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Analytical solution for descriptor system in partial differential equations

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Abstract We consider a first-order partial differential equation with constant irreversible coefficients in a Banach space in the regular case. The equation is split into equations in subspaces, in which non-degenerate subsystems are obtained. We obtain an analytical solution of each system with Showalter-type conditions. Finally, an example is given to illustrate the theoretical results.

Keywords. Banach space, Descriptor system, Differential algebraic equations, 0-normal eigenvalue, Showalter-type conditions.

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1. Introduction

We consider the following equation

$$A(\frac{\partial U}{\partial t} + \alpha U) = B(\frac{\partial U}{\partial x} + \beta U) + f(t, x), \quad t \in T, x \in X, \tag{1.1}$$

where $A, B : E \to E$, E Banach space, A-closed linear operator with a dense domain $D(A)$ in E. Zero is the normal eigenvalue of the operator A, $B \in L(E, E)$, B^{-1} does not exist, $f(t, x)$ - continuous vector-function where his range is a subset of the banach space E, $U = U(t, x) \in E$, α, β are scaler functions; $\alpha = \alpha(t, x), \beta = \beta(t, x)$, $(t, x) \in T \times X$, where $T = [0, t_0], X = [0, x_0]$.

Solving equation (1.1) leads to the understanding that function $U(t, x)$ is continuously differentiable $\forall (t, x) \in T \times X$, which is satisfying (1.1) for all t, x.

Due to the wide applications of analytical solutions [4, 5, 6, 7], we attempt to find the analytical solution of equation (1.1) with some boundary conditions.

Unsolvable equations with respect to derivative were studied in many works. The first one was Poincare in 1885. This type of equations are called differential algebraic equations or descriptor systems. Significant results for unsolvable differential

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equations were obtained in Voronezh school (S.G. Krein, A.G. Baskakov, S.P. Zubova and others), in Chelyabinsk school (G. A. Sviridyuk, V. E. Fedorov, and their students), in Irkutsk school (Yu. E. Boyarintsev, N. A. Sidorov, A. A. Shcheglova, V. F. Chistyakov, M. F. Falaleev, and their students), in Yekaterinburg school (I. V. Melnikova and her students)[16]. Ongoing researches have been conducted abroad by A. Favini, A. Yagi, S.Campbell, P. Kunkel, V. Mehrmann, R. Marz, S. M. Wade, I. B. Paul, and others[16].

Degenerate systems of partial differential equations have applications in various fields: hydrodynamics (Navier-Stokes equations), thermal engineering, electrical engineering, etc. [1, 11, 3, 8, 2, 9].

The regular case has been taken into consideration: when λ is sufficiently small in modulus but not zero ($\lambda \in \dot{U}(0)$), the operator pencil $(A - \lambda B)$ is invertible. When Operator A has 0 normal eigenvalue, Banach space E can be decomposed into the direct sum of two subspaces [15]

$$E = M \oplus N, \tag{1.2}$$

N is the root subspace of operator A, M is invariant subspace with respect to operator A. Restriction $\tilde{A}$ of operator A in M has a bounded inverse operator $\tilde{A}^{-1}$ [10]. In this paper, we consider dim ker A = 1.

2. Preliminaries

Assume that the null space of operator A is one-dimensional; $e_1 \in KerA$, $\{e_i\}$ - jordan chain, where $Ae_1 = 0, Ae_{i-1} = e_i$, $i = 2, ..., n$. This means
$N = Lin\{e_1, e_2, e_3, ..., e_n\}$.

In the subspace N, we introduced inner product $<(\cdot),(\cdot)>$, to get orthonormal basis $\{e_1, e_2, e_3, ..., e_n\}$. Projectors on M and N corresponding to decomposition (1.2) are represented with Q and P, respectively.

In this section, the following obtained lemmas and theorem 2.4 are in [12, 14, 13].

Lemma 2.1. *Equation $Av = w, \quad v, w \in E$, is solvable with respect to v if and only if*

$$< Pw, e_n >= 0, \tag{2.1}$$

wherein

$$v = A^{-}w + ce_1, \quad \forall c \in \mathbb{C}, \tag{2.2}$$

where $A^{-} = \tilde{A}^{-1}Q + A_1P$,

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & ... & 0 & 0 \\ 1 & 0 & 0 & ... & 0 & 0 \\ 0 & 1 & 0 & ... & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & ... & 1 & 0 \end{pmatrix},$$

$A_1P : N \to N$, $\tilde{A}^{-1}Q : M \to M$.

