Numerical Method for the Solution of Algebraic Fuzzy Complex Equations

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Abstract
In this paper, the numerical solution of an algebraic complex fuzzy equation of degree \( n \), based on the parametric fuzzy numbers, is discussed. The unknown variable and right-hand side of the equation are considered as fuzzy complex numbers, whereas, the coefficients of the equation, are considered to be real crisp numbers. The given method is a numerical method and proposed based on the separation of the real and imaginary parts of the equation and using the parametric forms of the fuzzy numbers in the form of polynomials of degree at most \( m \). In this case, a system of nonlinear equations achieved. To get the solutions of the system, we used the Gauss-Newton iterative method. We also very briefly explain the conjugate of the solution of such equations. Finally, the efficiency and quality of the given method are tested by applying it to some numerical examples.

Keywords. fuzzy numbers, fuzzy complex numbers, fuzzy polynomial, algebraic fuzzy complex equation.

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1. Introduction

The problem of solving equations has many applications in the applied sciences like finance [11, 27], economy [12, 13, 28, 35], mechanics [19], and etc. Buckley and Qu have done studied on solving the fuzzy equations [14, 15]. They have also investigated analytically, first degree equations as \( ax + b = c \) and second degree equations as \( ax^2 + bx = c \), by considering the coefficients of the equation as real, or complex fuzzy numbers, as well as the unknown of the equation as real, or complex fuzzy number [16]. Fuzzy equations \( a + x = b \), where each component of the equation is fuzzy numbers, have been solved in [25] by \( \alpha \)-cut method and also in [29] using the superimposition by analytically. In [3, 5, 6, 30] researchers have solved the fuzzy equations numerically. The numerical solution of the fuzzy polynomial equations by the neural network was investigated in [4], as well as was investigated using the ranking method [33]. Abbasbandy and Amirfakhrian, considered the application of
fuzzy polynomial equation [2]. They proposed a method for acquiring the best fuzzy function approximation on a set of points. Amirfakhrian presented the numerical solution of the algebraic fuzzy equation of degree $n$ with a crisp variable in [8] and so with crisp coefficients in [9]. Researchers have studied on the solutions of fully fuzzy equations as $\tilde{a}\tilde{x}^2 + \tilde{b}\tilde{x} + \tilde{c} = \tilde{d}$, based on the restricted variation method and based on the optimization theory [7, 31]. The primary purpose of this study is solving an $n$th degree algebraic fuzzy complex equation with fuzzy complex unknown and real crisp coefficients. The given method is a numerical method that transforms an equation into a system of nonlinear equations. The obtained system is solved by the Gauss-Newton iterative method. The organization of this article is as follows: We first define the parametric form as a complex fuzzy number, conjugate of it, and some basic properties of them, in Section 2. Then, we introduce an $n$th degree algebraic fuzzy complex equation, with complex fuzzy unknown and crisp coefficients. Section 3 contains the Gauss-Newton method to solve a system of nonlinear equations. In section 4, we introduce an algorithm to get the solutions of an $n$th degree algebraic fuzzy complex equation, for both cases nonnegative or negative root, by a numerical method. Finally, some examples are presented in Section 5.

2. Preliminaries

Definition 2.1. A fuzzy number $\tilde{u}$ is a convex normalized fuzzy set of the real line $\mathbb{R}$ such that there exists exactly one $x_0 \in \mathbb{R}$ with $\mu_{\tilde{u}}(x_0) = 1$ ($\mu_{\tilde{u}}$ is called the membership function which is upper semi-continuous and compactly supported. We denote by $F(\mathbb{R})$ the set of all real fuzzy numbers [10].

Definition 2.2. A fuzzy number in the parametric form is presented by $\tilde{u} = (\underline{u}, \overline{u})$, where functions $\underline{u}$ and $\overline{u}$; $0 \leq r \leq 1$ apply the following conditions [9, 23, 24, 32, 34]:

1. $\underline{u}$ is monotonically increasing left continuous function on $[0, 1]$.
2. $\overline{u}$ is monotonically decreasing left continuous function on $[0, 1]$.
3. $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

Suppose $\tilde{u} = (\underline{u}, \overline{u})$ and $\tilde{v} = (\underline{v}, \overline{v})$ belong to $F(\mathbb{R})$. Some operations on fuzzy numbers are defined as the below forms [9]:

\[
\begin{align*}
\tilde{x} > 0 &: \tilde{x}\tilde{u} = (\underline{x\overline{u}}, \overline{x\underline{u}}), \\
\tilde{x} < 0 &: \tilde{x}\tilde{u} = (\underline{x\overline{u}}, \overline{x\underline{u}}), \\
\tilde{u} + \tilde{v} &: (\underline{u} + \underline{v}, \overline{u} + \overline{v}), \\
\tilde{u} - \tilde{v} &: (\underline{u} - \overline{v}, \overline{u} - \underline{v}), \\
\tilde{u}:\tilde{v} &: (\min\{\underline{u}, \overline{v}, \overline{u}, \underline{v}, \underline{u}, \overline{v}\}, \max\{\underline{u}, \overline{v}, \overline{u}, \underline{v}, \underline{u}, \overline{v}\}).
\end{align*}
\]

A fuzzy number $\tilde{u}$ is nonnegative (non-positive) if for $x < 0$ ($x > 0$), we have $\mu_{\tilde{u}}(x) = 0$, equivalently if on $[0, 1]$ we have $\underline{u} \geq 0$ ($\overline{u} \leq 0$). Also a fuzzy number $\tilde{u}$ is positive (negative) if for $x \leq 0$ ($x \geq 0$), we have $\mu_{\tilde{u}}(x) = 0$, equivalently if on $[0, 1]$ we have $\underline{u} > 0$ ($\overline{u} < 0$). So we have
• If \( \tilde{u} \) and \( \tilde{v} \) are non-negative fuzzy numbers, then \( \tilde{u} \tilde{v} = (u, v, \tilde{u}, \tilde{v}) \).
• If \( \tilde{u} \) and \( \tilde{v} \) are non-positive fuzzy numbers, then \( \tilde{u} \tilde{v} = (\tilde{u}, \tilde{v}, u, v) \).
• If \( \tilde{u} \) be a non-negative fuzzy number and \( \tilde{v} \) be a non-positive fuzzy number, then \( \tilde{u} \tilde{v} = (u, \tilde{v}, \tilde{u}, v) \).

Definition 2.3. Inherit from the real complex numbers, a fuzzy complex number \( \tilde{z} \), is defined in [22] in the form of \( \tilde{z} = \tilde{x} + i\tilde{y} \), where \( \tilde{x} \) and \( \tilde{y} \) are two fuzzy real numbers. \( \tilde{x} \) is the fuzzy real part of \( \tilde{z} \) while \( \tilde{y} \) represents the fuzzy imaginary part.

We denote by \( F(\mathbb{C}) \), the set of all fuzzy complex numbers.

Definition 2.4. Since the real and imaginary parts are both fuzzy numbers, we can define a parametric representation for fuzzy complex number as

\[
\tilde{z} = \tilde{x} + i\tilde{y} = (x, \overline{\pi}) + i(y, \overline{\pi}),
\]

(2.6)

in which \( x, \overline{\pi}, y, \overline{\pi} \) satisfy in requirements of Definition 2.2.

