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Studying the thermal analysis of rectangular cross section porous fin: A numerical approach

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Abstract

In this work, a direct computational method has been developed for solving the thermal analysis of porous fins with a rectangular cross-section with the aid of Chebyshev polynomials. The method transforms the nonlinear differential equation into a system of nonlinear algebraic equations and then solved using a novel technique. The solution of the system gives the unknown Chebyshev coefficients. An algorithm for solving this nonlinear system is presented. The results are obtained for different values of the variables and a comparison with other methods is made to demonstrate the effectiveness of the method.

Keywords. Numerical, Porous Fin, Thermal analysis, Collocation method.

2010 Mathematics Subject Classification. 65L05, 65L60.

1. Introduction

The development of porous material over the past few years has grown tremendously due to its physical, mechanical, and biological properties that are of interest to a wide of researchers [13, 32, 33] The use of these materials has enriched the industry section in many ways due to its ability to provide different utilization. For example, the car industry, electronic industry, medical equipment, etc. have benefited from this in its requirements [11, 12, 18, 40–44]. Numerical and analytical methods come in handy when dealing with such types of problems and that motivates researchers from different disciplines especially mechanical ones to investigate solutions to the model of heat transfer in fins. There have been numerous works in this field which include the investigation of the extended surface for different cases in one and two-dimensional forms with multiple references. Chamkha et.al [1, 6, 14, 23, 24] proposed a simple yet efficient method for studying the performance of porous fins in a medium with natural convection. The stud includes three types of fins named long, finite length with adiabatic and non-adiabatic ends with specific convection coefficients. Ganji et.al [27, 31, 34, 37, 38]investigated another form of porous fins with the effect of convection radiative heat transfer with different shapes and fins materials. They proved that the heat transfer from the fins is increased by increasing the porosity of the fins. Also, they used the finite difference method in another article to investigate the performance of different profiles. The use of the Homotopy perturbation method (HAM) and the variational iteration method (VIM) as an example for the use of semi-analytical methods for solving the fin equation has been introduced by Sheikholeslami et al. [17, 25, 28, 29, 35, 36]. They studied the rectangle profiles of the fin and studied this profile with the help of the finite difference method that succeeds to simulate this. Other researchers applied different methods for solving different forms of this model with different profiles. One may find more details about the model and the methods used to solve it in [26, 30, 39] and references therein. Collocation methods are considered one of the most effective methods

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for solving different types of model equations. In this paper, we consider solving the equation that simulates the heat transfer through a porous fin with a rectangular shape in the following form [9]

$$\frac{d^2\theta}{dx^2} + 4R_d \frac{d^2\theta}{dx^2} + G\,(1 + e\ \theta) - S_h\ \theta^2 - (N_c + N_r)\,\theta = 0, \qquad 0 \le x \le 1, \tag{1.1}$$

where G is the dimensionless number of generated heat, R_d is the conduction radiation parameter, e is the parameter of the internal heat generation, S_h is the porosity parameter, convection parameter, and radiation parameter. Eq. (1.1) is subjected to the boundary conditions in the form

$$\theta'(0) = 0, \qquad \theta(1) = 1. \tag{1.2}$$

Here we are considered with solving Eq. (1.1) along with its boundary conditions in Eq. (1.2)using the Chebyshev collocation method.

Chebyshev collocation methods have been used in solving different types of equations both differential and integral due to their ability to provide accurate results. For example, Fokker Blank equations of fractional order in time and space have been solved using a two-dimensional Chebyshev wavelet method in [45]. Also, the Chebyshev wavelet method has been used for solving the non-uniform Euler Bernoulli beam along with some error analysis for the proposed technique [5]. The Fractional Bagley –Torvik equation has been solved with the aid of the Chebyshev collocation method achieving accurate results for this type of problem. Other types of problems including two-dimensional chemical engineering [3], diffusion equation with distributed order [15], optimal control problems [19], pantograph type equations [46], fractional Volterra-Fredholm integrodifferential equations [47], nonlinear integrodifferential equation [8], system of differential equation [20], fractional advection-dispersion equation [22], singular Volterra integral equations [21] and multi-order fractional differential equation [7]. Chebyshev polynomials along with their operational matrices have proven to be a valuable tool to construct accurate approximate solutions for a wide variety of problems. In the next section, we will introduce the basic definitions of the Chebyshev polynomials that will be used later in the paper.

