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Analysis of non-hyperbolic equilibria for Caputo fractional system

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Abstract

In this manuscript, a center manifold reduction of the flow of a non-hyperbolic equilibrium point on a planar dynamical system with the Caputo derivative is proposed. The stability of the non-hyperbolic equilibrium point is shown to be locally asymptotically stable, under suitable conditions, by using the fractional Lyapunov direct method.

Keywords. Caputo derivative, Stability, Center manifold.

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1. Introduction

Dynamical systems are an effective method for modeling the evolution of processes. In fact, they are wildly used in, and have formed, the foundation of Mathematical Biology. They have proven to be effective in determining the qualitative behavior of disease models or ecological models. Traditionally, a wide array of problems in mathematical biology are modeled using a first order dynamical system, where the classical (first derivative) is employed. In recent years the Caputo fractional derivative, a derivative of fractional order, has gained popularity in the application of Mathematical Biology see [1, 2, 15, 16]. The Caputo fractional order derivative is considered to be more effective, as opposed the classical order, in modeling an evolution process. Indeed, the classical derivative is local, that is to say, it tells you information of the change of the process in a neighborhood of a point in time, as opposed to the Caputo fractional order derivative which captures the change in its entire domain. This is considered to be a ”memory effect” of the Caputo derivative.

Though, the Caputo dynamical systems (dynamical systems employing the Caputo derivative) are much more difficult to analyze, and determining the qualitative behavior is not as straightforward as the classical derivative. Considerable amount of work has been done in the analysis of the Caputo derivative operator, see for example [3–5, 9–11, 13, 14, 18]. Furthermore, in [6] the authors provided a linearization theorem that shows that the solutions in a neighborhood of a hyperbolic equilibrium point, translated to the origin, are topologically equivalent to the behavior of the solutions of the linear system at the origin. However, if the equilibrium point if non-hyperbolic, then there is no method, currently, in the literature to determine the qualitative behavior of the non-linear system near it.

In this manuscript we only consider the fractional order $\alpha \in (0, 1)$. We propose a method to determine the qualitative behavior of a planar Caputo fractional system in a neighborhood of a non-hyperbolic equilibrium point, with one-zero eigenvalue and the other eigenvalue being negative. The results in [17] show that a center manifold does exist for a Caputo fractional system, and that the flow can be reduced accordingly, see Lemma 3.5. We show that the flow of the planar system can be determined, in a neighborhood of the non-hyperbolic equilibrium point, by the following expression

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$$cD_0^\alpha x(t) = a_m x^m + ...,$$

where cD_0^α, is the Caputo derivative operator, $\alpha \in (0,1)$, and $x \in \mathbb{R}$. It is shown using the results in [7] that, if $m \geq 2$, and $a_m < 0$, then the non-hyperbolic equilibrium point is locally asymptotically stable, see Theorems 4.1 and 4.3. These results are new, and it is, to the authors best knowledge, they are the only results that provide a method of dealing with the qualitative behaviour of a planar Caputo system with a non-hyperbolic equilibrium point.

2. Preliminaries

Definition 2.1. Let $\alpha \geq 0$. The operator J_a^α, defined on $L^1[a,b]$ by

$$J_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} f(x) dx \tag{2.1}$$

for $a \leq t \leq b$, is called the Riemann-Liouville fractional integral operator of order α. Here and in what follows $\Gamma(\cdot)$ is the Gamma function.

Remark 2.2. For $\alpha = 0$, we set $J_a^0 := I$, the identity operator.

Definition 2.3. Let $0 < \alpha < 1$. Then, we define the Caputo fractional differential operator cD_a^α as

$$cD_a^\alpha f(t) := J_a^{1-\alpha} f'(t) \tag{2.2}$$

whenever $f, f' \in L^1[a,b]$.

Definition 2.4. Let $0 < \alpha < 1$. Then, we define the Riemann-Liouville fractional differential operator D_a^α as

$$D_a^\alpha f(t) := \left(J_a^{1-\alpha} f(t)\right)' \tag{2.3}$$

whenever $f \in L^1[a,b]$.