Let \( \tilde{z}_1 = x_1 + i\tilde{y}_1 = (x_1, \overline{\pi}_1) + i(y_1, \overline{\pi}_1) \) and \( \tilde{z}_2 = x_2 + i\tilde{y}_2 = (x_2, \overline{\pi}_2) + i(y_2, \overline{\pi}_2) \) are belonging to \( F(\mathbb{C}) \). We defined some operations on fuzzy complex numbers as follows:

Addition:

\[
\tilde{z}_1 + \tilde{z}_2 = (x_1 + x_2, \overline{\pi}_1 + \overline{\pi}_2) + i(y_1 + y_2, \overline{\pi}_1 + \overline{\pi}_2).
\]

(2.7)

Subtraction:

\[
\tilde{z}_1 - \tilde{z}_2 = (x_1 - x_2, \overline{\pi}_1 - \overline{\pi}_2) + i(y_1 - y_2, \overline{\pi}_1 - \overline{\pi}_2).
\]

(2.8)

Multiplication:

\[
\tilde{z}_1 \tilde{z}_2 = (x_1 x_2 - y_1 \overline{y}_2, \overline{\pi}_1 \overline{\pi}_2 - y_1 \overline{\pi}_2) + i(x_1 \overline{y}_2 + y_1 x_2, \overline{\pi}_1 \overline{\pi}_2) + i(y_1 x_2 + y_1 \overline{x}_2).
\]

(2.9)

Let \( F(\mathbb{R})^+ \) be the set of all non-negative fuzzy real numbers and \( F(\mathbb{R})^- \) be the set of all non-positive fuzzy real numbers. If \( x_1, \tilde{x}_2, \tilde{y}_1 \) and \( \tilde{y}_2 \) belong to \( F(\mathbb{R})^+ \), then

\[
\tilde{z}_1 \tilde{z}_2 = (x_1 x_2 - y_1 \overline{y}_2, \overline{\pi}_1 \overline{\pi}_2 - y_1 \overline{\pi}_2) + i(x_1 \overline{y}_2 + y_1 x_2, \overline{\pi}_1 \overline{\pi}_2) + i(y_1 x_2 + y_1 \overline{x}_2).
\]

(2.10)

If \( x_1, \tilde{x}_2, \tilde{y}_1 \) and \( \tilde{y}_2 \) belong to \( F(\mathbb{R})^- \), then

\[
\tilde{z}_1 \tilde{z}_2 = (x_1 x_2 - y_1 \overline{y}_2, \overline{\pi}_1 \overline{\pi}_2 - y_1 \overline{\pi}_2) + i(x_1 \overline{y}_2 + y_1 x_2, \overline{\pi}_1 \overline{\pi}_2) + i(y_1 x_2 + y_1 \overline{x}_2).
\]

(2.11)

Definition 2.5. Conjugate of a fuzzy complex number \( \tilde{z} = \tilde{x} + i\tilde{y} \) is given by

\[
\tilde{z} = \tilde{x} - i\tilde{y} = (x, \overline{\pi}) + i(-\overline{y}, -\overline{\pi}).
\]

(2.12)

The relation between \( \tilde{z} \) and \( \tilde{z} \) are as the below form:

1. \( \tilde{z} + \overline{\tilde{z}} = 2\tilde{x} + i\tilde{0}_y \),
2. \( \tilde{z} - \overline{\tilde{z}} = 0_x + 2i\tilde{0}_y \),
3. \( \tilde{z}\overline{\tilde{z}} = \tilde{x}^2 + \tilde{y}^2 + \tilde{0}_{xy} \),

where for a fuzzy number \( \tilde{u}, \tilde{0}_u \) is a fuzzy zero with respect to \( \tilde{u} \tilde{0}_u = \tilde{u} - \tilde{u} \).
2.1. **Fuzzy polynomial over set of fuzzy complex numbers.** In this section, we generalize the concept of fuzzy algebraic equation to complex form based on \[9\]. Let \(a_1, \ldots, a_n \in \mathbb{R}, \ a_n \neq 0\) and \(n\) be a natural number. A complex fuzzy polynomial of degree at most \(n\) with fuzzy complex variable is shown as

\[
\tilde{P}_n(z) = \sum_{j=1}^{n} a_j \tilde{z}^j.
\]  

(2.13)

An \(n\)th degree algebraic fuzzy complex equation with fuzzy complex unknown and crisp coefficients, is

\[
a_n \tilde{z}^n + \cdots + a_1 \tilde{z} = \tilde{b},
\]

(2.14)

where \(\tilde{b} \in F(\mathbb{C}).\)

**Definition 2.6.** A fuzzy number \(\tilde{u} (\tilde{u} \in F(\mathbb{R}))\) has ”\(m\)–degree polynomial form” if there exist two polynomials of degree at most \(m\), \(p_m\) and \(q_m\), such that \(\tilde{u} = (p_m, q_m)[9]\).

Let \(F_m(\mathbb{R})\) be the set of all \(m\)–degree polynomial form fuzzy real numbers. By \(F_m(\mathbb{C})\) we also denote the set of all fuzzy complex numbers with \(m\)–degree polynomial form:

\[
F_m(\mathbb{C}) = \{ \tilde{z} = \tilde{x} + i\tilde{y} : \tilde{x}, \tilde{y} \in F_m(\mathbb{R}) \}.
\]  

(2.15)

The conjugate of a root of an algebraic fuzzy complex equation is not a root of the equation, necessarily. because \(\tilde{z}^n\) and \(\bar{\tilde{z}}^n\) are not the same, even if real and image parts of \(\tilde{z}\) are non-negative or non-positive and even for \(n = 2\).

### 3. Solving the system of equations

The system of linear equations containing \(s\) variables and \(d\) equations (\(d \geq s\)) can be shown like \(F(X) = (F_1(X), \ldots, F_d(X))^T = 0\), in which \(X = [x_i]_{s \times 1}\). For the solution of the system, the iterative process of Gauss-Newton \([20, 21]\), is as the following:

First it is required to use the Jacobian matrix \(J(X) = [\frac{\partial F_i(X)}{\partial x_j}]\), where \(i = 1, \ldots, d\) and \(j = 1, \ldots, s\), then we select an arbitrary starting vector \(X^{(0)}\) as an approximated solution. For gain the desired solution one have to solve the system \(J(X^{(k)})H^k = -F(X^{(k)})\), and in the next iterate is replaced \(X^{(k)}\) by \(X^{(k)} + H^{(k)}\) in other words \(X^{(k+1)} = X^{(k)} + H^{(k)}\). This system is evaluated by the least squares method as the following form

\[
J(X^{(k)})^T J(X^{(k)}) H^{(k)} = -J(X^{(k)})^T F(X^{(k)}).
\]  

(3.1)

The conditions on convergence and uniqueness of the solution are studied in \([17, 18, 26]\).