Condition (2.1) is the condition for the correctness of equation $Av = w$.

Lemma 2.2. $\exists(A - \lambda B)^{-1}, \forall\lambda \in \dot{U}(0)$ *if and only if* $\exists q \in \mathbb{N}$ *such that,* $< PB(A^{-}B)^{q-1}e_1, e_n >\neq 0$.

p - is the minimum of q, such that

$$< PB(A^{-}B)^{q-1}e_1, e_n >\neq 0.$$

v_i is the B-adjoint elements to e_1:

$$Av_1 = 0, \quad Av_i = Bv_{i-1}, \quad i = 2, 3, ... \quad . \tag{2.3}$$

It is known that in [15] the pencil $(A - \lambda B)$ is regular if and only if the B-Jordan chain v_i has finite length. The following results were obtained from [12, 14, 13].

Lemma 2.3. *The length of the chain* v_i *is equal to p; therefore, we can take elements* v_i *as the following:*

$$v_1 = e_1, \quad v_i = (A^{-}B)^{i-1}e_1, \quad i = 2, 3, ..., p \quad . \tag{2.4}$$

Theorem 2.4. *Operator* $A_\lambda = (A - \lambda B)^{-1}A$ *for some* $\lambda \in \dot{U}(0)$ *has 0 normal eigenvalue, so we can write* E *as the direct sum of two subcpaces:*

$$E = M_1 \oplus N_1, \tag{2.5}$$

where N_1 *is the root subspace of* A_λ*, the invariant subspace with respect to operator* A_λ *is the subspace* M_1*, where:*

$$M_1 = \{z \in E :< PB(A^{-}B)^{i}z, e_n >= 0, < PB(A^{-}B)^{p}z, e_n >\neq 0\}, \tag{2.6}$$

$i = 1, 2, ..., p - 1$.

Lemma 2.5. *Elements of the subspace* N_1 *are linear combinations of* v_i *elements, i = 1, 2, 3, ..., p.*

The projectors on N_1 and M_1 corresponding to decomposition (2.5) are represented with P_1 and Q_1, respectively. $P_1 + Q_1 = I$. The restriction $\tilde{A_\lambda}$ of the operator A_λ on M_1 has a bounded inverse operator $(\tilde{A_\lambda})^{-1}$.

3. Equation splitting

substitute $B = \dfrac{1}{\lambda}(\lambda B - A + A)$ and multiply equation (1.1) from the left by $(A - \lambda B)^{-1}$ to get the following equation

$$A_\lambda(\frac{\partial U}{\partial t} + \alpha U) = \frac{A_\lambda - I}{\lambda}\frac{\partial U}{\partial x} + \frac{A_\lambda - I}{\lambda}\beta U + (A - \lambda B)^{-1}f(t, x). \tag{3.1}$$

Substitute $U(t, x)$ in the form $Q_1U(t, x) + P_1U(t, x)$ in the last equation. As a result the vector function $(A - \lambda B)^{-1}f(t, x)$ splits into functions in subspaces N_1 and M_1.

Since subspaces N_1 and M_1 are invariant with respect to A_λ, and $N_1 \cap M_1 = \{0\}$ we get the following equations. In subspace N_1

$$A_\lambda(\frac{\partial P_1 U}{\partial t} + \alpha P_1 U) = \frac{A_\lambda - P_1}{\lambda}\frac{\partial P_1 U}{\partial x} + \frac{A_\lambda - P_1}{\lambda}\beta P_1 U \\ + P_1(A-\lambda B)^{-1} f(t,x), \tag{3.2}$$

in subspace M_1

$$\tilde{A}_\lambda(\frac{\partial Q_1 U}{\partial t} + \alpha Q_1 U) = \frac{\tilde{A}_\lambda - Q_1}{\lambda}\frac{\partial Q_1 U}{\partial x} + \frac{\tilde{A}_\lambda - Q_1}{\lambda}\beta Q_1 U \\ + Q_1(A-\lambda B)^{-1} f(t,x). \tag{3.3}$$