Definition 2.5. Let $\alpha > 0$. The function E_α, defined by

$$E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \tag{2.4}$$

whenever the series converges is called the Mittag-Leffler function.

3. Local Stability Theory of Planar Fractional System

In this section we present some background theory, without loss of generality we can take the equilibrium point to be the origin $(0,0)$, where an equilibrium point is considered to be a constant solution. Consider the Caputo planar system below

$$\begin{cases} cD_0^\alpha x(t) = f(x,y), \\ cD_0^\alpha y(t) = g(x,y) \end{cases} \tag{3.1}$$

subject to the initial condition:

$$(x(0), y(0)) = (x_0, y_0)$$

where $\alpha \in (0,1)$, $f, g \in C^1(\mathbb{R}^2)$, and x, y are assumed to be absolutely continuous.

Since, $f, g \in C^1(\mathbb{R}^2)$, it is well known that for any $(x_0, y_0) \in \mathbb{R}^2$ the initial value problem (3.1) has a unique solution, see [12].

We denote by $A(x,y)$ the Jacobian matrix of f and g at (x,y), that is,

$$A(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \tag{3.2}$$

and by $|A(x,y)|$ and $\mathrm{tr}(A(x,y))$ the determinant and trace of $A(x,y)$, respectively.

Definition 3.1. A point $(x^*, y^*) \in \mathbb{R}^2$ is called an equilibrium point of (3.1) if $f(x^*, y^*) = g(x^*, y^*) = 0$.

Below we define the linearized system of (3.1) about the equilibrium point (x^*, y^*).

Definition 3.2. Let A be the matrix defined in (3.2) is evaluated at the equilibrium point (x^*, y^*). Then,

$$cD_0^\alpha X = A^* X, \tag{3.3}$$

where $X = (x,y)^T$, is the linearization of system (3.1) at the equilibrium point (x^*, y^*).

Below we give the definition of the Mittag-Leffler stability, which a special case of the definition found in [7].

Definition 3.3. The solution of (3.1) is said to be Mittag-Leffler stable if

$$|x(t)| \leq \Big(mE_\alpha(-\lambda t^\alpha) \Big)^{\frac{1}{a}},$$

where $\alpha \in (0,1)$, $\lambda > 0$, $m(0) = 0$, $m \geq 0$.

Lemma 3.4. *Let $x = 0$ be an equilibrium point of (3.1) and $\mathbb{D} \subset \mathbb{R}$ be a domain containing the origin. Let*

$$V : [0,\infty) \times \mathbb{D} \to \mathbb{R}$$

be a continuously differentiable functions and locally Lipschitz with respect to x such that

$$\alpha_1 \|x\|^a \leq V(t, x(t)) \leq \alpha_2 \|x\|^{ab}, \tag{3.4}$$

$$cD_0^\alpha V(t, x(t)) \leq -\alpha_3 \|x\|^{ab}, \tag{3.5}$$

where $t \geq 0$, $x \in \mathbb{D}$, $\alpha \in (0,1)$, α_1, α_2, α_3, a, and b are arbitrary positive constants. Then, $x = 0$ is Mittag-leffler stable with x satisfying

$$|x(t)| \leq \left[mE_\alpha\Big(-\frac{\alpha_3}{\alpha_2} t^\alpha \Big) \right]^{\frac{1}{a}},$$

where $m = \frac{V(0,x(0))}{\alpha_1} \geq 0$, and $m(0) = 0$ if and only if $x_0 = 0$,.

The following Lemma, and a proof can be found in [17].