**Theorem 3.1.** \([9]\) Let \(N\) be a \(d \times s\) matrix where \(d > s\) and the linear system of equations \(NX = C\) has a unique solution. With any initial vector, the sequence of Gauss-Newton method will be converged to the exact solution of a linear system of equations at the first iterations.
4. SOLVING AN ALGEBRAIC FUZZY COMPLEX EQUATION

We consider the following nth degree algebraic complex fuzzy equation with complex fuzzy variable as

$$\sum_{j=1}^{n} a_j \tilde{z}^j = \tilde{b} \tag{4.1}$$

where \(a_n \neq 0\) and \(\tilde{b} \in F(\mathbb{C})\). By inserting \(\tilde{z} = \tilde{x} + i\tilde{y}\) and \(\tilde{b} = \tilde{c} + i\tilde{d}\) in “(4.1)”, we acquire

$$\sum_{j=1}^{n} a_j (\tilde{x} + i\tilde{y})^j = \tilde{c} + i\tilde{d}, \tag{4.2}$$

where \(\tilde{x}\) and \(\tilde{y}\) are single modal value fuzzy number in \(F_m(\mathbb{R})\), and \(\tilde{c}, \tilde{d}\) are known fuzzy numbers in parametric form belonging to \(F_l(\mathbb{R})\), such that \(l \leq nm\).

From “(4.2)”, according to binomial expansion, we have

$$\sum_{j=1}^{n} a_j \left(\sum_{k=0}^{j} \binom{j}{k} \tilde{x}^k (i\tilde{y})^{j-k}\right) =$$

$$\left(\sum_{q=0}^{l} c_q r^q, \sum_{q=0}^{l} d_q r^q\right) + i\left(\sum_{q=0}^{l} d_q r^q, \sum_{q=0}^{l} d_q r^q\right). \tag{4.3}$$

Left hand side of “(4.3)” is equal to

$$\sum_{j=1}^{n} a_j \sum_{k=0}^{[j/2]} (-1)^k \binom{j}{2k} \tilde{x}^{j-2k} \tilde{y}^{2k}$$

$$+ i \sum_{k=0}^{j-[j/2]-1} (-1)^k \binom{j}{j-(2k+1)} \tilde{x}^{j-(2k+1)} \tilde{y}^{2k+1}. \tag{4.4}$$

Since the multiplication of fuzzy numbers are dependent on the sign of numbers, so we consider the “Eq. (4.4)” in two special cases, i.e first case is that \(\tilde{x}\) and \(\tilde{y}\) are nonnegative fuzzy numbers belonging to \(F_m(\mathbb{R})\), and another case is that both of them are negative fuzzy numbers belonging to \(F_m(\mathbb{R})\).

**First case:**

By taking \(\tilde{x} = (x, x)\), \(\tilde{y} = (y, y)\) as non-negative single modal value fuzzy numbers belonging to \(F_m(\mathbb{R})\) and by considering “(4.4)”, “Eq. (4.3)” is equivalent to following two equations:

$$\sum_{j=1}^{n} a_j \sum_{k=0}^{[j/2]} (-1)^k \binom{j}{j-2k} \tilde{x}^{j-2k} \tilde{y}^{2k} = \left(\sum_{q=0}^{l} c_q r^q, \sum_{q=0}^{l} d_q r^q\right), \tag{4.5}$$
\[
\sum_{j=1}^{n} a_j \sum_{k=0}^{j-[j/2]-1} (-1)^k \left( j \right) (x_j^{-(2k+1)} y_j^{2k+1}, y_j^{-(2k+1)} y_j^{2k+1}) = \\
\left( \sum_{q=0}^{l} d_q r^q, \sum_{q=0}^{l} d_q r^q \right).
\]

By defining
\[
\alpha_{j,k} = a_j (-1)^k (j_{-2k}), \quad \beta_{j,k} = a_j (-1)^k (j_{-(2k+1)}),
\]
and considering
\[
\hat{\alpha}_{j,k} = \left\{ \begin{array}{ll}
\alpha_{j,k}, & \alpha_{j,k} \geq 0, \\
0, & \alpha_{j,k} < 0,
\end{array} \right. \quad \hat{\beta}_{j,k} = \left\{ \begin{array}{ll}
\beta_{j,k}, & \beta_{j,k} \geq 0, \\
0, & \beta_{j,k} < 0,
\end{array} \right.
\]

“Eq. (4.5)” reduces to the following form:
\[
\begin{cases}
\sum_{j=0}^{[j/2]} \sum_{k=0}^{j} \left( \hat{\alpha}_{j,k} x_j^{2k} + \hat{\alpha}_{j,k} x_j^{-(2k+2)} y_j^{2k+1} \right) = \sum_{q=0}^{l} c_q r^q, \\
\sum_{j=0}^{[j/2]} \sum_{k=0}^{j} \left( \hat{\beta}_{j,k} x_j^{2k} + \hat{\beta}_{j,k} x_j^{-(2k+2)} y_j^{2k+1} \right) = \sum_{q=0}^{l} c_q r^q.
\end{cases}
\]

Similarly by considering
\[
\hat{\beta}_{j,k} = \left\{ \begin{array}{ll}
\beta_{j,k}, & \beta_{j,k} \geq 0, \\
0, & \beta_{j,k} < 0,
\end{array} \right. \quad \hat{\beta}_{j,k} = \left\{ \begin{array}{ll}
\beta_{j,k}, & \beta_{j,k} \geq 0, \\
0, & \beta_{j,k} < 0,
\end{array} \right.
\]

Equation “Eq. (4.6)” will be of the following form:
\[
\begin{cases}
\sum_{j=0}^{\lfloor j/2 \rfloor} \sum_{k=0}^{j} \left( \hat{\beta}_{j,k} x_j^{2k} + \hat{\beta}_{j,k} x_j^{-(2k+2)} y_j^{2k+1} \right) = \sum_{q=0}^{l} d_q r^q, \\
\sum_{j=0}^{\lfloor j/2 \rfloor} \sum_{k=0}^{j} \left( \hat{\beta}_{j,k} x_j^{2k} + \hat{\beta}_{j,k} x_j^{-(2k+2)} y_j^{2k+1} \right) = \sum_{q=0}^{l} d_q r^q.
\end{cases}
\]

Let
\[
\begin{align*}
\bar{x}(r) &= \sum_{s=0}^{m} x_s r^s, \quad \bar{x}(r) = \sum_{s=0}^{m} x_s r^s, \\
\bar{y}(r) &= \sum_{s=0}^{m} y_s r^s, \quad \bar{y}(r) = \sum_{s=0}^{m} y_s r^s.
\end{align*}
\]
“Eq. (4.9)” and “Eq. (4.11),” are equivalence to the following equations

\[
\begin{align*}
\sum_{j=1}^{n} \sum_{k=0}^{[j/2]} (\hat{\alpha}_{j,k} (\sum_{s=0}^{m} x_s r^s)^{j-2k} (\sum_{s=0}^{m} y_s r^s)^{2k} + \hat{\alpha}_{j,k} (\sum_{s=0}^{m} x_s r^s)^{j-2k} (\sum_{s=0}^{m} y_s r^s)^{2k}) &= \sum_{q=0}^{l} c_q r^q, \\
\sum_{j=1}^{n} \sum_{k=0}^{[j/2]} (\hat{\beta}_{j,k} (\sum_{s=0}^{m} x_s r^s)^{j-(2k+1)} (\sum_{s=0}^{m} y_s r^s)^{2k+1} + \hat{\beta}_{j,k} (\sum_{s=0}^{m} x_s r^s)^{j-(2k+1)} (\sum_{s=0}^{m} y_s r^s)^{2k+1}) &= \sum_{q=0}^{l} d_q r^q.
\end{align*}
\]