4. Formulation of the problem

Equation (1.1) splits into equations in subspaces N_1 and M_1. Each one is solvable under the Showalter-type conditions:

in subspace N_1:

$$P_1 U(t,0) = \psi(t) \in N_1, \tag{4.1}$$

in subspace M_1:

$$Q_1 U(0,x) = \phi(x) \in M_1. \tag{4.2}$$

Our goal is to construct the solution of equation (1.1) with boundary conditions (4.1), (4.2) in the following form

$$U(t,x) = Q_1 U(t,x) + P_1 U(t,x). \tag{4.3}$$

5. Solution of the problem in the root subspace

To solve the problem (3.2), (4.1) elements $P_1 U(t,x)$ and $P_1(A-\lambda B)^{-1} f(t,x)$ are linear combinations of the basis $\{v_i\}, i = 1,2,...,p$, therefore we can write them as the following:

$$P_1 U(t,x) = \sum_{i=1}^{p} u_i v_i, \quad u_i = u_i(t,x), \tag{5.1}$$

$$P_1(A-\lambda B)^{-1} f(t,x) = \sum_{i=1}^{p} F_i v_i, \quad F_i = F_i(t,x,\lambda). \tag{5.2}$$

From (5.1), (5.2) equation (3.2) can be written as the following:

$$\sum_{i=1}^{p} \frac{\partial u_i}{\partial t} A_\lambda v_i + \alpha \sum_{i=1}^{p} u_i A_\lambda v_i = \frac{1}{\lambda}\sum_{i=1}^{p} \frac{\partial u_i}{\partial x}(A_\lambda - P_1) v_i \\ + \frac{\beta}{\lambda}\sum_{i=1}^{p} u_i (A_\lambda - P_1) v_i + \sum_{i=1}^{p} F_i v_i. \tag{5.3}$$

Lemma 5.1.

$$A_\lambda v_i = -\sum_{k=1}^{i-1} \frac{1}{\lambda^{i-k}} v_k. \tag{5.4}$$

Proof. We can write the element $A_\lambda v_i$ as it belongs to N_1 as in the following

$$A_\lambda v_i = \sum_{j=1}^{p} c_{ij} v_j. \tag{5.5}$$

From the above equation, we get

$$Av_i = \sum_{j=1}^{p} c_{ij}(A - \lambda B) v_j. \tag{5.6}$$

Elements Av_j in (5.6) are replaced with Bv_{j-1}, $j = 2, 3, ..., p$, $Av_1 = 0$, therefore relation (5.5) is equivalent to the next relation

$$Av_i = \sum_{j=1}^{p-1} (c_{ij+1} - \lambda c_{ij}) Bv_j - \lambda c_{ip} Bv_p. \tag{5.7}$$

By using lemma 2.1, equation (5.7) is true if and only if

$$\sum_{j=1}^{p-1} (c_{ij+1} - \lambda c_{ij}) < PBv_j, e_n > -\lambda c_{ip} < PBv_p, e_n >= 0, \tag{5.8}$$

wherein

$$v_i = \sum_{j=1}^{p-1} (c_{ij+1} - \lambda c_{ij}) A^- Bv_j - \lambda c_{ip} A^- Bv_p + ce_1, \forall c \in \mathbb{C}. \tag{5.9}$$

By condition (2.4) and the definition of p in equation (5.8), $< PBv_j, e_n >= 0, \forall j = 1, 2, ..., p-1$, so $c_{ip} = 0$. Now (5.9) has the form:

$$v_i = \sum_{j=1}^{p-1} (c_{ij+1} - \lambda c_{ij}) v_{j+1} + c.e_1. \tag{5.10}$$

By comparing the equal coefficients of v_i in (5.10), we obtain:
for $i = 2$:

$$c_{2j} = 0, (j = 2, 3, ..., p-1) \to c_{21} = \frac{-1}{\lambda},$$

for $i = 3$:

$$c_{3j} = 0, (j = 3, 4, ..., p-1), \to c_{32} = \frac{-1}{\lambda}, c_{31} = \frac{1}{\lambda} c_{32},$$

$$\cdots\cdots\cdots\cdots$$

for $i = k$:

$$c_{kj} = 0, (j = k, k+1, ..., p-1), \to c_{kk-1} = \frac{-1}{\lambda}, c_{kk-2} = \frac{1}{\lambda} c_{kk-1}, ..., c_{k1} = \frac{1}{\lambda} c_{k2},$$

from the above and (5.5):

$$A_\lambda v_i = \frac{-1}{\lambda^{i-1}}v_1 - \frac{1}{\lambda^{i-2}}v_2 - \cdots - \frac{1}{\lambda}v_{i-1} = -\sum_{k=1}^{i-1}\frac{1}{\lambda^{i-k}}v_k.$$