2. Basic Definitions of Chebyshev Polynomials

Chebyshev polynomials are one of the most important bases to be used using different types of spectral methods. These bases are defined on the interval $[-1, 1]$. The basic definitions for these polynomials are as follows

Definition 2.1. Chebyshev polynomials of degree n can be defined as

$$T_n\ (x) = cos(n\ arcos(x)), \qquad n = 0, 1, \ldots.., \qquad x \in [-1, 1].$$

Alternatively, it can be informed in another way as

$$T_n\,(x) = \cos\,(n\ \theta), \quad x = \cos\,(\theta), \qquad \theta \in [0, \pi]\,.$$

The properties of these polynomials on the interval $[-1, 1]$ are as
1- T_n has distinct zeros such that

$$T_n\,(x_k) = 0, \quad t_k = cos\left(\frac{(2k-1)\pi}{2n}\right), \quad k = 1, 2, \ldots.., n.$$

2- $T_n(x)$ has an alternating maximal value with $(n + 1)$ times sign as

$$T_n = 1, T_n(x_k) = -1^k,$$
$$x_k = cos\left(\frac{\pi\ k}{n}\right), \quad k = 0, 1, \ldots.., n.$$

Lemma 2.2. *Chebyshev polynomials T_n satisfies the following relation*

$$T_0\,(x) = 1, \quad T_1\,(x) = x,$$
$$T_{n+1} = 2xT_n\,(x) - T_{n-1}\,(x)\,, \qquad n \ge 1,$$

which provides that T_n is an algebraic polynomial of degree n with the leading coefficient $2^{(n-1)}$. Also, the orthogonal property of Chebyshev polynomials is

$$\int_{-1}^{1} T_m(x) T_n(x) \frac{dx}{\sqrt{1-x^2}} = 0, \qquad n \neq m,$$

$$\int_{-1}^{1} T_n^2(x) \frac{dx}{\sqrt{1-x^2}} = \begin{cases} \frac{\pi}{2} & n > 0 \\ \pi & n = 0 \end{cases}.$$

In this paper, we will use the orthonormal Chebyshev polynomial. The next section is devoted to illustrating how to approximate functions using Chebyshev bases.

3. Function Approximation

In this section, we will illustrate how to use the Chebyshev basis defined on the interval $[-1, 1]$ for approximating functions. We consider the approximate solution in the form

$$\theta_N(x) = \sum_{r=0}^{N} \eta'_r \, T_r(x), \quad -1 \le x \le 1 \tag{3.1}$$

where $T_r(x)$ is the first kind Chebyshev polynomial of order r, η'_r is the Chebyshev coefficients that are needed to be determined as mentioned in [16] and N is a positive integer and chosen so as $N \geq 2$. The first-order derivative can be expressed in the form

$$\theta_N^{(k)}(x) = \sum_{r=0}^{N} \eta'_r \, T_r^{(k)}(x), \qquad k = 1, 2. \tag{3.2}$$

Then, the solution and its derivatives can be written in the following matrix form

$$A^{(k)} = 2^k M^k \eta \tag{3.3}$$

$$\theta^{(k)} = 2^k T M^k \eta \tag{3.4}$$

where

$$\mathbf{T} = \begin{pmatrix} T_0(x_0) & T_1(x_0) & T_2(x_0) & \dots & T_N(x_0) \\ T_0(x_1) & T_1(x_1) & T_2(x_1) & \dots & T_N(x_1) \\ T_0(x_2) & T_1(x_2) & T_2(x_2) & \dots & T_N(x_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T_0(x_N) & T_1(x_N) & T_2(x_N) & \dots & T_N(x_N) \end{pmatrix},$$

