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Optimal control of satellite attitude and its stability based on quaternion parameters

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Abstract

This paper proposes an optimal control method for the chaotic attitude of the satellite when it is exposed to external disturbances. When there is no control over the satellite, its chaotic attitude is investigated using Lyapunov exponents (LEs), Poincare diagrams, and bifurcation diagrams. In order to overcome the problem of singularity in the great maneuvers of satellite, we consider the kinematic equations based on quaternion parameters instead of Euler angles, and obtain control functions by using the Pontryagin maximum principle (PMP). These functions are able to reach the satellite attitude to its equilibrium point. Also the asymptotic stability of these control functions is investigated by Lyapunov's stability theorem. Some simulation results are given to visualize the effectiveness and feasibility of the proposed method.

Keywords. Optimal control, Stability, Quaternion, Chaotic systems.

2010 Mathematics Subject Classification. 37N35, 49J15, 49-XX..

1. Introduction

Satellites are purposely located in orbit around the Earth, other planets, or the Sun. They should put themselves in the right direction relative to the Sun and Earth. Especially they have to maintain, their solar panels toward the Sun and their antennas toward the Earth.

It is important for us to control the position of the satellites due to the gradual deviation of their orientation as well as placing them in the new desired position. Many control have been introduced for this purpose so far. In general, they are classified as active or passive methods. Passive method has been studied in [13]. Some of the active methods include, generalized predictive control method [11], sliding-mode method [2], control by Lyapunov function [6], nonlinear control via linear matrix inequality [25], linear time-delay feedback control [9], finite-time stabilization of satellite quaternion attitude [17], and impulsive control of satellite attitude based on kinematic equations by Euler angels [10]. In this research, chaotic control of the system is formulated as a nonlinear optimal control problem, and the HJB equation corresponding to it is obtained, then by considering an appropriate Lyapunov function, its stability is proved.

In connection with the satellites some optimal control methods of attitude include optimal sliding mode control under stochastic disturbances [26], optimal control formulation [24], fuzzy optimal control [22], inverse optimal adaptive control [15], optimal control using redundant kinematics parameterizations [3], optimal sliding mode manoeuvring control and active vibration reduction [8], optimal control using Euler parameters without angular velocity measurements [4], constrained optimal PID-like controller design [14], optimal nonlinear feedback control [29], robust and optimal attitude control of spacecraft with disturbances [21], and triaxial optimal control of satellite attitude based on PMP [18].

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When a satellite is influenced by big disturbances due to a given torque, or when performing the great maneuvers is required, there deviation of Euler angles from its desired attitude is large. This leitmotiv causes singularity problem in solving kinematic equations with Euler angles. With regard to this matter, as well as control of satellite attitude in this situation, this paper proposes a control function on each of the three satellite's kinetic equations along with quaternion parameters instead of Euler angles in the satellite's kinematic equations.

The optimal attitude stabilization of spacecraft has been investigated in [20]. In this study a robust and optimal three-axis attitude control scheme has been presented for the rigid body motion with external disturbances. Also the optimality property of the robust attitude control law has been analyzed based on the minimax approach and the inverse optimal approach.

Attitude and vibration control of a satellite with a flexible solar panel using LQR tracking with infinite time has been investigated in [7]. In this research, the LQR method is applied to the nonlinear system by using Jacobi linearization. This method is described ineffective when the satellite's angular velocities are higher. So, this method is most useful for linear systems. Also optimal control via SDRE has been studied in [19, 23]. The SDRE method needs a pseudo-linear form of the nonlinear system, so its controller has the same structure as the LQR controller, except that all the coefficients are state-dependent. The SDRE controller, by its structure, ensures that there is a near-optimal solution for the system, and works better than The LQR method. However, due to linearization, if the deviation is large in the around the equilibrium point (especially in the nonlinear complex systems), this method doesn't work well. Our proposed method, the optimal control method, works with nonlinear equations and is free from this defect.

The basis of the research on optimal control derives from the Pontryagin maximum principle (PMP). The result this principle are derived using ideas from the calculus of variations. This principle has already been used by various researchers, See for example [16, 18]. But its use in controlling satellite attitude, when quaternion parameters accompany the system and also the stability of the controls obtained is analyzed, is one of the unique advantages and also the novelty of our research.