Lemma 3.5. *Let $f, g \in C^r(E)$ where E is an open subset of $\mathbb{R}$ containing the origin and $r = 1, 2, 3, ...,$. Suppose that $f(0) = 0$ and that $Df(0)$ has 1 eigenvalue with zero real parts and 1 eigenvalue with negative real parts. The system (3.1) can be written in the diagonal form*

$$\begin{cases} cD_0^\alpha x(t) = F(x,y), \\ cD_0^\alpha y(t) = -y + G(x,y), \end{cases} \tag{3.6}$$

where $\alpha \in (0,1)$, $(x,y) \in \mathbb{R}^2$, and $F(0) = G(0) = 0$, $DF(0) = DG(0) = 0$; furthermore, there exists a $\delta > 0$ and a function $h \in C^r_\alpha(N_\delta(0))$ that defines the local center manifold

$$W^c_{loc} := \{(x,y) \in \mathbb{R}^c \times \mathbb{R}^s : y = h(x) \quad \textit{for } |x| < \delta\},$$

and satisfies

$$cD_0^{\alpha}h(x)F(x,h(x)) + h(x) - G(x,h(x)) = 0,$$

for $|x| < \delta$; *and the flow on the center manifold* $W^c(0)$ *is defined by the system of differential equations*

$$cD_0^{\alpha}x(t) = F(x,h(x)),$$

for all $x \in \mathbb{R}^c$ *with* $|x| < \delta$.

4. Stability of Non-Hyperbolic Equilibria

In this section our main results are stated. Similairly, as in section 3, the equilibrium point is assumed to be at the origin. The Lyapunov direct method is used to determine the stability of the non-hyperbolic equilibrium point. Where the extension of the Lyapunov direct method to fractional derivatives of the Caputo type can be found in [7].

Theorem 4.1. *Let* $f, g \in C^r(E)$ *where* E *is an open subset of* $\mathbb{R}$ *containing the origin, and* $r \in \mathbb{N}$. *Suppose that* $f(0,0)$, *and* $g(0,0) = 0$, *and that* f, g *are non-linear in* x *and* y. *Let the origin* $(0,0)$ *be a non-hyperbolic equilibrium point of* (3.1) *with only one eigenvalue of* (3.3) *having a zero real part, and the other eigenvalue being negative. Then, the flow on the center manifold* W^c *can be given by*

$$cD_0^{\alpha}x(t) = a_m x^m + ..., \tag{4.1}$$

where $m \geq 2$, *and* $a_m \neq 0$.

Proof. Since, the origin is a non-hyperbolic equilibrium point of (3.1), with one eigenvalue of (3.3) having a zero real part, and the other being negative, then by Lemma 3.5 and (3.6), we have

$$\begin{cases} cD_0^{\alpha}x(t) = F(x,y), \\ cD_0^{\alpha}y(t) = -y + G(x,y), \end{cases} \tag{4.2}$$

where $(x,y) \in \mathbb{R}^2$ and F and G are both nonlinear in x and y. Thus, $F(0,0) = G(0,0) = 0$, and by Theorem 3.5, we have that for each $r = 1, 2, 3, ...$, there exists a $\delta > 0$ and a function $h \in C_{\alpha}^r(N_{\delta}(0))$ that defines the local center manifold

$$W_{loc}^c := \{(x,y) \in \mathbb{R}^c \times \mathbb{R}^s : y = h(x) \quad \text{for } |x| < \delta\},$$

and satisfies

$$cD_0^{\alpha}h(x)F(x,h(x)) + h(x) - G(x,h(x)) = 0,$$

for $|x| < \delta$. Then, if we take r to be sufficiently large, we can approximate the function h by

$$h(x) = a_2x^2 + a_3x^3 + O(x^4),$$

in a neighbourhood of the origin, where $O(x^4)$ are polynomial terms with degree 4 or higher. It will become self evident, from (4.3) below, why $a_0 = a_1 = 0$ are taken. Since $h(x) = a_2x^2 + a_3x^3 + O(x^4)$, then

$$\begin{aligned} cD_0^{\alpha}h(x) &= a_2 cD_0^{\alpha}x^2 + a_3 cD_0^{\alpha}x^3 + cD_0^{\alpha}O(x^4) \\ &= a_2 J_0^{\alpha}(x^2)' + a_3 J_0^{\alpha}(x^3)' + J_0^{\alpha}O((x^4)') \\ &= 2a_2 J_0^{\alpha}x + 3a_3 J_0^{\alpha}x^2 + 4a_4 J_0^{\alpha}x^3 + ... \\ &= 2a_2\frac{\Gamma(2)}{\Gamma(3)}x^2 + 3a_3\frac{\Gamma(3)}{\Gamma(4)}x^3 + 4a_4\frac{\Gamma(4)}{\Gamma(5)}x^4 + ... \end{aligned}$$