$$\boldsymbol{\theta^{(k)}} = \begin{pmatrix} \theta^{(k)}(x_0) \\ \theta^{(k)}(x_1) \\ \vdots \\ \theta^{(k)}(x_N) \end{pmatrix}, \qquad \boldsymbol{\theta} = \begin{pmatrix} \theta(x_0) \\ \theta(x_1) \\ \vdots \\ \theta(x_N) \end{pmatrix}, \qquad \boldsymbol{\eta} = \begin{pmatrix} \frac{\eta_0}{2} \\ \eta_1 \\ \vdots \\ \eta_N \end{pmatrix},$$

$$\boldsymbol{M} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{5}{2} & \dots & m_1 \\ 0 & 0 & 2 & 0 & 4 & 0 & \dots & m_2 \\ 0 & 0 & 0 & 3 & 0 & 5 & \dots & m_3 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & N \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

,

where

$$m_1 = \frac{N}{2}, \quad m_2 = 0, \quad m_3 = N, \quad if\ N\ is\ odd,$$

$$m_1 = 0, \quad m_2 = N, \quad m_3 = 0, \quad if\ N\ is\ even.$$

This method can be extended for solving problems defined on the interval $[a, b]$. We need the following theorem.

Theorem 3.1. *The shifted Chebyshev polynomials defined on the interval* $[a, b]$ *can be defined as*

$$\overline{T_n}\,(x) = T_n\,(t)\,, \qquad t = \frac{2}{b-a}\left(x - \frac{a+b}{2}\right).$$

The coefficients are equal to $2^{(n-1)}\frac{2}{(b-a)^n}$.

The shifted Chebyshev polynomials can be used to approximate the solution $\theta(x)$ in the form

$$\theta_N\,(x) = \sum_{r=0}^{N} \eta_r^*\,\overline{T_r}\,(x)\,, \quad a \le x \le b \tag{3.5}$$

where $\overline{T_r}\,(x) = T_r\left(\frac{2}{b-a}\left(x - \left(\frac{a+b}{2}\right)\right)\right)$. The shifted Chebyshev collocation points that are used with these polynomials are defined as

$$x_j = \frac{b-a}{2}\left[\left(\frac{a+b}{b-a}\right) + \cos\left(\frac{j\pi}{N}\right)\ \right], \quad j = 0, 1, \ldots, N-1, \tag{3.6}$$

and the relation

$$\eta^{*(k)} = \left(\frac{4}{b-a}\right)^k M^k\,\eta^*, \qquad k = 0, 1, 2,$$

where

$$\boldsymbol{\eta^*} = \left[\frac{\eta_0^*}{2}, \eta_1^*,\ \ldots, \eta_N^*\right]^t.$$

We shall use the shifted Chebyshev polynomials along with the collocation points to solve Eq. (1.1) in the next section.

4. Description of the Collocation Method

In this part, the proposed method based on the shifted Chebyshev polynomials is used to solve Eq. (1.1) in the form

$$\frac{d^2\theta}{dx^2} + 4R_d\frac{d^2\theta}{dx^2} + G\,(1 + e\,\theta) - S_h\,\theta^2 - (N_c + N_r)\,\theta = 0, \qquad 0 \le x \le 1. \tag{4.1}$$

We first begin by substituting the approximate solution and its derivative represented in the form

$$\theta_N\,(x) = \sum_{r=0}^{N} \eta_r^*\,\overline{T_r}\,(x)\,, \quad 0 \le x \le 1 \tag{4.2}$$

where $\overline{T_r}(x) = T_r(2x-1)$is the shifted Chebyshev of the first kind and order r and the shifted Chybybychev collocation points defined on the interval $[0, 1]$ are

$$x_j = \frac{1}{2}\left[1 + \cos\left(\frac{j\pi}{N}\right)\ \right], \quad j = 0, 1, \ldots, N. \tag{4.3}$$

The kth derivative of have truncated Chebyshev series expansion in the form

$$\theta_N^{(k)}\,(x) = \sum_{r=0}^{N} (\eta_r^*)^{(k)}\overline{T_r}\,(x)\,, \quad k = 0, 1, 2. \tag{4.4}$$