In this paper we obtain the optimal control functions by using the Pontryagin maximum principle in the framework of minimization an objective function that be expressed in terms of control functions and attitude error vector of system. Therefore obtained control functions are able to minimize the difference between attitude of system with its equilibrium point. Quaternion parameters are used to overcome singularity problem in the numerical solution of system. The asymptotic stability of control functions is investigated by Lyapunov's stability theorem. The simulation results show that these controllers are able to return back the satellite's attitude to its equilibrium point, when satellite attitude is tilted of this point.

This paper is organized as follows: Section 2, expresses the quaternion and equations of satellite motion. Also, in this section a terse introduction of the quaternion is provided. Section 3, describes chaotic behavior of a system using LEs. Also, optimal control functions are extracted in this section via PMP. The stability analysis is investigated by Lyapunov's stability theorem in section 4. The simulation results are shown in section 5. Finally, our concluding remarks are given.

2. Quaternion and motion equations

2.1. ***Quaternion.*** The unit quaternion vector provides a non-singular representation of satellite kinematic equations. The four-component quaternion vector is defined as [28]

$$q = iq_1 + jq_2 + kq_3 + q_4, \tag{2.1}$$

where i, j, and k are imaginary numbers satisfying the condition

$$\begin{aligned} &i^2 = j^2 = k^2 = -1, \\ &ij = -ji = k, \\ &jk = -kj = i, \\ &ki = -ik = j. \end{aligned} \tag{2.2}$$

Table 1. Initial conditions and constant values of the SA system.

q_{10}	q_{20}	q_{30}	q_{40}	$w_{x_0}(r/s)$	w_{y_0}	w_{z_0}	$I_x(kgm^2)$	I_y	I_z
0.2425	0.0491	0.4645	0.8503	0.2	0.1	0.2	3000	2000	1000

In this definition q_4 is a scalar part, and $Q = [q_1 \;\; q_2 \;\; q_3]^T$ form a vector part. Thus the quaternion $q = [q_1 \;\; q_2 \;\; q_3 \;\; q_4]^T$ may be written as $q = [Q^T \;\; q_4]^T$. The norm of q is defined as

$$|q| = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2}. \tag{2.3}$$

2.2. ***Equations of motion.*** The mathematical model of a satellite is described by kinetic and kinematic equations of motion.

2.2.1. *Kinetic Equations.* The relationship between angular velocity and torque in the body frame is expressed by kinetic equations. If we consider the satellite as a rigid object and also the inertia of its body is diagonal and along to the actuators, then kinetic equations can be obtained from a Newton-Euler formula [5]

$$\begin{aligned} I_x\dot{w}_x &= [(I_y - I_z)w_y w_z + c_x]\,, \\ I_y\dot{w}_y &= [(I_z - I_x)w_x w_z + c_y]\,, \\ I_z\dot{w}_z &= [(I_x - I_y)w_x w_y + c_z]\,, \end{aligned} \tag{2.4}$$

where w_x, w_y, w_z are angular velocities around axes of the body, I_x, I_y, I_z are the inertial moments of satellite around its principal axes, and c_x, c_y, c_z are torques around these axes.

2.2.2. *Kinematic Equations.* By regarding the satellite as an rigid object, the kinematics equations based on quaternion parameters are expressed as follows

$$\begin{aligned} \dot{q_1} &= \tfrac{1}{2}\left(w_x q_4 - w_y q_3 + w_z q_2\right), \\ \dot{q_2} &= \tfrac{1}{2}\left(w_x q_3 + w_y q_4 - w_z q_1\right), \\ \dot{q_3} &= \tfrac{1}{2}\left(w_y q_1 - w_x q_2 + w_z q_4\right), \\ \dot{q_4} &= -\tfrac{1}{2}\left(w_x q_1 + w_y q_2 + w_z q_3\right). \end{aligned} \tag{2.5}$$

The relationship between attitude and angular velocity is explained by the kinematic equations. In the following, we use the notation SA to refer to the equations (2.4) and (2.5).

3. Analysis of chaos and optimal control

In this section, at first, chaotic behavior of the nonlinear SA system is investigated using LEs, phase portraits, Poincare diagrams, and bifurcation diagrams. Then some theoretical results obtained from the use of PMP [1] on a given dynamical system in accordance with what has been investigated in [18], are presented. These results are applied on the system SA to acquire control functions and the corresponding control system.