Hence we obtain,

$$\left(2a_2\frac{\Gamma(2)}{\Gamma(3)}x^2 + 3a_3\frac{\Gamma(3)}{\Gamma(4)}x^3 + 4a_4\frac{\Gamma(4)}{\Gamma(5)}x^4 + ...\right)F(x,h(x)) + h(x) - G(x,h(x)) = 0. \tag{4.3}$$

Note that, h is approximated as a power series expansion near the origin, with polynomial terms of degree at least 2. Then, equation (4.3) can be solved by setting the coefficients of the polynomial terms with like powers to zero. Thus, from (4.3) it follows that setting the last two terms in the algebraic expression to zero is sufficient. Namely,

$$h(x) - G(x,h(x)) = 0. \tag{4.4}$$

Therefore, we obtain that $h(x) = G(x,h(x))$. Since, $G(x,h(x))$ contains polynomial terms of order 2 or greater, then the coefficients $a_0 = a_1 = 0$, and there exists $i = 2,3,...$, such that $a_i \neq 0$, and

$$h(x) = a_i x^i + O(x^{i+1}). \tag{4.5}$$

Lastly, from Lemma 3.5, the flow on the center manifold is given by

$$cD_0^\alpha x(t) = F(x,h(x)) = F(x, a_i x^i + O(x^{i+1})).$$

Then, from above it is easy to verify that

$$cD_0^\alpha x(t) = a_m x^m + ...,$$

where m is the smallest power of the polynomial terms. The result follows. □

Remark 4.2. Theorem 4.1 is new. It shows that the flow in a neighbourhood of the non-hyperbolic equilibrium $(0,0)$ can be studied by reducing system (3.1) to the form of

$$cD_0^\alpha x(t) = a_m x^m + ...,$$

where $x \in \mathbb{R}$. Then, instead of determining the stability of the non-linear system (3.1), it is sufficient to determine the stability of the flow on the Center manifold $W^c(0)$, given by the above expression. Furthermore, if an equilibrium point of (3.1) is not on the origin, then it can simply be translated to the origin.

Below we give a result to determine the behaviour of the flow on the Center manifold, $W^c(0)$, and ultimately stability of the non-hyperbolic equilibrium point $(0,0)$.

Theorem 4.3. *Let the conditions in Theorem 4.1 hold. If m is odd, $a_m < 0$, and $x_0 \neq 0$, where a_m is given in (4.1), then the origin $(0,0)$ is locally asymptotically stable; If m is even, $a_m < 0$ and $x_0 > 0$, then the origin is locally asymptotically stable.*

Proof. We prove the case when m is odd, as the case when m is even followed in the exact same manner. We proceed to prove the result for the case when m is odd. By Theorem (4.1), we have that the flow of the non-hyperbolic equilibrium point can be described by

$$cD_0^\alpha x(t) = a_m x^m + ...,$$

where the first term $a_m x^m$ dominates the behaviour of the flow since we are in a neighborhood of the origin $(0,0)$. Thus, we study the behaviour of

$$cD_0^\alpha x(t) = a_m x^m, \tag{4.6}$$

near the origin $(0,0)$. Let $V(x(t)) = x(t)^{m+1}$ be the Lyapunov candidate function. Then, $\dot{V}(x(t)) = (m+1)x^m(t)\dot{x}(t)$. Let $\epsilon_0 > 0$ be a positive constant. Then,

$$\int_t^{t+\epsilon_0} \dot{V}(x(\tau))d\tau = \int_t^{t+\epsilon_0} (m+1)x^m(\tau)\dot{x}(\tau)d\tau \\ = x^{m+1}(t+\epsilon_0) - x^{m+1}(t). \tag{4.7}$$