Then by substituting the approximations in Eq. (4.2) and (4.4) into Eq. (4.1) we have

$$\begin{aligned}&(1+4R_d)\sum_{r=0}^{N}(\eta_r^*)^{(2)}\overline{T_r}\,(x)\ +G\left(1+e\ \sum_{r=0}^{N}\eta_r^*\ \overline{T_r}\,(x)\right)-S_h\\&\left(\sum_{r=0}^{N}\eta_r^*\ \overline{T_r}\,(x)\right)^2-(N_c+N_r)\sum_{r=0}^{N}\eta_r^*\ \overline{T_r}\,(x)=0.\end{aligned}\tag{4.5}$$

As can be seen from Eq. (4.5) there is a nonlinear part represented by the term $\theta^2(x)$, which needed to be dealt with first. For that, we need the following theorem.

Theorem 4.1. *[44] The approximation of the function $\theta^\nu(x), \nu=0,1,2,...$ can be represented according to the following*

$$\begin{pmatrix}\theta^\nu(x_0)\\ \theta^\nu(x_1)\\ \vdots\\ \theta^\nu(x_N)\end{pmatrix}=\begin{pmatrix}\theta(x_0)&0&\cdots&0\\0&\theta(x_1)&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\0&0&\cdots&\theta(x_N)\end{pmatrix}^{\nu-1}\begin{pmatrix}\theta(x_0)\\ \theta(x_1)\\ \vdots\\ \theta(x_N)\end{pmatrix}\tag{4.6}$$

$$=(\Theta)^{\nu-1}\Theta=\left(\boldsymbol{\overline{T}^*\overline{\eta}^*}\right)^{\boldsymbol{\nu-1}}\overline{T}\eta^*$$

where

$$\boldsymbol{\overline{T_r}^*}=\begin{pmatrix}\overline{T}(x_0)&0&\cdots&0\\0&\overline{T}(x_1)&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\0&0&\cdots&\overline{T}(x_N)\end{pmatrix},\quad \boldsymbol{\overline{\eta}^*}=\begin{pmatrix}\eta^*&0&\cdots&0\\0&\eta^*&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\0&0&\cdots&\eta^*\end{pmatrix},$$

substituting $x=x_j$ of shifted Chebyshev collocation points in Eq. (4.3), we reach the following theorem.

Theorem 4.2. *If the assumed approximate solution of the problem represented in Eq. (4.1) is (4.2), then the discrete Chebyshev system can be given in the form*

$$\begin{aligned}&(1+4R_d)\sum_{r=0}^{N}(\eta_r^*)^{(2)}\overline{T_r}\,(x_j)\ +G\left(1+e\ \sum_{r=0}^{N}\eta_r^*\ \overline{T_r}\,(x_j)\right)-S_h\\&\left(\sum_{r=0}^{N}\eta_r^*\ \overline{T_r}\,(x_j)\right)^2-(N_c+N_r)\sum_{r=0}^{N}\eta_r^*\ \overline{T_r}\,(x_j)=0.\end{aligned}\tag{4.7}$$

The matrix form for the system (4.7) is

$$\boldsymbol{\Psi}\ \eta^*=\boldsymbol{F}\ ,\tag{4.8}$$

where

$$\boldsymbol{\Psi=}(1+4R_d)\,\overline{T}M^2+G\left(1+e\ \overline{T}\right)-S_h\left(\boldsymbol{\overline{T}^*\overline{\eta}^*}\right)\overline{T}-(N_c+N_r)\,\overline{T}.$$

Replacing the rows of the augmented matrix $[\Psi;F]$ by the equations of the boundary conditions, we acquire new augmented matrix

$$\widetilde{\boldsymbol{\Psi}}\ \eta^*=\widetilde{\boldsymbol{F}}.$$

Now we have an $N+1$ equations in $N+1$ unknowns. We obtain the unknowns by solving the above system of equations using the same algorithm mentioned in [2].

- Input (integer) N.
- input (double) tol.