3.1. ***Analysis of chaos.*** By assuming initial conditions and constant values given in Table.1, the LEs of the SA system are obtained under the perturbing torque

$$\begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} = \begin{bmatrix} -1200 & 0 & (1000)\frac{\sqrt{6}}{2} \\ 0 & 350 & 0 \\ -(1000)\sqrt{6} & 0 & -400 \end{bmatrix} \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}, \tag{3.1}$$

for more detail the reader can see [12, 27]. The LEs are values that determine the degree of separation of infinitesimally close trajectories of a dynamical system. The LEs are illustrated in Figure.1. The chaos of the system is understood from the existence of positive LEs such as $\lambda_{w_x=0.13533}$. Furthermore the phase portraits in Figure.2, and Poincare diagram in Figure.3 illustrate chaotic behaviour the SA system under initial conditions and constants. Also bifurcation diagrams for the external torque parameters c_x, c_y, c_z in equations (2.4) are presented in Figures.4 and 5.

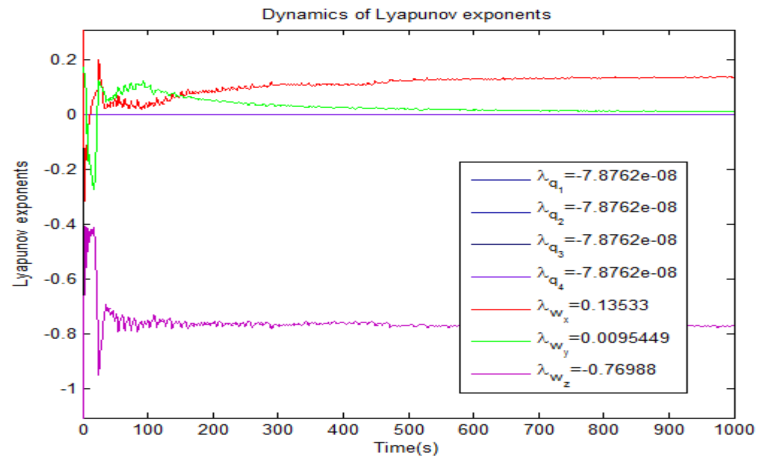

FIGURE 1. LEs of the SA system under the initial conditions and constant values in Table.1.

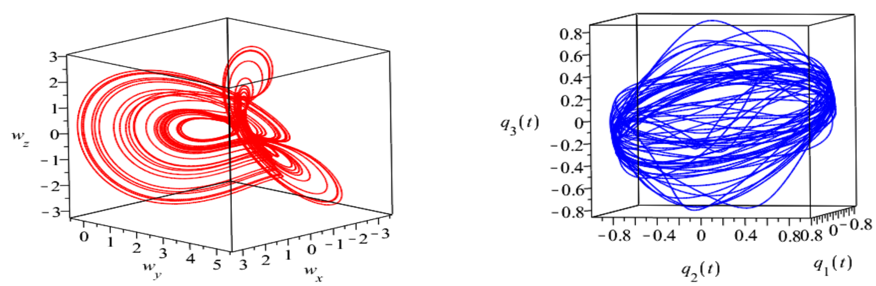

FIGURE 2. Phase portraits of the angular velocities and quaternion parameters of SA system, whit perturbing torque (3.1).

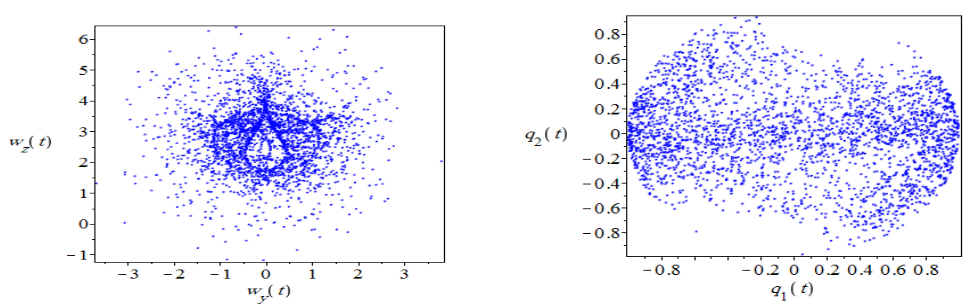

FIGURE 3. Poincare diagrams of the angular velocities and quaternion parameters of SA system, whit perturbing torque (3.1).