Applying, the Riemann-Liouville derivative to (4.6), yields

$$\dot{x}(t) = -D_t^{1-\alpha}x^m(t) \\ = -\frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-\tau)^{\alpha-1}x^m(\tau)d\tau\right)' \\ = -\frac{1}{\Gamma(\alpha)}\left(\frac{x^m(0)}{(t-\tau)^{1-\alpha}}\right)_{t=\tau} + \frac{1-\alpha}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-2}x^m(\tau)d\tau.$$

It then follows from $x_0 \neq 0$, and $x = 0$ being the equilibrium point that $x_0x(t) > 0$ for all $t \in (0,\infty)$. Furthermore, since $a_m < 0$, then $x(t) \leq x_0$ for all $t \in [0,\infty)$. Thus, $x_0x(t) \leq x_0^2$ for all $t \in [0,\infty)$, where the equality holds for $t = 0$. Then,

$$x_0\dot{x}(t) \leq -\frac{x^{m+1}(0)t^{\alpha-1}}{\Gamma(\alpha)} < 0.$$

Indeed,

$$x_0\dot{x}(t) = -\frac{1}{\Gamma(\alpha)}\left(\frac{x^{m+1}(0)}{(t-\tau)^{1-\alpha}}\right)_{t=\tau} + \frac{1-\alpha}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-2}x_0x(\tau)x^{m-1}(\tau)d\tau \\ \leq -\frac{1}{\Gamma(\alpha)}\left(\frac{x^{m+1}(0)}{(t-\tau)^{1-\alpha}}\right)_{t=\tau} + \frac{(1-\alpha)x_0^{2+i}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-2}x^{m-1-i}(\tau)d\tau,$$

where $0 \leq i \leq m-1$. Note that, the second line in the above can be repeated, by bringing in an x_0 term into the integral term, until $x^{m-1-i}(\tau) = 1$ or $m-1-i = 0$, thus

$$x_0\dot{x}(t) \leq -\frac{1}{\Gamma(\alpha)}\left(\frac{x^{m+1}(0)}{(t-\tau)^{1-\alpha}}\right)_{t=\tau} + \frac{(1-\alpha)x_0^{2+(m-1)}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-2}d\tau \\ = -\frac{1}{\Gamma(\alpha)}\left(\frac{x^{m+1}(0)}{(t-\tau)^{1-\alpha}}\right)_{t=\tau} + \frac{(1-\alpha)x_0^{m+1}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-2}d\tau \\ = -\frac{x^{m+1}(0)t^{\alpha-1}}{\Gamma(\alpha)}.$$

Then, from (4.7), we have

$$\int_t^{t+\epsilon_0} \dot{V}(x(\tau))d\tau = \int_t^{t+\epsilon_0} (m+1)x^m(\tau)\dot{x}(\tau)d\tau \\ = x^{m+1}(t+\epsilon_0) - x^{m+1}(t) < 0.$$

Hence, V is a decreasing function.

Next, we show $\lim_{t\to\infty} V(x(t)) = 0$. Indeed, suppose that there exists a positive constant $\delta_0 > 0$ such that $x_0x(t) \geq \delta_0$ for all $t \geq 0$, we have

$$\begin{aligned}
cD_0^\alpha V &= (m+1)J_0^{1-\alpha}x^m\dot{x} \le \frac{(m+1)\delta_0^m}{x_0^{m+1}}J_0^{1-\alpha}x_0\dot{x} \\
&\le -\frac{(m+1)\delta_0^m}{\Gamma(\alpha)}J_0^{1-\alpha}t^{\alpha-1} \\
&= -\frac{(m+1)\delta_0^m}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^t (t-x)^{-\alpha}x^{\alpha-1}dx \\
&= -\frac{(m+1)\delta_0^m}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^1 [t(1-u)]^{-\alpha}(tu)^{\alpha-1}tdu \\
&= -\frac{(m+1)\delta_0^m}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^1 u^{\alpha-1}(1-u)^{-\alpha} \\
&= -\frac{(m+1)\delta_0^m}{\Gamma(\alpha)\Gamma(1-\alpha)}\frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(1)} = -(m+1)\delta_0^m \\
&= -\frac{(m+1)\delta_0^m x_0^{m+1}}{x_0^{m+1}} \le -\alpha_3 x^{m+1} = -\alpha_3 V.
\end{aligned}$$