- Input (array) $\eta^*_{old}=\eta^*_0$ (initial approximation, η^*_0 with $N+1$ dimension, can be chosen so that the boundary conditions are satisfied.)
- $\widetilde{\boldsymbol{\Psi}}\ (\eta^*_{\boldsymbol{old}})\,\eta^*_{\boldsymbol{new}}=\widetilde{\boldsymbol{F}}$, is a linear algebraic equation system. This system is solved and η^*_{new} is found.
- if $|\eta^*_{old}-\eta^*_{new}|<tol$ then $\eta^*_{new}=\eta^*$, break (the program is terminated).
- Else then $\eta^*_{old}\leftarrow\eta^*_{new}$.

5. Error Analysis

5.1. **Error Bound.** In this section, we shall illustrate the convergence analysis of the proposed method. We assume that $\theta(x)$ is a sufficiently smooth function on the interval $[0,1]$ and consider $H_N(x)$ is the interpolating polynomials to θ at the grid points where $\theta_j, j=0,1,...,N$, are the Chybybychev grid points, then we have the Cauchy interpolation error theorem [4, 16]

$$\theta\left(x\right)-H_N\left(x\right)=\frac{\theta^{(n+1)}\left(\upsilon\right)}{(n+1)!}\prod_{j=0}^{N}\left(x-x_j\right),\quad \upsilon\in\left[0,1\right]. \tag{5.1}$$

Then, we have [45,46]

$$\left|\theta\left(x\right)-H_N\left(x\right)\right|\le\frac{1}{2^{2N+1}}\left\|\theta^{(N+1)}\left(x\right)\right\|_{\infty}. \tag{5.2}$$

According to the previous definitions, we reach the following theorem.

Theorem 5.1. *Suppose that the known functions for Eq.(1.1) are $(N+1)$-times continuously differentiable function on the interval $[0,1]$ and $\theta_N\left(x\right)=\sum_{r=0}^{N}\eta^*_r\,\overline{T_r}\left(x\right)$, are the shifted Chebyshev polynomials expansion of the exact solution. Also, let $\widetilde{\theta}_N\left(x\right)=\sum_{r=0}^{N}\widetilde{\eta^*_r}\,\overline{T_r}\left(x\right)$, be the approximate solution obtained by the proposed method. Then, there exists a real number δ such that*

$$\left\|\theta\left(x\right)-\theta_N\left(x\right)\right\|_2\le\delta\frac{1}{2^{N+1}}\left\|\theta^{(N+1)}\left(x\right)\right\|_{\infty}+\sqrt{\frac{3\pi}{8}}\left\|\eta^*-\widetilde{\eta}^*\right\| \tag{5.3}$$

where $\eta^*=[\eta_0,\ \eta_1,\ \dots,\ \eta_n]$ and $\widetilde{\eta}^*=[\widetilde{\eta_0},\ \widetilde{\eta_1},\ \dots,\ \widetilde{\eta_n}]$.

Proof. Let $\theta_N\left(x\right)$ be the real-valued polynomial of degree $\le N$ and $\theta_N(x)$ be the best approximation of $\theta(x)$. Then, we can write

$$\left\|\theta\left(x\right)-\theta_N\left(x\right)\right\|_2\le\left\|\theta\left(x\right)-\widetilde{\theta}_N\left(x\right)\right\|_2+\left\|\widetilde{\theta}_N\left(x\right)-\theta_N\left(x\right)\right\|_2. \tag{5.4}$$

We can define the least square norm to $\theta(x)$ in the form

$$\left\|\theta\right\|_2=\left(\int_0^1\left|\theta\left(x\right)-\theta_N\left(x\right)\right|^2dx\right)^{\frac{1}{2}}. \tag{5.5}$$

Using Eqs. (5.2) and (5.4), we obtain

$$\begin{aligned}\left\|\theta\left(x\right)-\theta_N\left(x\right)\right\|_2&=\left(\int_0^1\left|\theta\left(x\right)-\theta_N\left(x\right)\right|^2dx\right)^{\frac{1}{2}}\\&\le\left(\int_0^1\left[\frac{1}{(N+1)!\ 2^{2N+1}}\left\|\theta^{(N+1)}\left(x\right)\right\|_{\infty}\right]^2dx\right)^{\frac{1}{2}}\\&=\sqrt{\omega}\frac{1}{(N+1)!\ 2^{2N+1}}\left(\left\|\theta^{(N+1)}\left(x\right)\right\|_{\infty}\right)^{N+1}.\end{aligned}$$