□

By considering formula (5.4), equation (5.3) can be written as the following:

$$\begin{aligned} &-\sum_{i=2}^{p}\Big(\frac{\partial u_i}{\partial t} - \frac{1}{\lambda}\frac{\partial u_i}{\partial x}\Big)\sum_{k=1}^{i-1}\frac{1}{\lambda^{i-k}}v_k + \frac{1}{\lambda}\sum_{i=1}^{p}\frac{\partial u_i}{\partial x}v_i \\ &= -\sum_{i=2}^{p}\Big(\frac{\beta}{\lambda}u_i - \alpha u_i\Big)\sum_{k=1}^{i-1}\frac{1}{\lambda^{i-k}}v_k + \sum_{i=1}^{p}\Big(F_i - \frac{\beta}{\lambda}u_i\Big)v_i. \end{aligned} \tag{5.11}$$

In equation (5.11), a comparison between the equal coefficients of v_i leads to the following relations:

$$\begin{aligned} &\frac{\partial u_p}{\partial x} = \lambda F_p - \beta(t,x)u_p, \\ &\frac{\partial u_i}{\partial x} = \frac{\partial u_{i+1}}{\partial t} + \alpha(t,x)u_{i+1} - \beta(t,x)u_i + (\lambda F_i - F_{i+1}), \end{aligned} \tag{5.12}$$

i = 1, 2, ..., p-1.

Lemma 5.2. *Functions $\lambda F_p, \lambda F_i - F_{i+1}, i = 1, ..., p-1$ are independent of λ.*

Proof. Expression $(A-\lambda B)^{-1}f(t,x)$ standing on the right side of equation (3.1) has the following form:

$$(A-\lambda B)^{-1}f(t,x) = \sum_{i=1}^{p}F_i v_i + Q_1F, \quad Q_1F = Q_1(A-\lambda B)^{-1}f(t,x).$$

From here

$$f(t,x) = \sum_{i=1}^{p}F_i(A-\lambda B)v_i + (A-\lambda B)Q_1F. \tag{5.13}$$

By the definition of the basis $v_i, i = 1, 2, ..., p$, we get the following:

$$\sum_{i=1}^{p}F_i(A-\lambda B)v_i = \sum_{i=1}^{p-1}(F_{i+1}-\lambda F_i)Bv_i - \lambda F_pBv_p. \tag{5.14}$$

Consequently, expression (5.13) is:

$$f(t,x) = \sum_{i=1}^{p-1}(F_{i+1}-\lambda F_i)Bv_i - \lambda F_pBv_p + (A-\lambda B)Q_1F,$$

therefore

$$AQ_1F = f(t,x) + \lambda BQ_1F - \sum_{i=1}^{p-1}(F_{i+1}-\lambda F_i)Bv_i + \lambda F_pBv_p. \tag{5.15}$$

By lemma 2.1, equation (5.15) is solvable with respect to Q_1F if and only if

$$\begin{aligned} < Pf, e_n > = \sum_{i=1}^{p-1}(F_{i+1} - \lambda F_i) < PBv_i, e_n > \\ - \lambda < PBQ_1F, e_n > -\lambda F_p < PBv_p, e_n >, \end{aligned} \tag{5.16}$$

wherein

$$\begin{aligned} Q_1F = A^- f(t,x) + \lambda A^- BQ_1F - \sum_{i=1}^{p-1}(F_{i+1} - \lambda F_i)A^- Bv_i \\ + \lambda F_p A^- Bv_p + cA^- e_1, \end{aligned} \tag{5.17}$$

so that

$$\begin{aligned} A^- f(t,x) = \sum_{i=1}^{p-1}(F_{i+1} - \lambda F_i)A^- Bv_i \\ - \lambda F_p A^- Bv_p + (I - \lambda A^- B)Q_1F + ce_1, \forall c \in \mathbb{C}. \end{aligned} \tag{5.18}$$