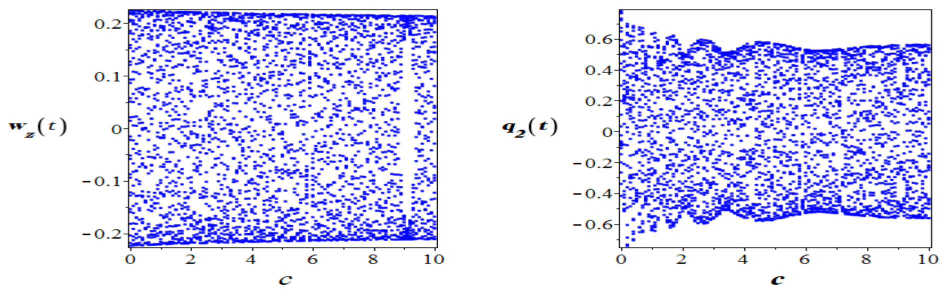

FIGURE 4. Bifurcation diagrams corresponding to angular velocitie w_z and quaternion parameter q_2 under torques $c_x = c_y = c_z = c \in [0, 10]$ of the system (2.4) and (2.5).

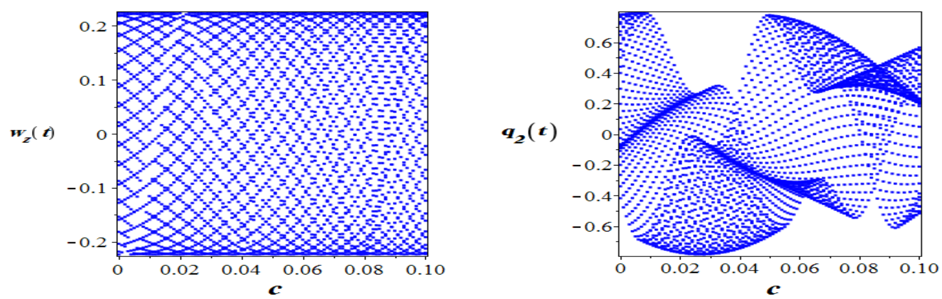

FIGURE 5. Bifurcation diagrams corresponding to angular velocitie w_z and quaternion parameter q_2 under torques $c_x = c_y = c_z = c \in [0, 0.1]$ of the system (2.4) and (2.5).

3.2. ***Theoretical results.*** Considering the system

$$\dot{x}_i(t) = f_i(x_1(t), x_2(t), ..., x_n(t)), \qquad i = 1, 2, ..., n \tag{3.2}$$

where, $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}, (i = 1, 2, ..., n)$ are continuous nonlinear functions, and $x_1, x_2, ..., x_n$ are the state variables, the controlled system with initial and final conditions is

$$\dot{x}_i(t) = f_i(x_1(t), x_2(t), ..., x_n(t)) + u_i(t), \qquad i = 1, 2, ...n \tag{3.3}$$

$$x_i(0) = x_{i,0}, \;\; x_i(t_f) = \bar{x}_i, \tag{3.4}$$

where, $\bar{x}_i, (i = 1, 2, ..., n)$ are components equilibrium point $\bar{X}$, t_f is a constant final time and $u_i(t), (i = 1, 2, ..., n)$ are controllers which minimize the objective function

$$J = \int_0^{t_f} \sum_{i=1}^{n} \frac{1}{2}[\alpha_i(x_i(t) - \bar{x}_i)^2 + \beta_i u_i^2(t)]dt, \tag{3.5}$$

where, $\alpha_i, \beta_i, (i = 1, 2, ..., n)$ are positive constants and J as a function of variables x_i and u_i. Control functions are designed to bring the system to equilibrium point in time t_f. The corresponding Hamiltonian is

$$H(x, u, \lambda) = \sum_{i=1}^{n} -\frac{1}{2}[\alpha_i(x_i(t) - \bar{x}_i)^2 + \beta_i {u_i}^2(t)] + \sum_{i=1}^{n} \lambda_i(f_i + u_i(t)), \tag{3.6}$$

where, $x = (x_1(t), x_2(t), ..., x_n(t))$, $u = (u_1(t), u_2(t), ..., u_n(t))$, $\lambda = \lambda_1(t), \lambda_2(t), ..., \lambda_n(t))$, and $\lambda_i, (i = 1, 2, ..., n)$ are costate variables. According to PMP, optimal conditions require that

$$\begin{aligned} &\dot{x_i}(t) = \frac{\partial H}{\partial \lambda_i}, \\ &\dot{\lambda_i}(t) = -\frac{\partial H}{\partial x_i}, \\ &\frac{\partial H}{\partial u_i} = 0. \end{aligned} \tag{3.7}$$