Then we have

$$\left\|\theta\left(x\right)-\widetilde{\theta_N\left(x\right)}\right\|_2=\left[\sum_{r=0}^{N}\left(\eta_r-\widetilde{\eta_r}\right)^2\right]^{\frac{1}{2}}\left(\left[\sum_{r=0}^{N}\int_0^1\left|\overline{T_r}\left(x\right)\right|^2dx\right]\right)^{1/2}$$
$$=\sqrt{\frac{3\pi}{8}}\ \left\|\eta^*-\widetilde{\eta^*}\right\|. \tag{5.6}$$

□

In the next section, we shall illustrate the accuracy of the proposed method based on the residual error function.

5.2. **Residual Error Analysis.** We can easily check the accuracy of the proposed method as follows. Since the truncated Chebyshev series represented in Eq. (4.2) is the approximate solutions of Eq. (1.1) when the function $\theta_N(x)$ and its derivatives and the approximate solution are substituted in Eq. (1.1), the resulting equation approximately satisfies; that is; for

$$x=x_k\in[0,1]\,,\quad k=0,1,\ldots,N$$

$$\mathcal{R}_N\left(x_k\right)=\left|(1+4R_d)\theta_N^{''}(x)+G\left(1+e\ \theta_N\left(x\right)\right)-S_h\ \theta_N^2(x)-\left(N_c+N_r\right)\theta_N(x)\right|, \tag{5.7}$$

or

$$\mathcal{R}_N\left(x_k\right)\leq 10^{-q_k}$$

where q_k is any positive integer. If max $10^{-q_k}=10^{-q}$is prescribed, then the truncation limit N is increased until the difference $R_N(x_k)$ at each of the points become smaller than the prescribed 10^{-q}. Also, the error function can be estimated to through the following form

$$\mathcal{R}_N\left(x_k\right)=(1+4R_d)\theta_N^{''}(x)+G\left(1+e\ \theta_N\left(x\right)\right)-S_h\ \theta_N^2(x)-\left(N_c+N_r\right)\theta_N(x). \tag{5.8}$$

If $R_N(x)\to 0$, as N has sufficiently enough value, then the error decreases.

6. Numerical Simulations

In this section, we will illustrate the use of our method based on the Chebyshev collocation method for solving Eq. (1.1)for different values of parameters. To illustrate the performance of our method we have compared our method with the Homotopy Method, Homotopy perturbation method, and Rung Kutta method [10], which proves that our method is effective and gives good results. Our method is in good agreement with the Rung Kutta method more than the other two methods. Table. 1 and Table. 2 give a comparison between the presented method and the last-mentioned methods for comparison. Also, in the same tables, the residual error function is provided to prove the efficiency of the proposed method. Results indicate that our method is effective compared to other methods for solving the present model. The results depicted in this table prove that our method provides accurate and reliable results. From these tables, we conclude that our method provides good agreement with the other mentioned methods. Also, the effect of changing different parameters is illustrated in Figures. (1-3) which presents the values of the dimensionless temperature distribution $\theta(x)$ in a porous fin with adiabatic end for different convection N_c, radiation surrounding N_r, porosity S_h and radiation surface R_d. For example, according to these figures, increasing some of these parameters including convection, radiation surrounding, and porosity and with the decline of the internal heat generation, the dimensionless temperature distribution curve is decreased which will result in increasing the slope of the temperature profiles. Also, the temperature drops and heat flow from the base increases as the other parameters increase.