By (2.4), (2.6), and definition of p, all terms in the right side of equation (5.16) equal zero except for the last one, therefore

$$< Pf, e_n >= -\lambda F_p < PBv_p, e_n > . \tag{5.19}$$

Applying the inner product $< PB(\cdot), e_n >$ to both sides of equation (5.18) gives the following

$$< PBA^- f, e_n >= (F_p - \lambda F_{p-1}) < PBv_p, e_n > -\lambda F_p < PBA^- Bv_p, e_n > .$$

Applying the inner product $< PB(A^-B)^{k-1}(\cdot), e_n >$ to both sides of equation (5.18) gives the relation

$$\begin{aligned} < P(BA^-)^k f, e_n > = \sum_{i=1}^{k}(F_{p-j+1} - \lambda F_{p-j}) < PB(A^-B)^{k-j} v_p, e_n > \\ - \lambda F_p < PB(A^-B)^k v_p, e_n >, \quad k = 2, 3, .., p-1. \end{aligned} \tag{5.20}$$

From equations (5.19), (5.20) for $k = 1, 2, ..., p-1$ the functions $\lambda F_p, \lambda F_i - F_{i+1}, i = 1, ..., p-1$ do not depend on λ. □

As a result, we can write these functions as the following

$$\begin{aligned} \lambda F_p &= \Phi_p(t,x), \\ \lambda F_i - F_{i+1} &= \Phi_i(t,x). \end{aligned} \tag{5.21}$$

Equations in (5.12) have the following forms

$$\begin{aligned} \frac{\partial u_p}{\partial x} &= \Phi_p - \beta(t,x)u_p, \\ \frac{\partial u_i}{\partial x} &= \frac{\partial u_{i+1}}{\partial t} + \alpha(t,x)u_{i+1} - \beta(t,x)u_i + \Phi_i, i = 1, 2, ..., p-1. \end{aligned} \tag{5.22}$$

To find $P_1 U(t,x) = \sum_{i=1}^{p} u_i v_i$, substitute $\psi(t) \in N_1$ in equation (4.1) as the form

$$\psi(t) = \sum_{i=1}^{p} \psi_i(t) v_i,$$

then

$$u_i(t,0) = \psi_i(t), i = 1, 2, ..., p. \tag{5.23}$$

By the first equation of (5.22) and condition (5.23) with $i = p$, we define

$$\begin{aligned} u_p(t,x) = & \int_0^x \Phi_p(t,z) exp\left(\int_x^z \beta(t,s)ds\right) dz \\ & + exp\left(-\int_0^x \beta(t,s)ds\right) \psi_p(t). \end{aligned} \tag{5.24}$$

Next, we substitute (5.24) in the second equation of (5.22) with $i = p-1$ to get $u_{p-1}(t,x)$. Repeat this process for $i = p-2, p-3, ..., 1$ to get all elements $u_i(t,x)$. Consequently, the solution of system (5.12) with conditions in (5.23) leads to the following result.

Lemma 5.3. *Assume that $f(t,x)$ is continuously differentiable $(p+1)$ times with respect to t, and $\psi(t)$ is continuously differentiable $(p+1)$ times. As a result, the solution of (3.2), (4.1) does exist and is unique, independent of λ and is determined from (5.24), equations of (5.22), $i = 1, 2, ..., p-1$, and condition (5.23) with $i = 1, 2, ..., p-1$.*

Remark 5.4. The smoothness conditions for the functions $f(t,x)$ and $\psi(t)$ can be weakened by the fact that different smoothness is required from different components of them in N_1.

6. The solution of the problem in the complementary subspace

Equation (3.3) is solvable with respect to the partial derivative of t due to the existence of the inverse of operator $\tilde{A}_\lambda$ in subspace M_1:

$$\begin{aligned} \frac{\partial Q_1 U}{\partial t} + \alpha Q_1 U = & \frac{Q_1 - \tilde{A_\lambda}^{-1}}{\lambda} \frac{\partial Q_1 U}{\partial x} + \frac{Q_1 - \tilde{A_\lambda}^{-1}}{\lambda} \beta Q_1 U \\ & + \tilde{A_\lambda}^{-1} Q_1 (A - \lambda B)^{-1} f(t,x). \end{aligned} \tag{6.1}$$

The right-hand side of this equation and the operator $\dfrac{Q_1 - \tilde{A_\lambda}^{-1}}{\lambda}$ at the derivative with respect to x are independent of λ [16]. We earlier proved that the operator $P_1(A - \lambda B)^{-1}$ does not depend on λ, but operator $Q_1(A - \lambda B)^{-1}$ does depend on λ. However, we get the following lemma.