By placing the Hamiltonian function (3.6) in (3.7), we get

$$\begin{aligned} &\dot{x_i}(t) = f_i(x_1(t), x_2(t), ..., x_n(t)) + u_i(t), \\ &\dot{\lambda_i}(t) = \alpha_i(x_i(t) - \bar{x_i}) - \frac{\partial(\sum_{i=1}^{n} \lambda_i(t)(f_i + u_i(t)))}{\partial x_i}, \end{aligned} \tag{3.8}$$

with optimal control functions

$$u_i^*(t) = \frac{\lambda_i(t)}{\beta_i}. \qquad i = 1, 2, ..., n. \tag{3.9}$$

From (3.4), (3.8) and (3.9) we obtain

$$\begin{aligned} &\dot{x_i}(t) = f_i(x_1(t), x_2(t), ..., x_n(t)) + \frac{\lambda_i(t)}{\beta_i}, \\ &\dot{\lambda_i}(t) = \alpha_i(x_i(t) - \bar{x_i}) - \frac{\partial(\sum_{i=1}^{n} \lambda_i(t)(f_i + u_i(t)))}{\partial x_i}, \\ &x_i(0) = x_{i,0}, \ x_i(t_f) = \bar{x_i}. \end{aligned} \tag{3.10}$$

System (3.10) is a set of first order nonlinear ODEs with boundary conditions. Now, we apply system (3.10) on SA equations, so that the system corresponding to optimal control of satellite attitude is obtained

$$\begin{aligned} &\dot{q_1} = \frac{1}{2}\left(w_x q_4 - w_y q_3 + w_z q_2\right), \\ &\dot{q_2} = \frac{1}{2}\left(w_x q_3 + w_y q_4 - w_z q_1\right), \\ &\dot{q_3} = \frac{1}{2}\left(w_y q_1 - w_x q_2 + w_z q_4\right), \\ &\dot{q_4} = -\frac{1}{2}\left(w_x q_1 + w_y q_2 + w_z q_3\right), \\ &I_x \dot{w_x} = [(I_y - I_z) w_y w_z - 1200 w_x + 1225 w_z + \tfrac{\lambda_5}{\beta_5}], \\ &I_y \dot{w_y} = [(I_z - I_x) w_x w_z + 350 w_y + \tfrac{\lambda_6}{\beta_6}], \\ &I_z \dot{w_z} = [(I_x - I_y) w_x w_y - 2450 w_x - 400 w_z + \tfrac{\lambda_7}{\beta_7}], \\ &\dot{\lambda_1} = \alpha_1(q_1 - \bar{q_1}) + \lambda_2(.5w_z) - \lambda_3(.5w_y) + \lambda_4(.5w_x), \\ &\dot{\lambda_2} = \alpha_2(q_2 - \bar{q_2}) - \lambda_1(.5w_z) + \lambda_3(.5w_x) + \lambda_4(.5w_y), \\ &\dot{\lambda_3} = \alpha_3(q_3 - \bar{q_3}) + \lambda_1(.5w_y) - \lambda_2(.5w_x) + \lambda_4(.5w_z), \\ &\dot{\lambda_4} = \alpha_4(q_4 - \bar{q_4}) - \lambda_1(.5w_x) - \lambda_2(.5w_y) - \lambda_3(.5w_z), \\ &\dot{\lambda_5} = \alpha_5(w_x - \bar{w_x}) - \lambda_1(.5q_4) - \lambda_2(.5q_3) + \lambda_3(.5q_2) + \lambda_4(.5q_1) \\ &+\lambda_5 \tfrac{1200}{I_x} - \lambda_6 \tfrac{I_z - I_x}{I_y} w_z - \lambda_7 \tfrac{1}{I_z}[(I_x - I_y) w_y - 2450], \\ &\dot{\lambda_6} = \alpha_6(w_y - \bar{w_y}) + \lambda_1(.5q_3) - \lambda_2(.5q_4) - \lambda_3(.5q_1) + \lambda_4(.5q_2) \\ &-\lambda_5 \tfrac{I_y - I_z}{I_x} w_z - \lambda_6 \tfrac{350}{I_y} - \lambda_7 \tfrac{I_x - I_y}{I_z} w_x, \\ &\dot{\lambda_7} = \alpha_7(w_z - \bar{w_z}) - \lambda_1(.5q_2) + \lambda_2(.5q_1) - \lambda_3(.5q_4) + \lambda_4(.5q_3) \\ &-\lambda_5 \tfrac{1}{I_x}[(I_y - I_z) w_y + 1225] - \lambda_6 \tfrac{I_z - I_x}{I_y} w_x + \lambda_7 \tfrac{400}{I_z}, \\ &X(0) = X_o, \qquad X(t_f) = \bar{X}, \end{aligned} \tag{3.11}$$