TABLE 1. Comparison of the numerical and analytical solutions along with residual error function for Eq. (1.1)for $R_d = 0.6, S_h = 0.4, G = 0, N_c = 0.3, N_r = 0.1$.

x	Presented method	HAM [45]	HPM [45]	Rung Kutta [45]	$R_N(x)$
0.0	0.89707909023	0.897079121	0.897077821	0.897079127	1.46145E-13
0.05	0.89732937649	0.897329407	0.897328112	0.897329391	1.13597E-13
0.1	0.89808044099	0.898080472	0.898079189	0.898080471	1.20794E-13
0.15	0.89933290136	0.899332932	0.899331670	0.899332937	1.14697E-13
0.2	0.90108778823	0.901087819	0.901086585	0.901087821	1.09770E-13
0.25	0.90334654726	0.903346578	0.903345380	0.903346581	1.09965E-13
0.3	0.90611104183	0.906111073	0.906109919	0.906111076	1.09146E-13
0.35	0.90938355665	0.909383588	0.909382484	0.909383589	1.03615E-13
0.4	0.91316680208	0.913166834	0.913165787	0.913166833	9.57384E-14
0.45	0.91746391931	0.917463951	0.917462968	0.917463950	8.96149E-14
0.5	0.92227848645	0.922278518	0.922277604	0.922278514	8.60670E-14
0.55	0.92761452535	0.927614557	0.927613717	0.927614551	8.20386E-14
0.6	0.93347650943	0.933476540	0.933475780	0.933476535	7.43161E-14
0.65	0.93986937233	0.939869402	0.939868726	0.939869395	6.35146E-14
0.7	0.94679851755	0.946798546	0.946797958	0.946798537	5.39000E-14
0.75	0.95426982904	0.954269855	0.954269358	0.954269849	4.82522E-14
0.8	0.96228968280	0.962289706	0.962289304	0.962289701	4.27580E-14
0.85	0.97086495948	0.970864979	0.970864674	0.970864971	3.04338E-14
0.9	0.98000305815	0.980003073	0.980002867	0.980003079	1.55208E-14
0.95	0.98971191113	0.989711919	0.989711815	0.989711953	1.51793E-14
1.0	1.00000000000	1.0000000000	1.0000000000	1.0000000000	3.54570E-14

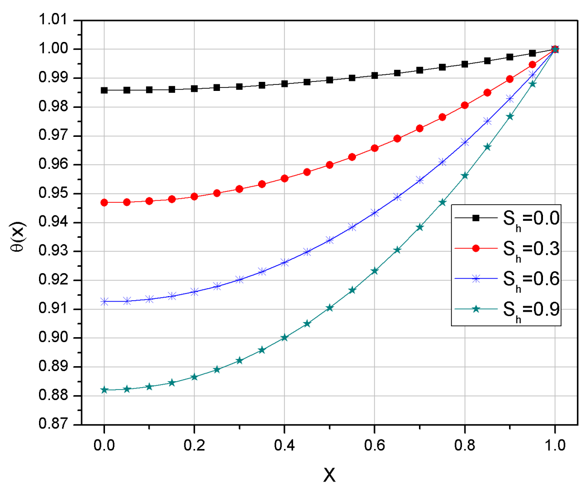

FIGURE 1. Dimensionless temperature distribution for different porosity parameters and for $R_d = 0.6, e = 0.5, G = 0.2, N_c = 0.3, N_r = 0.1$.

7. CONCLUSION

In this paper, the Chebyshev collocation method has been applied to study the heat transfer of porous fins with a rectangular cross-section. The method is based on the use of Chebyshev polynomials along with its collocation points of converting the proposed model into a system of nonlinear algebraic equations. The system is then solved using a

TABLE 2. Comparison of the numerical and analytical solutions along with residual error function for Eq. (1.1) for $R_d = 0.7, S_h = 0.2, G = 0, N_c = 0.4, N_r = 0.3$