Lemma 6.1. *Operator $\tilde{A_\lambda}^{-1} Q_1 (A - \lambda B)^{-1}$ does not depend on λ.*

Proof. All elements of equation (6.1) except for $\tilde{A_\lambda}^{-1}Q_1(A-\lambda B)^{-1}f(t,x)$ do not depend on λ and $f(t,x)$ does not depend on λ; therefore the operator $\tilde{A_\lambda}^{-1}Q_1(A-\lambda B)^{-1}$ does not depend on λ. $\square$

Let's move on to solve the problem (6.1), (4.2). We introduce the following notations:

$$G=\frac{Q_1-\tilde{A_\lambda}^{-1}}{\lambda},$$
$$L(t,x)=\beta(t,x)G-\alpha(t,x)Q_1,$$
$$h(t,x)=\tilde{A_\lambda}^{-1}Q_1(A-\lambda B)^{-1}f(t,x).$$

As a result, equation (6.1) can be written as the following:

$$\frac{\partial Q_1U}{\partial t}=G\frac{\partial Q_1U}{\partial x}+L(t,x)Q_1U+h(t,x). \tag{6.2}$$

The solution of this equation with the condition (4.2) is:

$$\begin{aligned}Q_1U&=exp\left(\int_0^t L(s,D_1)ds\right)\phi(D_2)\\ &+\int_0^t exp\left(\int_\tau^t L(s,D_1)ds\right)h(\tau,D_3)d\tau,\end{aligned}\tag{6.3}$$

where, $D_1=(t-s)G+xQ_1,\quad D_2=tG+xQ_1,\quad D_3=(t-\tau)G+xQ_1,$

$$L(s,D_1)=\frac{-1}{2\pi i}\oint_\Gamma\Big((t-s)G+(x-\mu)Q_1\Big)^{-1}L(s,\mu)d\mu,$$
$$\phi(D_2)=\frac{-1}{2\pi i}\oint_\Gamma\Big(tG+(x-\mu)Q_1\Big)^{-1}\phi(\mu)d\mu,$$
$$h(\tau,D_3)=\frac{-1}{2\pi i}\oint_\Gamma\Big((t-\tau)G+(x-\mu)Q_1\Big)^{-1}h(\tau,\mu)d\mu,$$

where, Γ - closed rectifiable Jordan contour, surrounding spectrum of bounded operators D_1, D_2, and D_3.

Theorem 6.2. *Under the conditions of lemma 5.3, solution of the problem (1.1), (4.1), (4.2) exists, is unique in E, determined by (4.3), (5.1), (5.24) and solutions of (5.22) with conditions (5.23) as well as formula (6.3).*

Example 6.3. *In the space ℓ_2, the following system is given*

$$\begin{aligned}&0=\frac{\partial U_1}{\partial x}+\frac{\partial U_2}{\partial x}+\beta U_1+\beta U_2,\quad \frac{\partial U_2}{\partial t}+\alpha U_2=\frac{\partial U_3}{\partial x}+\beta U_3,\\ &\frac{\partial U_3}{\partial t}+\alpha U_3=\frac{\partial U_4}{\partial x}+\beta U_4,\quad \frac{\partial U_i}{\partial t}+\alpha U_i=\frac{\partial U_{i+1}}{\partial x}+\beta U_{i+1},\end{aligned}\tag{6.4}$$

$i=4,5,...,$ with conditions (4.1), (4.2) under the assumption of infinite differentiability with respect to x of the functions $\alpha=\alpha(t,x)$, $\beta=\beta(t,x)$ and $\phi(x)$. Operator $(A-\lambda B)$ is regular since the chain $v_1,v_2,...,$ is finite; where $Av_1=0, Av_i=Bv_{i-1}, i=$

$2, 3, \dots$. *Therefore,* $v_1 = (1, 0, 0, \dots), v_2$ *does not exist. Here* $dim N_1 = 1$*.The space* ℓ_2 *decomposes among the direct sum of two subspaces* $\ell_2 = M_1 + N_1$*, i.e.:*