Where the substitution $u := u(x) = \frac{x}{t}$ was used, and $\alpha_3 = \frac{(m+1)\delta_0^m}{x_0^{m+1}} > 0$. It follows from Lemma 3.4 that $\lim_{t\to\infty} V(x(t)) = \lim_{t\to\infty} x(t)^{m+1} = 0$, contradicting the assumption that $x_0x(t) \ge \delta_0$. Thus, the equilibrium point $x = 0$ is locally asymptotically stable, and the result follows. □

Remark 4.4. The case for m being even follows in the same manner, though we need to restrict $x_0 > 0$, to ensure that (3.4) is satisfied. Indeed, if $m \ge 2$ is even and $x_0 < 0$, choosing the Lyapunov candidate function to be $V(t,x) = x^{m+1}$, as above Then, $V(0,x_0) = x_0^{m+1} < 0$, and (3.4) is not satisfied.

Below we provide an example illustrating Theorem (4.1), and Theorem (4.3), **and FDE12 solver is used to generate the numerical simulation**.

Example 4.1. Consider the following planar Caputo system

$$\begin{cases} cD_0^\alpha x(t) = -xy, \\ cD_0^\alpha y(t) = -y + x^2, \end{cases} \tag{4.8}$$

then the origin $(0,0)$ is locally asymptotically stable.

Proof. The linearized part of (4.8) is given by

$$A(x,y) = \begin{pmatrix} -y & -x \\ 2x & -1 \end{pmatrix}. \tag{4.9}$$

It is clear to see that for $A(0,0)$ that the determinant $|A(0,0)| = 0$, and $\text{tr}(A(0,0)) = -1 < 0$. Thus, we have that the eigenvalues are 0 and -1. Furthermore, (4.8) is already in the form of (3.6). By (4.4) we have that the local center manifold can be given by $y(x) = h(x) = x^2$. and by Theorem 4.1, we have that the flow on the center manifold is given by

$$cD_0^\alpha x(t) = -x^3,$$

where $m = 3$, and $a_3 = -1 < 0$. Then, by Theorem 4.3, we have that the origin $(0,0)$ is locally asymptotically stable. □

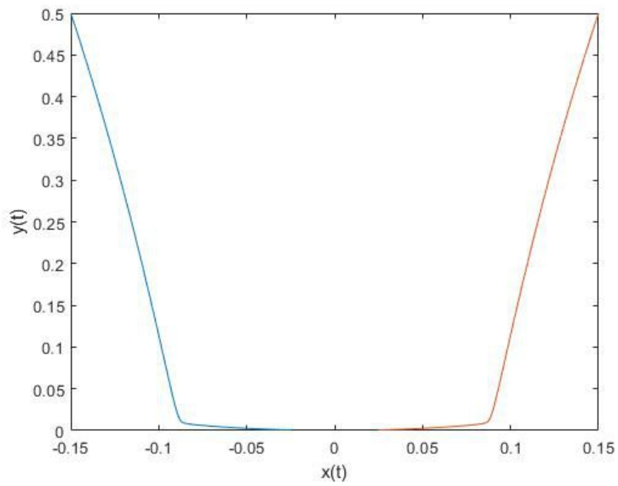

FIGURE 1. Trajectories of (4.8) illustrating that the origin $(0, 0)$ is locally asymptotically stable, with $(x_0, y_0) = (-0.15, 0.5)$ and $(x_0, y_0) = (0.15, 0.5)$. Here $\alpha = 0.98$.