x	Presented method	HAM [45]	HPM [45]	Rung Kutta [45]	$R_N(x)$
0.0	0.89416278844	0.894162789	0.894162588	0.894162814	2.82638E-13
0.05	0.89442129664	0.894421297	0.894421096	0.894421308	9.00986E-14
0.1	0.89519700116	0.895197002	0.895196803	0.895197022	1.37498E-13
0.15	0.89649044194	0.896490442	0.896490247	0.896490467	1.09649E-13
0.2	0.89830251967	0.898302520	0.898302329	0.898302543	9.16666E-14
0.25	0.90063449693	0.900634497	0.900634312	0.900634520	1.07092E-13
0.3	0.90348799982	0.903488000	0.903487822	0.903488023	1.19778E-13
0.35	0.90686502003	0.906865021	0.906864850	0.906865042	1.07794E-13
0.4	0.91076791745	0.910767918	0.910767757	0.910767939	8.50431E-14
0.45	0.91519942320	0.915199424	0.915199272	0.915199444	7.56753E-14
0.5	0.92016264313	0.920162644	0.920162503	0.920162662	8.44258E-14
0.55	0.92566106191	0.925661063	0.925660934	0.925661080	9.30271E-14
0.6	0.93169854754	0.931698548	0.931698432	0.931698565	8.23001E-14
0.65	0.93827935641	0.938279357	0.938279254	0.938279372	5.58641E-14
0.7	0.94540813885	0.945408140	0.945408050	0.945408152	3.90625E-14
0.75	0.95308994533	0.953089946	0.953089871	0.953089959	4.82824E-14
0.8	0.96133023304	0.961330234	0.961330173	0.961330245	6.04180E-14
0.85	0.97013487323	0.970134874	0.970134828	0.970134881	3.34303E-14
0.9	0.97951015896	0.979510159	0.979510129	0.979510173	7.12078E-15
0.95	0.98946281363	0.989462814	0.989462798	0.989462841	4.22140E-14
1.0	1.00000000000	1.000000000	1.000000000	1.000000000	2.09627E-13

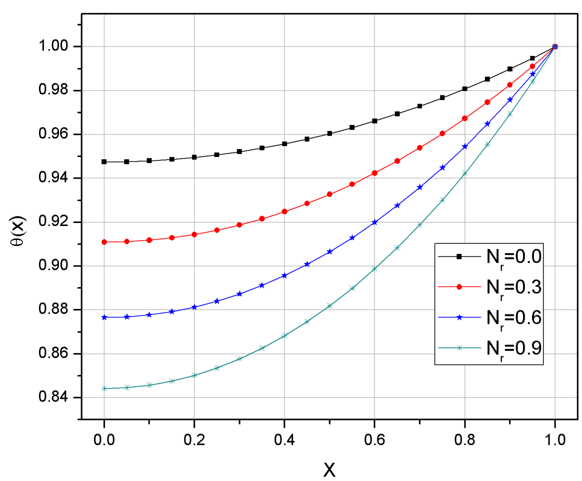

FIGURE 2. Dimensionless temperature distribution for different value of N_r and for $R_d = 0.6, S_h = 0.4, G = 0.2, N_c = 0.3, e = 0.2$.

novel technique. The proposed method is effective for solving this type of problem when compared to other existing methods from the literature. Calculated results and comparisons with other methods are illustrated in Tables (1- 4). These calculations demonstrated that the accuracy of the proposed method is quite good even when using a small number of grid points. The method is proven to be a fast, simple, reliable, and convenient alternative method.

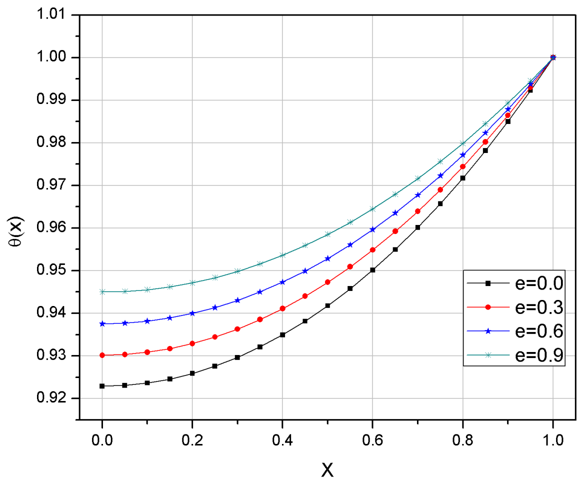

FIGURE 3. Dimensionless temperature distribution for different value of e and for $R_d = 0.6, S_h = 0.4, G = 0.2, N_c = 0.3, N_r = 0.1$.

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The authors declare that they have no competing interests.

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