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \dots \end{pmatrix} = \begin{pmatrix} -u_2 \\ u_2 \\ u_3 \\ u_4 \\ \dots \end{pmatrix} + \begin{pmatrix} u_1 + u_2 \\ 0 \\ 0 \\ 0 \\ \dots \end{pmatrix}. \tag{6.5}$$

Condition (4.1) *is:*

$$u_1(t, 0) + u_2(t, 0) = \psi_1(t). \tag{6.6}$$

Condition (4.2) *is:*

$$u_i(0, x) = \phi_i(x), \quad i = 2, 3, \dots . \tag{6.7}$$

The system in N_1 *satisfies system* (5.12)*:*

$$0 = \frac{-1}{\lambda} \frac{\partial(u_1 + u_2)}{\partial x} - \frac{1}{\lambda} \beta(u_1 + u_2), \tag{6.8}$$

hence, the solution of (6.8) *with the condition* (6.6) *is*

$$u_1 + u_2 = \psi_1(t) exp\Big(- \int_0^x \beta(t, s) ds \Big). \tag{6.9}$$

Operator G *and projectors* P_1, Q_1 *are*

$$G = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, P_1 = \begin{pmatrix} 1 & 1 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \end{pmatrix}, Q_1 = I - P_1.$$

The solution of the problem in M_1 *is constructed by formula* (6.3) *as the following:*

$$Q_1 U = e^{W(t,x)} \phi(D_2), \tag{6.10}$$

where,

$$W(t, x) = \int_0^t L(s, D_1) ds = \begin{pmatrix} 0 & -a_0 & -b_1 & -b_2 & -b_3 & \dots & \dots \\ 0 & a_0 & b_1 & b_2 & b_3 & \dots & \dots \\ 0 & 0 & a_0 & b_1 & b_2 & b_3 & \dots \\ 0 & 0 & 0 & a_0 & b_1 & b_2 & \dots \\ 0 & 0 & 0 & 0 & a_0 & b_1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \tag{6.11}$$

$$a_0 = \int_0^t \frac{-1}{2\pi i} \oint_\Gamma \frac{-\alpha(s, \mu)}{x - \mu} d\mu ds,$$

$$b_j = \int_0^t \frac{(-1)^j}{2\pi i} \oint_\Gamma \Big(\frac{\beta(s, \mu)}{(x - \mu)^j} (t - s)^{j-1} + \frac{\alpha(s, \mu)}{(x - \mu)^{j+1}} (t - s)^j \Big) d\mu ds, \quad j = 1, 2, \dots,$$

$\alpha(t,x)$ and $\beta(t,x)$ are smooth at x, therefore a_0, b_j have the following forms:

$$a_0 = -\int_0^t \alpha(s,x)ds,$$

$$b_1 = \int_0^t \big(\beta(s,x) - (t-s)\frac{\partial}{\partial x}\alpha(s,x)\big)ds,$$

$$b_j = \int_0^t \Big(\frac{(t-s)^{j-1}}{(j-1)!}\frac{\partial^{j-1}}{\partial x^{j-1}}\beta(s,x) - \frac{(t-s)^j}{(j)!}\frac{\partial^j}{\partial x^j}\alpha(s,x)\Big)ds, \quad j = 2,3,\dots,$$

$$\phi(D_2) = \begin{pmatrix} -\sum_{i=1}^{\infty} t^{i-1}\frac{\phi_{i+1}^{(i-1)}(x)}{(i-1)!} \\ \sum_{i=1}^{\infty} t^{i-1}\frac{\phi_{i+1}^{(i-1)}(x)}{(i-1)!} \\ \sum_{i=1}^{\infty} t^{i-1}\frac{\phi_{i+2}^{(i-1)}(x)}{(i-1)!} \\ \sum_{i=1}^{\infty} t^{i-1}\frac{\phi_{i+3}^{(i-1)}(x)}{(i-1)!} \\ \dots \end{pmatrix}. \tag{6.12}$$