(4.13)

and

\[
\begin{align*}
\sum_{j=1}^{n} \sum_{k=0}^{[j/2]-1} (\hat{\alpha}_{j,k} (\sum_{s=0}^{m} x_s r^s)^{j-2k} (\sum_{s=0}^{m} y_s r^s)^{2k+1} + \hat{\alpha}_{j,k} (\sum_{s=0}^{m} x_s r^s)^{j-2k} (\sum_{s=0}^{m} y_s r^s)^{2k+1}) &= \sum_{q=0}^{l} c_q r^q, \\
\sum_{j=1}^{n} \sum_{k=0}^{[j/2]-1} (\hat{\beta}_{j,k} (\sum_{s=0}^{m} x_s r^s)^{j-(2k+1)} (\sum_{s=0}^{m} y_s r^s)^{2k+1} + \hat{\beta}_{j,k} (\sum_{s=0}^{m} x_s r^s)^{j-(2k+1)} (\sum_{s=0}^{m} y_s r^s)^{2k+1}) &= \sum_{q=0}^{l} d_q r^q.
\end{align*}
\]

(4.14)

Let for \(i = 0, 1, \ldots, nm; L_i, U_i, M_i\) and \(N_i\) be the coefficients of \(r^i\) in “(4.13)” and “(4.14),” respectively, thus

\[
\begin{align*}
\sum_{i=0}^{nm} L_i (x_0, \ldots, x_m, x_0, \ldots, x_m, y_0, \ldots, y_m) r^i &= 0, \\
\sum_{i=0}^{nm} U_i (x_0, \ldots, x_m, x_0, \ldots, x_m, y_0, \ldots, y_m) r^i &= 0, \\
\sum_{i=0}^{nm} M_i (x_0, \ldots, x_m, x_0, \ldots, x_m, y_0, \ldots, y_m) r^i &= 0, \\
\sum_{i=0}^{nm} N_i (x_0, \ldots, x_m, x_0, \ldots, x_m, y_0, \ldots, y_m) r^i &= 0.
\end{align*}
\]

(4.15)

Since the fuzzy numbers are one modal fuzzy numbers, we consider the following equations

\[
Q_1(x_0, \ldots, x_m, x_0, \ldots, x_m) = \mu(1) - \nu(1) = 0
\]

(4.16)

and

\[
Q_2(y_0, \ldots, y_m, y_0, \ldots, y_m) = \eta(1) - \tau(1) = 0
\]

(4.17)
“Eq. (4.15)”, “Eq. (4.16)” and “Eq. (4.17)” yield that

\[
\begin{align*}
L_i(x_0, \ldots, x_m, y_0, \ldots, y_m) = 0, & \quad i = 0, \ldots, nm \\
U_i(x_0, \ldots, x_m, y_0, \ldots, y_m) = 0, & \quad i = 0, \ldots, nm \\
M_i(x_0, \ldots, x_m, y_0, \ldots, y_m) = 0, & \quad i = 0, \ldots, nm \\
N_i(x_0, \ldots, x_m, y_0, \ldots, y_m) = 0, & \quad i = 0, \ldots, nm \\
Q_i(x_0, \ldots, x_m, y_0, \ldots, y_m) = 0, & \\
Q_2(y_0, \ldots, y_m) = 0.
\end{align*}
\]

(4.18)

Which is a system of the nonlinear equations with \(4nm + 6\) equations and \(4m + 4\) unknowns.

We solve the system “(4.18)” by an iterative Gauss-Newton method that introduced in Section (3). Defining

\[
J = \begin{pmatrix}
\frac{\partial L_i}{\partial x_i} & \frac{\partial L_i}{\partial y_j} & \frac{\partial L_i}{\partial y_j} & \frac{\partial L_i}{\partial y_j} \\
\frac{\partial U_i}{\partial x_i} & \frac{\partial U_i}{\partial y_j} & \frac{\partial U_i}{\partial y_j} & \frac{\partial U_i}{\partial y_j} \\
\frac{\partial M_i}{\partial x_i} & \frac{\partial M_i}{\partial y_j} & \frac{\partial M_i}{\partial y_j} & \frac{\partial M_i}{\partial y_j} \\
\frac{\partial N_i}{\partial x_i} & \frac{\partial N_i}{\partial y_j} & \frac{\partial N_i}{\partial y_j} & \frac{\partial N_i}{\partial y_j} \\
1 & -1 & 0 & 0 \\
0 & 0 & 1 & -1
\end{pmatrix},
\]

(4.19)

and

\[
B = \begin{pmatrix}
L_i \\
U_i \\
M_i \\
N_i \\
Q_1 \\
Q_2
\end{pmatrix}, \quad X = \begin{pmatrix}
x_i \\
y_i \\
y_i \\
y_i \\
y_i \\
y_i
\end{pmatrix}, \quad H = \begin{pmatrix}
h_1 \\
h_2 \\
\vdots \\
h_{4m+4}
\end{pmatrix},
\]

(4.20)

where \(i = 0, 1, \ldots, m; j = 0, 1, \ldots, m\), we have \(JH = B \) (\(J = J(X), B = B(X), H = H(X)\)). We solve this system with an initial vector \(X^{(0)}\), by the least squares method and then we improve this guess. We find the sequence \(Y^{(k)}\) as follows. With an initial vector \(X^{(0)}\), in iteration \(k\), we compute \(J^{(k)}\) and \(B^{(k)}\) and solve \(J^{(k)}H^{(k)} = B^{(k)}\) by the least squares method as the following: \((J^{(k)} = J(X^{(k)}), B^{(k)} = B(X^{(k)})\)

\[
J^{(k)}^TJ^{(k)}H^{(k)} = J^{(k)}^TB^{(k)}.
\]

(4.21)

The solution will be improved by

\[
X^{(k+1)} = X^{(k)} + H^{(k)}.
\]

(4.22)

If \(X^{(k)}\) converges to a vector \(X^*\); then \(\hat{x}^*\) in the form of

\[
\hat{x}^* = \left( \sum_{i=0}^{m} x_i^* r^s, \sum_{i=0}^{m} x_i^* r^s \right)
\]

(4.23)
and \( \tilde{y}^* \) in the form of
\[
\tilde{y}^* = \left( \sum_{s=0}^{m} y^*_s \cdot r^s, \sum_{s=0}^{m} \bar{y}^*_s \cdot r^s \right) \quad (4.24)
\]
are the parts of the Gauss-Newton solution. If \( x^* \) and \( y^* \) are both increasing, \( \pi^* \) and \( \bar{y}^* \) are both decreasing, then \( \tilde{z}^* = \tilde{x}^* + i\tilde{y}^* \) is a solution of “(2.14)”.