where, X and $\bar{X}$ are satellite's attitude vector $[q_1 \ \ q_2 \ \ q_3 \ \ q_4 \ \ w_x \ \ w_y \ \ w_z]^T$ and equilibrium point of SA system respectively, and t_f is a given time. Note that in (3.11) the control functions is only considered for kinematic equations. Next, by solving the nonlinear system (3.11) with given boundary conditions (3.4), we obtain the optimal control functions and the optimal state trajectory of the SA system.

4. Stability analysis

In this section, at first a lemma is presented about angular velocities w_x, w_y, w_z, and control functions corresponding with dynamics of these velocities. Then Lyapunov's stability theorem is used to analyze the asymptotic stability of system under control functions corresponding to this method.

Lemma 1. *Consider the system (3.11) with terminal conditions* $X(t_0) = [q_{1_0} q_{2_0} q_{3_0} q_{4_0} w_{x_0} w_{y_0} w_{z_0}]^T$ *and* $[\lambda_1(t_f) ... \lambda_7(t_f)]^T = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T$. *For* $t \in (t_0, t_f)$ *we have*

$$\begin{aligned} &i)\ \ w_x(\tfrac{\lambda_5}{\beta_5}) < 0 \\ &ii)\ \ w_y(\tfrac{\lambda_6}{\beta_6}) < 0 \\ &iii)\ \ w_z(\tfrac{\lambda_7}{\beta_7}) < 0 \end{aligned} \tag{4.1}$$

where $\frac{\lambda_5}{\beta_5}$, $\frac{\lambda_6}{\beta_6}$ *and* $\frac{\lambda_7}{\beta_7}$ *are optimal control functions obtained in (3.9) corresponding to three kinetic equations (2.4).*

Proof. The performance of optimal control $\frac{\lambda_5}{\beta_5}$ versus the angular velocity w_x is similar to the performance of the control force caused by the spring and the damper versus the displacement of the spring from its equilibrium state. In fact, in each of the kinetic equations the control functions $\frac{\lambda_5}{\beta_5}$, $\frac{\lambda_6}{\beta_6}$, $\frac{\lambda_7}{\beta_7}$ act like a damper for angular velocities w_x, w_y, w_z, respectively. For example, in the dynamic equation

$$\dot{w_x} = \frac{1}{I_x}[(I_y - I_z)w_y w_z - 1200 w_x + 1225 w_z + \frac{\lambda_5}{\beta_5}], \tag{4.2}$$

the control function $\frac{\lambda_5}{\beta_5}$ acts like a damper for angular velocity w_x, so that the purpose of applying the controller $\frac{\lambda_5}{\beta_5}$ is to approach the value w_x to zero in this equation. In other words, if for $t \in (t_0,\ t_f)$, w_x is positive (right side of equilibrium value), control function $\frac{\lambda_5}{\beta_5}$ acts to reduce it, and regardless of the amount of other variables in this equation, $\frac{\lambda_5}{\beta_5}$ should be negative.; if w_x is negative(left side of equilibrium value), controller $\frac{\lambda_5}{\beta_5}$ acts to increase it, and consequently $\frac{\lambda_5}{\beta_5}$ should be positive, and having the same sign is inconsistent with this control process. The similar discussion is right for controller functions $\frac{\lambda_6}{\beta_6}$ and $\frac{\lambda_7}{\beta_7}$. Therefore, these functions are different with their corresponding angular velocities in the sign.

□

Theorem 1. *By choosing conveniently coefficients* β_5, β_6 *and* β_7, *the control functions* $\frac{\lambda_5}{\beta_5}$, $\frac{\lambda_6}{\beta_6}$ *and* $\frac{\lambda_7}{\beta_7}$ *asymptotically stabilize the system (3.11) around the equilibrium point* $\bar{X} = [0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0]^T$.