5. Conclusion

In this manuscript a method of determining the qualitative behavior of a non-hyperbolic equilibrium point of a non-linear planar system with the Caputo derivative is shown. Using Theorem 4.1, it is shown that, under suitable conditions, the flow on the center manifold W^c could be reduced to

$$cD_0^{\alpha} x(t) = a_m x^m + ..., \tag{5.1}$$

where $m \geq 2$, and $a_m \neq 0$. Furthermore, by using the Lyapunov direct method, it is shown, in Theorem 4.3, that (5.1) could be locally asymptotically stable. This type of result is new. Additionally, in this paper we require (3.3) to have one eigenvalue with a negative real part. Future work could consider the case when (3.3) has an eigenvalue with a zero real part, and no eigenvalues with a negative real part. The methods introduced in this paper would provide a strong insight into how to proceed in this direction.

References

[1] J. Alidousti and E. Ghafari, *Stability and bifurcation of fractional tumor-immune model with time delay*, Computational methods for differential equations, In Press.

[2] A. Boukhouima, K. Hattaf, and N. Yousif, *Dynamics of a Fractional Order HIV Infection Model with Specific Functional Response and Cure Rate*, International Journal of Differential Equations, *2017*(1) (2017), 1-8.

[3] M. H. Derakhshan, A. Ansari, and MA. Darani, *On asymptotic stability of weber fractional differential systems*, Computational Methods for Differential Equations, *6*(1) (2018), 30-39.

[4] A. S. Hendyt and Z. Mahmoud, *Global consistency analysis of L1-Galerkin spectral schemes for coupled nonlinear space-time fractional Schrödinger equations*, Applied Numerical Mathematics, *156*(1) (2020), 276-302.

[5] M. Kirane and B. T. Torebek, *A Lyapunov type inequality for a fractional boundary value problem with Caputo-Fabrizio Derivative*, Journal of Mathematical Inequalities, *12*(4) (2018), 1005-1012.

[6] C. Li and Y. Ma, *Fractional Dynamical Systems and Its Linearization Theorem*, Nonlinear Dynamics, *71*(4) (2013), 621-633.

[7] Y. Li, YQ. Chen, and I. Podlubny, *Mittag-Leffler Stability of fractional order nonlinear dynamic systems*, Automatica, *45*(8) (2009), 1965-1969.

[8] Z. Mahmoud, *Existence, uniqueness and numerical analysis of solutions of tempered fractional boundary value problems*, Applied numerical mathematics (2019), *145*(1), 429-457.

[9] D. Matignon, *Stability results for fractional differential equations with applications to control processing*, Computational Engineering in Systems and Application Multiconference, IMACS, IEEE-SMC, Lille, France, *133*(2) (1996), 963-968.

[10] Z. M. Odibat. *Analytical study on linear systems of fractional differential equations*, Computers and Mathematics with Applications *59*(1) (2010), 1171-1183.

[11] I. Podlubny, *Fractional Differential Equations*, Mathematics in Science and Engineering, *198* (1999).

[12] I. Podlubny, *Fractional Differential Equations*, Academic Press, 1999.

[13] D. Qian, C. Li, R. P. Agarwal, and P. J. Y. Wong, *Stability analysis of fractional differential system with Riemann-Liouville derivative*, Mathematical and Computer Modeling, *52*(5) (2010), 862-874.

[14] J. A. L. Renteria, B. A. Hernandez, and G. F. Anaya, *LMI Stability Test For Fractional Order Initilized Systems*, Applied and Computational Mathematics, *18*(1) (2019), 50-61.

[15] D. Rostamy and E. Mottaghi, *Stability analysis of a fractional order epidemics model with multiple equilibriums*, Advances in Difference Equations, *2016*(1) (2016).

[16] H. A. A. El-Saka, *The fractional order SIS-epidemic model with a variable population size. Journal of Egyptian Mathematical Society*, 22 (2014), 50-54.

[17] K. Sayevand, *Fractional Dynamical Systems: A fresh view on the local qualitative theorems*, Int. J. Nonlinear Anal. Appl, *7*(2) (2016), 303-318.

[18] J. R. L. Webb, *Initial value problems for Caputo fractional equations with singular nonlinearities*, Electronic Journal of Differential Equations, *117*(1) (2019), 1-32.