**Second case:**
For a negative fuzzy number such as \( \tilde{x} = (x; \pi) \), we have
\[
\tilde{x}^{2j} = (x^{2j}; \pi^{2j}), \quad \tilde{x}^{2j+1} = (x^{2j+1}; \pi^{2j+1}) \quad (4.25)
\]
therefore, if \( \tilde{x} \) and \( \tilde{y} \) be two negative fuzzy numbers, then real and imaginary parts of “(4.3)” will be rewrite to the following two relations:

Real part of “(4.3)” is:
\[
\sum_{j=1}^{n} a_j \sum_{k=0}^{[j/2]} (-1)^k \left( \frac{j}{j - 2k} \right) \tilde{x}^{j-2k} \tilde{y}^{2k} = \quad (4.26)
\]
\[
\sum_{j=1}^{l_1} a_{2j} \sum_{k=0}^{j} (-1)^k \left( \frac{2j}{2j - 2k} \right) (x^{2j-2k}; \pi^{2j-2k})(x^{2k}; \pi^{2k})
\]
\[
+ \sum_{j=0}^{l_2} a_{2j+1} \sum_{k=0}^{j} (-1)^k \left( \frac{2j+1}{2j + 1 - 2k} \right) (x^{2j+1-2k}; \pi^{2j+1-2k})(x^{2k}; \pi^{2k}).
\]

Imaginary part of “(4.3)” is:
\[
\sum_{j=1}^{n} a_j \sum_{k=0}^{[j/2]-1} (-1)^k \left( \frac{j}{j - (2k + 1)} \right) \tilde{x}^{j-(2k+1)} \tilde{y}^{2k+1} = \quad (4.27)
\]
\[
\sum_{j=1}^{l_1} a_{2j} \sum_{k=0}^{j-1} (-1)^k \left( \frac{2j}{2j - (2k + 1)} \right) (x^{2j-2k-1}; \pi^{2j-2k-1})(y^{2k+1}; \pi^{2k+1})
\]
\[
+ \sum_{j=0}^{l_2} a_{2j+1} \sum_{k=0}^{j} \left( \frac{2j+1}{2j + 1 - 2k} \right) (x^{2j-2k}; \pi^{2j-2k})(y^{2k+1}; \pi^{2k+1}).
\]

In “(4.26)” and “(4.27)” \( l_1 \) and \( l_2 \) are as follows:
\[
l_1 = \left\lfloor \frac{n}{2} \right\rfloor, \quad l_2 = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor & \text{if } n \text{ is odd} \\ \left\lfloor \frac{n}{2} \right\rfloor - 1 & \text{if } n \text{ is even} \end{cases} \quad (4.28)
\]
Thus “Eq. (4.5)” and “Eq. (4.6)” reduce to following: 
\[
\sum_{j=1}^{l_1} a_{2j} \sum_{k=0}^{j} (-1)^k \left( \frac{2j}{2j - 2k} \right) (x^{2j-2k}, y^{2k}) + \sum_{j=0}^{l_2} a_{2j+1} \sum_{k=0}^{j} (-1)^k \left( \frac{2j + 1}{2j + 1 - 2k} \right) (x^{2j+1-2k}, y^{2k}) 
\]
(4.29)
\[
= \left( \sum_{q=0}^{l} c_q r^q, \sum_{q=0}^{l} \bar{c}_q r^q \right),
\]
and
\[
\sum_{j=1}^{l_1} a_{2j} \sum_{k=0}^{j-1} (-1)^k \left( \frac{2j}{2j - 2k - 1} \right) (x^{2j-2k-1}, y^{2k+1}) + \sum_{j=0}^{l_2} a_{2j+1} \sum_{k=0}^{j} (-1)^k \left( \frac{2j + 1}{2j + 1 - 2k} \right) (x^{2j+1-2k}, y^{2k+1}) 
\]
(4.30)
\[
= \left( \sum_{q=0}^{l} d_q r^q, \sum_{q=0}^{l} \bar{d}_q r^q \right).
\]

By using
\[
\alpha_{j,k} = (-1)^k a_{2j} \left( \frac{2j}{2j - 2k} \right), \quad \beta_{j,k} = (-1)^k a_{2j+1} \left( \frac{2j + 1}{2j + 1 - 2k} \right),
\]
(4.31)
and considering
\[
\bar{\alpha}_{j,k} = \left\{ \begin{array}{l}
\alpha_{j,k}, \\
0,
\end{array} \right. \quad \bar{\beta}_{j,k} = \left\{ \begin{array}{l}
\alpha_{j,k}, \\
0,
\end{array} \right.,
\]
(4.32)
and
\[
\hat{\beta}_{j,k} = \left\{ \begin{array}{l}
\beta_{j,k}, \\
0,
\end{array} \right. \quad \bar{\beta}_{j,k} = \left\{ \begin{array}{l}
\beta_{j,k}, \\
0,
\end{array} \right.,
\]
(4.33)

Equation “Eq. (4.29)” is changes to following equations
\[
\left\{ \begin{array}{l}
\sum_{j=1}^{l_1} \sum_{k=0}^{j} (\hat{\alpha}_{j,k} \bar{x}^{(2j-2k)} y^{2k} + \bar{\alpha}_{j,k} \bar{x}^{(2j-2k)} y^{2k}) + \\
\sum_{j=0}^{l_2} \sum_{k=0}^{j} (\hat{\beta}_{j,k} \bar{y}^{2j-2k-1} y^{2k} + \bar{\beta}_{j,k} \bar{x}^{2j-2k+1} y^{2k}) = \sum_{q=0}^{l} c_q r^q,
\end{array} \right.
\]
(4.34)
\[
\left\{ \begin{array}{l}
\sum_{j=1}^{l_1} \sum_{k=0}^{j} (\bar{\hat{\alpha}}_{j,k} \bar{x}^{(2j-2k)} y^{2k} + \bar{\bar{\alpha}}_{j,k} \bar{x}^{2j-2k+1} y^{2k}) + \\
\sum_{j=0}^{l_2} \sum_{k=0}^{j} (\bar{\hat{\beta}}_{j,k} \bar{x}^{2j-2k+1} y^{2k} + \bar{\bar{\beta}}_{j,k} \bar{x}^{2j-2k+1} y^{2k}) = \sum_{q=0}^{l} \bar{c}_q r^q.
\end{array} \right.
\]

Also, by using
\[
\hat{\gamma}_{j,k} = \left\{ \begin{array}{l}
\gamma_{j,k}, \\
0,
\end{array} \right. \quad \bar{\gamma}_{j,k} = \left\{ \begin{array}{l}
\gamma_{j,k}, \\
0,
\end{array} \right.,
\]
(4.35)
and
\[
\hat{\eta}_{j,k} = \left\{ \begin{array}{l}
\eta_{j,k}, \\
0,
\end{array} \right. \quad \bar{\eta}_{j,k} = \left\{ \begin{array}{l}
\eta_{j,k}, \\
0,
\end{array} \right.,
\]
(4.36)
where
\[
\gamma_{j,k} = (-1)^k a_{2j} \left( \frac{2j}{2} \right)^{2k-1}, \quad \eta_{j,k} = (-1)^k a_{2j+1} \left( \frac{2j+1}{2} \right)^{2k-1},
\]
(4.37)

Equation “Eq. (4.30)” changes to the following system
\[
\begin{align*}
\sum_{j=1}^{l_1} \sum_{k=0}^{l_2} \gamma_{j,k} \bar{x}^{(2j-2k-1)} y^{(2k+1)} + \gamma_{j,k} \bar{x}^{(2j-2k-1)} y^{(2k+1)} + \eta_{j,k} \bar{x}^{(2j-2k-1)} y^{(2k+1)} &= \sum_{q=0}^{l_1} d_q r^q, \\
\sum_{j=1}^{l_1} \sum_{k=0}^{l_2} \gamma_{j,k} \bar{x}^{(2j-2k-1)} y^{(2k+1)} + \gamma_{j,k} \bar{x}^{(2j-2k-1)} y^{(2k+1)} + \eta_{j,k} \bar{x}^{(2j-2k-1)} y^{(2k+1)} &= \sum_{q=0}^{l_1} d_q r^q.
\end{align*}
\]
(4.38)

Equations of “(4.34)” and “(4.38)” yield that for \( i = 0, \cdots, mn \); \( L_i, U_i, M_i \) and \( N_i \) are the coefficients of \( r^i \), respectively, also \( Q_1, Q_2 \) are the same as “(4.17)”, “(4.18)”. Thus again we have \( 4nm + 6 \) equations.