Proof. Consider the following Lyapunov function candidate

$$V = \frac{1}{2}[q_1^2 + q_2^2 + q_3^2 + (1 - q_4)^2 + I_x w_x^2 + I_y w_y^2 + I_z w_z^2]. \tag{4.3}$$

Taking the time derivative of V along the trajectories of the system (3.11), we have

$$\begin{aligned} \dot{V} &= q_1\dot{q_1} + q_2\dot{q_2} + q_3\dot{q_3} + \dot{q_4}(q_4 - 1) + I_x w_x \dot{w_x} + I_y w_y \dot{w_y} + I_z w_z \dot{w_z} \\ &= \tfrac{1}{2}(q_1 w_x + q_2 w_y + q_3 w_z) + w_x\left[(I_y - I_z)w_y w_z + c_x\right] \\ &+ w_y\left[(I_z - I_x)w_x w_z + c_y\right] + w_z\left[(I_x - I_y)w_x w_y + c_z\right] \\ &+ \tfrac{\lambda_5}{\beta_5}w_x + \tfrac{\lambda_6}{\beta_6}w_y + \tfrac{\lambda_7}{\beta_7}w_z. \end{aligned} \tag{4.4}$$

or

$$\dot{V} = w_x(c_x + \frac{q_1}{2} + \tfrac{\lambda_5}{\beta_5}) + w_y(c_y + \frac{q_2}{2} + \tfrac{\lambda_6}{\beta_6}) + w_z(c_z + \frac{q_3}{2} + \tfrac{\lambda_7}{\beta_7}). \tag{4.5}$$

$$\dot{V} \leq |w_x(c_x + \frac{q_1}{2})| + w_x\tfrac{\lambda_5}{\beta_5} + |w_y(c_y + \frac{q_2}{2})| + w_y\tfrac{\lambda_6}{\beta_6} + |w_z(c_z + \frac{q_3}{2})| + w_z\tfrac{\lambda_7}{\beta_7}. \tag{4.6}$$

Suppose $-\varepsilon_1, -\varepsilon_2$ and $-\varepsilon_3$ are maximum of the functions $w_x\lambda_5, w_y\lambda_6$ and $w_z\lambda_7$, on the $(t_0,\ t_f)$, respectively. Since β_5, β_6 and β_7 are positive, it follows from Lemma 1 that $-\varepsilon_1, -\varepsilon_2$ and $-\varepsilon_3$ are negative on the $(t_0,\ t_f)$. Also suppose W_1, W_2, W_3, C_1, C_2 and C_3 are upper bounds of w_x, w_y, w_z, c_x, c_y and c_z on the $(t_0,\ t_f)$, respectively, then

$$\dot{V} \leq W_1(C_1 + \tfrac{1}{2}) - \frac{\varepsilon_1}{\beta_5} + W_2(C_2 + \tfrac{1}{2}) - \frac{\varepsilon_2}{\beta_6} + W_3(C_3 + \tfrac{1}{2}) - \frac{\varepsilon_3}{\beta_7}. \tag{4.7}$$

TABLE 2. Constant values of the SA system.

α_1	α_2	α_3	α_4	α_5	α_6	β_4	β_5	β_6	$I_x(kgm^2)$	$I_y(kgm^2)$	$I_z(kgm^2)$
15	15	15	10	10	10	0.2	0.4	0.2	3000	2000	1000

TABLE 3. Initial conditions of the SA system.

q_{10}	q_{20}	q_{30}	q_{40}	$w_{x_0}(r/s)$	$w_{y_0}(r/s)$	$w_{z_0}(r/s)$
0.2425	0.04915	0.4645	0.8503	0.2	0.6	0.8

Now by choosing $0 < \beta_5 < \frac{\varepsilon_1}{W_1(C_1+\frac{1}{2})}$, $0 < \beta_6 < \frac{\varepsilon_2}{W_2(C_2+\frac{1}{2})}$ and $0 < \beta_7 < \frac{\varepsilon_3}{W_3(C_3+\frac{1}{2})}$, we simply see that $\dot{V} < 0$, for $t \in (t_0,\, t_f)$. Figure.6 confirms mentioned discussion. Then asymptotic stability of the system (3.11) follows from Lyapunov's stability theorem. □

