 63-109.

[52] M. Shakee and S. T. Mohyud-Din, *New G'/G-expansion method and its application to the Zakharov-Kuznetsov- VBenjamin-Bona-Mahony (ZK- VBBM) equation*, J. Association Arab Uni. Basic Appl. Sci., *18* (2015), 66-81.

[53] S. Shen and Z. Pan, *A note on the Jacobi elliptic function expansion method*, Phys. Lett. A., *308* (2003), 143-148.

[54] R. Silambarasan, H. M. Baskonus, R. V. Anand, M. Dinakaran, B. Balusamy, and W. Gao, *Longitudinal strain waves propagating in an infinitely long cylindrical rod composed of generally incompressible materials and its Jacobi elliptic function solutions*, Math. Comput. Simul., *182* (2021), 566-602.

[55] Y. A. Tandogan, Y. Pandir, and Y. Gurefe, *Solutions of the nonlinear differential equations by use of modified Kudryashov method*, Turkish J. Math. Comput. Sci., *1* (2013), 54-60.

[56] E. T. Tebue, Z.I. Djoufack, E. F. Donfack, A. K. Jiotsa, and T. C. Konfane, *Exact solutions of the unstable nonlinear Schrödinger equation with the new Jacobi elliptic function rational expansion method and the exponential rational function method*, Optik, *127* (2016), 11124-11130.

[57] M. Wang and X. Li, *Applications of F-expansion to periodic wave solutions for a new Hamiltonian amplitude equation*, Chaos Solitons Frac., 24 (2005), 1257-1268.

[58] K. Yang and J. Liu, *The extended F-expansion method and exact solutions of nonlinear PDEs*, Chaos Solitons Frac., *22* (2004), 111-121.

[59] G. Yel, H. M. Baskonus, and W. Gao, *New dark-bright soliton in the shallow water wave model*, AIMS Math., *5* (2020) 4027-4044.

[60] J. Zhang, F. Jiang, and X. Zhaok, *An improved an improved (G'/G)-expansion method for solving nonlinear evolution equations*, Int. J. Comput. Math., *87* (2010), 1716-1725.