**Algorithm**

(1) Input an initial solution \( X^{(0)} \).
(2) Evaluate \( J^{(k)} \) and \( B^{(k)} \) from “(4.19)” and “(4.20)”.
(3) Solve \( H^{(k)} \) from the system of linear equations \( J^{(k)^T} J^{(k)} H^{(k)} = J^{(k)^T} B^{(k)} \).
(4) \( X^{(k+1)} = X^{(k)} + H^{(k)} \).
(5) If the convergence condition is yield, then end.
(6) \( k = k + 1 \), and go to step 3.

Theorem and Lemmas of this part are extended from Theorem and Lemmas related to [9].

**Lemma 4.1.** Let \( n = 1 \) (i.e. \( a \hat{z} = \hat{b} \)), where real and imaginary parts of \( \hat{z} \) and \( \hat{b} \) belong to \( F_m(\mathbb{R}) \). If \( a \neq 0 \), then in each iteration, we have
\[
J^T J = a^2 I_{4m+4} + V,
\]
and
\[
(J^T J)^{-1} = \frac{1}{a^2} I_{4m+4} - \frac{1}{a^2(a^2 + 2m + 2)} V
\]
such that
\[
V = \begin{pmatrix}
\psi & -\psi & O & O \\
-\psi & \psi & O & O \\
O & O & \psi & -\psi \\
O & O & -\psi & \psi
\end{pmatrix},
\]
(4.41)
where \( \psi_{ij} = 1 \), for \( i, j = 1, 2, \cdots, m + 1 \) [9].
Proof. For equation of degree 1, i.e. \( a(x, \bar{x}) + ia(y, \bar{y}) = (b, \bar{b}) \), we have

\[
\begin{aligned}
&\sum_{k=0}^{m} (\hat{a}x_k + \hat{a}x_k - \bar{b}_k)r^k = 0 \\
&\sum_{k=0}^{m} (\hat{a}y_k + \hat{a}y_k - \bar{b}_k)r^k = 0 \\
&\sum_{k=0}^{m} (\bar{a}y_k + \bar{a}y_k)r^k = 0 \\
&\sum_{k=0}^{m} (\bar{a}y_k + \bar{a}y_k)r^k = 0
\end{aligned}
\]

(4.42)

where

\[
\hat{a} = \begin{cases} 
    a, & a \geq 0 \\
    0, & a < 0
\end{cases}, \quad \hat{a} = \begin{cases} 
    0, & a \geq 0 \\
    a, & a < 0
\end{cases}
\]

(4.43)

thus for \( k = 0, \cdots, m+1 \) we have

\[
\begin{aligned}
L_k &= \hat{a}x_k + \hat{a}x_k - \bar{b}_k = 0 \\
U_k &= \hat{a}x_k + \hat{a}x_k - \bar{b}_k = 0 \\
M_k &= \hat{a}y_k + \hat{a}y_k = 0 \\
N_k &= \hat{a}y_k + \hat{a}y_k = 0 \\
Q_1 &= \sum_{k=0}^{m} (\hat{x}_k - \bar{x}_k) = 0 \\
Q_2 &= \sum_{k=0}^{m} (\hat{y}_k - \bar{y}_k) = 0
\end{aligned}
\]

(4.44)

thus Jacobian matrix has the following form

\[
J = \begin{pmatrix}
T_1 & T_2 & 0 & 0 \\
T_2 & T_1 & 0 & 0 \\
0 & 0 & T_1 & T_2 \\
O & O & S^T & -S^T \\
\end{pmatrix}
\]

(4.45)

where \( T_1 = \hat{a}I_{m+1}, T_2 = \hat{a}I_{m+1} \) and \( S^T \) is an \( 1 \times (m+1) \) matrix such that \( S_i = 1 \) for \( i = 1, 2, \cdots, m+1 \). This yields that

\[
J^T J = \begin{pmatrix}
a^2I_{m+1} + \psi & -\psi & 0 & 0 \\
-\psi & a^2I_{m+1} + \psi & 0 & 0 \\
0 & O & a^2I_{m+1} + \psi & -\psi \\
0 & O & -\psi & a^2I_{m+1} + \psi
\end{pmatrix}
\]

(4.46)

\[= a^2I_{4m+4} + V.\]
Let

$$B = \frac{1}{a^2} I_{4m+4} - \frac{V}{a^2(a^2 + 2m + 2)}. \tag{4.47}$$

we show that $BJ^TJ = J^TJB = I$. Consequently

$$J^TJB = (a^2 I_{4m+4} + V) \left( \frac{1}{a^2} I_{4m+4} - \frac{V}{a^2(a^2 + 2m + 2)} \right) \tag{4.48}$$

$$= I_{4m+4} + \frac{1}{a^2(a^2 + 2m + 2)}((2m + 2)V - V^2) = I_{4m+4}$$

because

$$V^2 = \begin{pmatrix} 2\psi^2 & -2\psi^2 & O & O \\ -2\psi^2 & 2\psi^2 & O & O \\ O & O & 2\psi^2 & -2\psi^2 \\ O & O & -2\psi^2 & 2\psi^2 \end{pmatrix} = 2(m+1)V. \tag{4.49}$$

Similarly we have $BA^T A = I_{4m+4}$.

It can be proofed for the matrix defined above we have

$$|J^TJ| = a^{4(2m+1)}((m+1)(4m+4) + (4m+4)a^2 + a^4). \tag{4.50}$$

\[\square\]

**Lemma 4.2.** For $n=1$ i.e. $a \approx b$, $a \neq 0$ the algorithm is convergent to the exact solution of equation, at the first step, with every initial vector.

**Proof:** According to Lemma “(4.1)”, $F(Y) = 0$ is a linear system of equations, and by using Theorem “(3.1)”, the proof is completed.