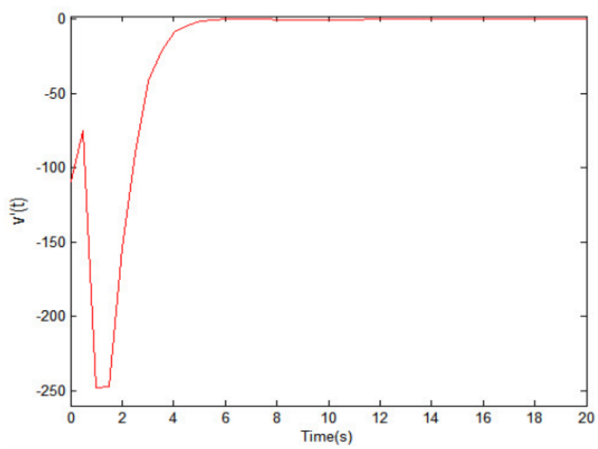

FIGURE 6. Time series response corresponding to the time derivative of V along the trajectories of the system (3.11).

5. Numerical simulation of optimal control

In this section, to verify the effectiveness of the theoretical results of section 4, we solve the system (3.11) with $t_f = 50s$, equilibrium point $\bar{X} = [0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 0]^T$ and the hypothetical constant values and the initial conditions are given in Table.2 and Table.3.

Numerical simulations are obtained using the Matlab's bvp4c solver. Figure.7 and Figure.8 illustrate the simulation results of the SA system based on the control functions (3.9). In these figures, time series responses corresponding to quaternion parameters and angular velocities demonstrates the appropriate performance of the optimal controllers with regard to the suppression of chaos. Also time series responses for optimal controllers $u_1 = \frac{\lambda_5}{\beta_5}, u_2 = \frac{\lambda_6}{\beta_6}, u_3 = \frac{\lambda_7}{\beta_7}$ are depicted in Figure.9.

6. Conclusions

In this paper, the problem of triaxial optimal control of the chaotic satellite attitude has been developed. Optimal control functions were proposed based on the Pontryagin maximum principle, and quaternion parameters were used to overcome singularity problem in the numerical solution of system. The control functions were powerful in order to align the body axes with the orbit axes when satellite attitude was confused to a disturbed torque. Moreover, angular

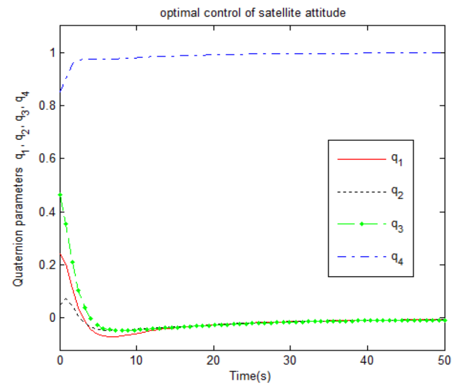

FIGURE 7. Time series responses corresponding to quaternion parameters in the system (3.11).

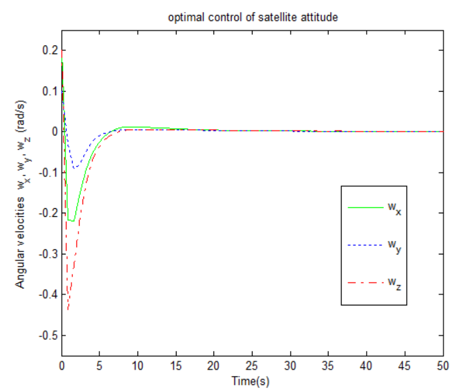

FIGURE 8. Time series responses corresponding to angular velocities in the system (3.11).

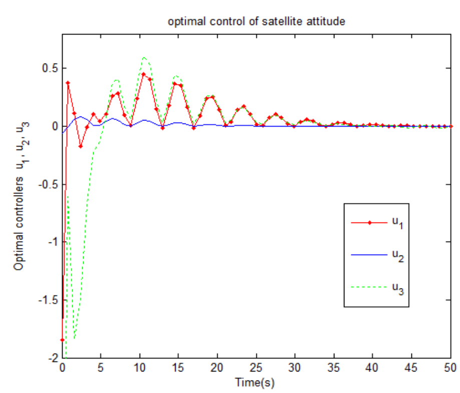

FIGURE 9. Time series responses corresponding to optimal controllers in the system (3.11).

velocities were diminished to zero by them. Also the asymptotic stability of optimal control functions was investigated by Lyapunov's stability theorem.

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