Example 6.4. *Assume that in example 6.3 $\alpha(t,x) = t, \beta(t,x) = x$. Then W(t,s) is constructed by (6.11) as the following:*

$$W(t,x) = \begin{pmatrix} 0 & \frac{t^2}{2} & -xt & -\frac{t^2}{2} & 0 & 0 & 0 & \dots \\ 0 & -\frac{t^2}{2} & xt & \frac{t^2}{2} & 0 & 0 & 0 & \dots \\ 0 & 0 & -\frac{t^2}{2} & xt & \frac{t^2}{2} & 0 & 0 & \dots \\ 0 & 0 & 0 & -\frac{t^2}{2} & xt & \frac{t^2}{2} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Function Q_1U takes the form of (6.10) with the matrix $W(t,x)$. In fact, the solution in M_1 satisfies system (6.4) and condition $Q_1U(0,x) = \phi(x)$.

From (6.9), (6.10), (6.12), the general solution of system (6.4) with conditions (6.6), (6.7) is:

$$U(t,x) = \begin{pmatrix} \psi_1(t)e^{-x^2/2} \\ 0 \\ 0 \\ 0 \\ \dots \end{pmatrix} + e^{W(t,x)} \begin{pmatrix} -\sum_{i=1}^{\infty} t^{i-1}\frac{\phi_{i+1}^{(i-1)}(x)}{(i-1)!} \\ \sum_{i=1}^{\infty} t^{i-1}\frac{\phi_{i+1}^{(i-1)}(x)}{(i-1)!} \\ \sum_{i=1}^{\infty} t^{i-1}\frac{\phi_{i+2}^{(i-1)}(x)}{(i-1)!} \\ \sum_{i=1}^{\infty} t^{i-1}\frac{\phi_{i+3}^{(i-1)}(x)}{(i-1)!} \\ \dots \end{pmatrix}. \tag{6.13}$$

Example 6.5. *If* $\alpha(t,x) = x, \beta(t,x) = t$. *Then:*

$$W(t,x) = \begin{pmatrix} 0 & xt & 0 & 0 & 0 & ... \\ 0 & -xt & 0 & 0 & 0 & ... \\ 0 & 0 & -xt & 0 & 0 & ... \\ 0 & 0 & 0 & -xt & 0 & ... \\ \dots & \dots & \dots & \dots & \dots & ... \end{pmatrix},$$

and

$$Q_1 U = e^{-tx} \begin{pmatrix} -\sum\limits_{i=1}^{\infty} t^{i-1} \dfrac{\phi_{i+1}^{(i-1)}(x)}{(i-1)!} \\ \sum\limits_{i=1}^{\infty} t^{i-1} \dfrac{\phi_{i+1}^{(i-1)}(x)}{(i-1)!} \\ \sum\limits_{i=1}^{\infty} t^{i-1} \dfrac{\phi_{i+2}^{(i-1)}(x)}{(i-1)!} \\ \sum\limits_{i=1}^{\infty} t^{i-1} \dfrac{\phi_{i+3}^{(i-1)}(x)}{(i-1)!} \\ \dots \\ \sum\limits_{i=1}^{\infty} t^{i-1} \dfrac{\phi_{i+j}^{(i-1)}(x)}{(i-1)!} \end{pmatrix}, \quad j = 4, 5, \dots .$$

Substituting $Q_1 U$ *into* (6.4) *and condition* (4.2) *proves the obtained result true. The general solution of system* (6.4) *with conditions* (6.6), (6.7) *in this case is:*

$$U_1(t,x) = \psi_1(t) e^{-tx} - e^{-tx} \sum_{i=1}^{\infty} t^{i-1} \frac{\phi_{i+1}^{(i-1)}(x)}{(i-1)!}, \tag{6.14}$$

$$U_j(t,x) = e^{-tx} \sum_{i=1}^{\infty} t^{i-1} \frac{\phi_{i+j-1}^{(i-1)}(x)}{(i-1)!}, \quad j = 2, 3, \dots . \tag{6.15}$$

7. Conclusion

In this study, we use an analytical method for solving equation (1.1) with Showalter-type conditions when the operator pencil $(A - \lambda B)$ is regular. Due to the regularity of the operator pencil, equation (1.1) splits into equations in disjoint subspaces. In each subspace the equation is differential, and the algebraic part is absent; therefore equation (1.1) cannot be called differential-algebraic. One of these equations is solved at $x = 0$, the other at $t = 0$; thus, the problem with such conditions can be called semi-boundary or semi-initial.

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