**Theorem 4.3.** If the equation is crisp then the sequence is given by the following recurrence equation:

$$z_{k+1} = z_k - \frac{a_0 z_k^2 + \ldots + a_1 z_k - b}{na_n z_k^{n-1} + \ldots + a_1} \tag{4.51}$$

**Proof.** We consider “Eq. (4.1)” at the following forms

$$F(x, y) = f_1(x, y) + if_2(x, y) = 0. \tag{4.52}$$

and

$$P(z) - b = 0; \quad z = x + iy. \tag{4.53}$$

By using Equation “(4.52)”, the system of $A^{(k)}H^k = B^{(k)}$, in iteration $k$ is as the following form

$$\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = - \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{4.54}$$
the solution of the system “(4.54)” is

\[
\begin{pmatrix}
  h_1 \\
  h_2
\end{pmatrix} = \begin{pmatrix}
  f_1 \frac{\partial f_2}{\partial y} - f_2 \frac{\partial f_1}{\partial y} \\
  -f_1 \frac{\partial f_2}{\partial x} + f_2 \frac{\partial f_1}{\partial x} \\
\end{pmatrix}.
\]

(4.55)

Therefore

\[
z_{k+1} = z_k - \frac{f_1 \left( \frac{\partial f_2}{\partial y} - i \frac{\partial f_2}{\partial x} \right) + f_2 \left( - \frac{\partial f_1}{\partial y} + i \frac{\partial f_1}{\partial x} \right) - \frac{\partial f_1}{\partial x} \cdot \frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y} \cdot \frac{\partial f_2}{\partial x}}{\frac{\partial f_1}{\partial x} \cdot \frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y} \cdot \frac{\partial f_2}{\partial x}}.
\]

(4.56)

From “(4.52)”, we have

\[
\frac{\partial F}{\partial x} = \frac{\partial f_1}{\partial x} + i \frac{\partial f_2}{\partial x},
\]

(4.57)

\[
\frac{\partial F}{\partial y} = \frac{\partial f_1}{\partial y} + i \frac{\partial f_2}{\partial y}.
\]

(4.58)

From “Eq. (4.52)” and “Eq. (4.53)” we have

\[
F(x, y) = P(z) - b, \quad z = x + iy.
\]

(4.59)

Now, from “Eq. (4.59)” we can write

\[
\frac{\partial F}{\partial x} = P'(z), \quad \frac{\partial F}{\partial y} = iP'(z),
\]

(4.60)

From “(4.57)”, “(4.58)” and “(4.60)”, we have

\[
\frac{\partial f_2}{\partial x} + i \frac{\partial f_2}{\partial y} = -\frac{\partial f_1}{\partial y} + i \frac{\partial f_1}{\partial x},
\]

(4.61)

by substituting “(4.61)” in “(4.56)”, we have

\[
z_{k+1} = z_k - \frac{P(z) - b \left( \frac{\partial f_2}{\partial y} - i \frac{\partial f_2}{\partial x} \right)}{\frac{\partial f_1}{\partial x} \cdot \frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y} \cdot \frac{\partial f_2}{\partial x}}.
\]

(4.62)

Now by multiplying \(P'(z) = \frac{\partial f_1}{\partial x} + i \frac{\partial f_2}{\partial y}\) by \(\frac{\partial f_2}{\partial x}\) and by multiplying \(iP'(z) = \frac{\partial f_1}{\partial y} + i \frac{\partial f_2}{\partial y}\) by \(\frac{\partial f_2}{\partial x}\), yield

\[
\frac{\partial f_1}{\partial x} \frac{\partial f_2}{\partial y} = \frac{\partial f_2}{\partial y} \left( P'(z) - \frac{\partial f_2}{\partial x} \right)
\]

(4.63)

\[
\frac{\partial f_2}{\partial x} \frac{\partial f_1}{\partial y} = i \frac{\partial f_2}{\partial x} \left( P'(z) - \frac{\partial f_2}{\partial y} \right).
\]

(4.64)

By substituting “(4.63)” and “(4.64)” in “(4.62)”, we have \(z_{k+1} = z_k - \frac{P(z_k)}{P'(z_k)}\). □
5. Numerical Examples

This section contains some numerical results about our investigations that have been solved by software Matlab. The numerical solutions of the following examples have not been provided up to now. They are not given from any specific references. A sequence of iterative solutions is given for Example 5.3. The process for all examples is the same.

**Example 5.1.** Suppose we want to solve the equation

\[ \tilde{z} = (-10 + 10r + 2r^2, 29 - 35r + 8r^2). \]  

(5.1)

For this equation we consider \( n = 1, m = 2 \) and \( l = 2 \). Taking \( \tilde{z} = (1+r^2, 4-r)+i(2, 2) \), after one iteration we have

\[ \tilde{z}^* = (-2 + 2r + \frac{2}{5}r^2, \frac{29}{5} - 7r + \frac{8}{5}r^2). \]  

(5.2)

**Example 5.2.** Let we want to solve the equation \( \tilde{z}^2 = -1 \). For this equation we have \( n = 2, l = 0 \).

We test our method for this equation with different values of \( m \). Table 1 presents the results.

**Table 1.** The solved results of the equation in Example 5.2.

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \tilde{z}^0 )</th>
<th>Number of iterations</th>
<th>( \tilde{z}^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>((3, 3) + i(-2, -2))</td>
<td>6</td>
<td>(i(-1, -1) = -i)</td>
</tr>
<tr>
<td></td>
<td>((4, 4) + i(3, 3))</td>
<td>6</td>
<td>(i(1, 1) = i)</td>
</tr>
<tr>
<td>1</td>
<td>((4 + r, 6 - r) + i(3, 3))</td>
<td>7</td>
<td>(i(-1, -1) = -i)</td>
</tr>
<tr>
<td></td>
<td>((1 + r, 3 - r) + i(-2, -2))</td>
<td>5</td>
<td>(i(1, 1) = i)</td>
</tr>
<tr>
<td>2</td>
<td>((-4 + r^2, -1 - 2r) + i(-4, -4))</td>
<td>5</td>
<td>(i(-1, -1) = -i)</td>
</tr>
<tr>
<td></td>
<td>((1, 1) + i(1 + r^2, 3 - r))</td>
<td>6</td>
<td>(i(1, 1) = i)</td>
</tr>
</tbody>
</table>

**Example 5.3.** We want to solve

\[ \tilde{z}^2 = (-4 + 4r - r^2, -r^2). \]  

(5.3)

For this equation we have \( n = 2, l = 2 \).

We examine the method for this equation with values of \( m = 1 \) and \( m = 2 \). More details are given in Tables 2 and 3.

**Example 5.4.** Let we want to solve the equation

\[ \tilde{z}^2 = (r^2 - 1, r^2 - 4r + 3) + 2i(r, 2 - r). \]  

(5.4)

For this equation we have \( n = 2 \) and \( l = 2 \). We solve this equation with values of \( m = 1, 3 \). Table 4 shows the results.
The iterative solutions of the equation in Example 5.3 for \( m = 1 \).

<table>
<thead>
<tr>
<th>Initial guess</th>
<th>((2 + r, 4 - r) + i(1 + r, 3 - r))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>((0.2308 + 1.0327r, 2.0327 - 0.2728r + i(0.8927 + 0.4450r, 1.5466 - 0.8054r))</td>
</tr>
<tr>
<td></td>
<td>((-0.5273 + 0.4499r, 0.8414 + 0.0369 + i(-0.1012 + 0.7001r, 1.5506 - 0.7971r))</td>
</tr>
<tr>
<td></td>
<td>((-0.0697 + 0.1099r, -0.0971 + 0.0361r + i(-0.0417 + 0.8638r, 1.9719 - 1.1541r))</td>
</tr>
<tr>
<td>Iterative solutions</td>
<td>((0.0084 - 0.0136r, -0.0002 + 0.0061r + i(0.0061 + 1.0063r, 1.9981 - 0.9966r))</td>
</tr>
<tr>
<td></td>
<td>((-0.0000 - 0.0001r, -0.0000 - 0.0000r + i(0.0000 + 1.0000r, 2.0000 - 0.9999r))</td>
</tr>
<tr>
<td></td>
<td>((-0.0000 + 0.0000r, -0.0000 - 0.0000r + i(-0.0000 + 1.0000r, 2.0000 - 1.0000r))</td>
</tr>
</tbody>
</table>

| Number of iterations | 7 |
| Final solution | \(i(r, 2 - r)\) |

The iterative solutions of the equation in Example 5.3 for \( m = 2 \).

<table>
<thead>
<tr>
<th>Initial guess of solution</th>
<th>((1 + r^2, 4 - 2r) + i(2 + r, 4 - r^2))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>((0.0622 + 0.2809r + 0.3379r^2, 1.9903 - 0.7550r - 0.0463r^2)+i(1.2007 + 0.7334r - 0.1418r^2, 2.1673 - 0.2783r - 0.5398r^2))</td>
</tr>
<tr>
<td></td>
<td>((-0.0018 + 0.0055r - 0.0103r^2, -0.0003 + 0.0007r - 0.00018r^2)+i(-0.0036 + 1.0071r - 0.0184r^2, 1.9995 - 1.0012r - 0.0010r^2))</td>
</tr>
<tr>
<td>Iterative solutions</td>
<td>((-0.0001 - 0.0000r - 0.0002r^2, -0.0000 - 0.0000r - 0.0000r^2)+i(0.0000 + 1.0000r - 0.0000r^2, 2.0000 - 1.0000r - 0.0000r^2))</td>
</tr>
<tr>
<td></td>
<td>((-0.0000 + 0.0000r + 0.0000r^2, -0.0000 - 0.0000r - 0.0000r^2)+i(-0.0000 + 1.0000r + 0.0000r^2, 2.0000 - 1.0000r + 0.0000r^2))</td>
</tr>
</tbody>
</table>

| Number of iterations | 7 |
| Final solution | \(i(r, 2 - r)\) |

Example 5.5. We consider the equation

\[
\ddot{z}^2 - \dot{z} = (7r - 6, -r^4 + r^2 - 7r + 8) + i(2r^3 + 2r^2 + r - 2, r^2 - 10r + 12)
\]

(5.5)

for \( n = 2 \) and \( l = 4 \). We solve this equation for \( m = 2, 3 \). The results of solving the equation are given in table 5.
Table 4. The solved results of the equation in Example 5.4.

<table>
<thead>
<tr>
<th>(m)</th>
<th>(\bar{z}^0)</th>
<th>Number of iterations</th>
<th>(\bar{z}^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((2 + r, 4 - r) + i(1 + r, 3 - r))</td>
<td>6</td>
<td>((r, 2 - r) + i(1, 1))</td>
</tr>
<tr>
<td>3</td>
<td>((r^2 + r^3, 3 - r) + i(r^4 + r^3, 3 - r))</td>
<td>6</td>
<td>((r, 2 - r) + i(1, 1))</td>
</tr>
</tbody>
</table>

Table 5. The solved results of the equation in Example 5.5.

<table>
<thead>
<tr>
<th>(m)</th>
<th>(\bar{z}^0)</th>
<th>Number of iterations</th>
<th>(\bar{z}^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>((1, 1) + i(0, 0))</td>
<td>5</td>
<td>((1 + r, 3 - r) + i(r^4, 2 - r))</td>
</tr>
<tr>
<td>3</td>
<td>((1, 1) + i(0, 0))</td>
<td>7</td>
<td>((1 + r, 3 - r) + i(r^4, 2 - r))</td>
</tr>
</tbody>
</table>

Example 5.6. We solve the equation

\[
\bar{z}^3 = ((r + 1)^3, (4 - 2r)^3); \quad n = 3, \quad l = 3.
\]

We examine the method for this equation with two values of \(m\). Table 6 presents the results.

Table 6. The solved results of the equation in Example 5.6.

<table>
<thead>
<tr>
<th>(m)</th>
<th>(\bar{z}^0)</th>
<th>Number of iterations</th>
<th>(\bar{z}^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((1 + r, 3 - r) + i(3r, 5 - 2r))</td>
<td>22</td>
<td>((1 + r, 4 - 2r))</td>
</tr>
<tr>
<td>2</td>
<td>((1 + r + r^2, 4 - r^2) + i(3r, 5 - 2r))</td>
<td>18</td>
<td>((1 + r, 4 - 2r))</td>
</tr>
</tbody>
</table>

Example 5.7. Suppose that we want to solve the following equation for \(n = 3\) and \(l = 3\).

\[
\bar{z}^3 + \bar{z}^2 - \bar{z} = (-r^3 + 3r^2 + 4r - 9, -r^3 + 7r^2 - 18r + 9) + i(3r^2 + 2r - 2, 3r^2 - 14r + 6).
\]

Table 7 presents the results of solving the equation by the method with the different values of \(m\).

Table 7. The solved results of the equation in Example 5.7.

<table>
<thead>
<tr>
<th>(m)</th>
<th>(\bar{z}^0)</th>
<th>Number of iterations</th>
<th>(\bar{z}^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((1, 1) + i(2, 2))</td>
<td>6</td>
<td>((r, 2 - r) + i(1, 1))</td>
</tr>
<tr>
<td>2</td>
<td>((4 + r, 6 - r) + i(3, 3))</td>
<td>8</td>
<td>((r, 2 - r) + i(1, 1))</td>
</tr>
</tbody>
</table>

Example 5.8. Suppose that we want to solve the following equation

\[
\bar{z}^2 + \bar{z} = (e^{2r} + e^r - 1, \quad e^4 - 2e^r + e^3 - r^2 - 1) + i(-2e^{2r - 1}, \quad -2e^r - 1), \quad (5.8)
\]

by taking \(\bar{x}\) as non-negative fuzzy number and \(\bar{y}\) as non-positive fuzzy number. Considering the nearest 18-degree polynomial form for the real and imaginary parts of
the right hand side [1], \( z^0 = (r, 2 - r) \) and \( m = 9 \), after 7 iterations we have
\[
\tilde{z} = (\tilde{\alpha}, \tilde{\beta}) + i(\tilde{\gamma}, \tilde{\delta}) \quad \text{where}
\]
\[
\tilde{\alpha} = 1.0000 + 1.0000r + 0.5000r^2 + 0.1667r^3 + 0.0417r^4 + 0.0083r^5 + 0.0014r^6 + 0.0002r^7 + 0.0000r^8 + 0.0000r^9,
\]
\[
\tilde{\beta} = 7.3891 - 7.3891r + 3.6945r^2 - 1.2315r^3 + 0.3079r^4 - 0.0616r^5 + 0.0103r^6 - 0.0015r^7 + 0.0002r^8 - 0.0000r^9,
\]
\[
\tilde{\gamma} = -1.0000, \quad \tilde{\delta} = -1.0000
\]
For this solution the infinity norm of the error functions for left and right sides of real part are
\( 2.135 \times 10^{-5}, 4.366 \times 10^{-5} \). The infinity norm of both the left and right sides of the imaginary part is 0 and 0, respectively.

6. Conclusion

Up to now, there are not any analytically and numerically method for solving the algebraic fuzzy complex equation of degree \( n \). In this article, the research has been done related to gain the roots of a particular algebraic fuzzy complex equation of degree \( n \), by a numerical method. We plan to consider the solutions of another algebraic fuzzy complex equation, shortly.

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REFERENCES