Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 10, No. 1, 2022, pp. 77-92
DOI:10.22034/cmde.2021.36796.1638

Numerical method for the solution of algebraic fuzzy complex equations

S. Robab Fayyaz Behrooz and Majid Amirfakhrian*
Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran.

Abstract

In this paper, the numerical solution of an algebraic complex fuzzy equation of degree n, based on the parametric fuzzy numbers, is discussed. The unknown variable and right-hand side of the equation are considered as fuzzy complex numbers, whereas, the coefficients of the equation, are considered to be real crisp numbers. The given method is a numerical method and proposed based on the separation of the real and imaginary parts of the equation and using the parametric forms of the fuzzy numbers in the form of polynomials of degree at most m. In this case, a system of nonlinear equations is achieved. To get the solutions of the system, we used the Gauss-Newton iterative method. We also very briefly explain the conjugate of the solution of such equations. Finally, the efficiency and quality of the given method are tested by applying it to some numerical examples.

Keywords. Fuzzy numbers, Fuzzy complex numbers, Fuzzy polynomial, Algebraic fuzzy complex equation.
2010 Mathematics Subject Classification. 03E72, 08A72,15B15, 34A07.

1. Introduction

The problem of solving equations has many applications in the applied sciences like finance [11, 27], economy [12, 13, 28, 35], mechanics [19], and etc. Buckley and Qu have done a study on solving the fuzzy equations [14, 15]. They have also investigated analytically, first degree equations as $ax+b=c$ and second degree equations as $ax^2+bx=c$, by considering the coefficients of the equation as real, or complex fuzzy numbers, as well as the unknown of the equation as real, or complex fuzzy number [16]. Fuzzy equations $a+x=b$, where each component of the equation is fuzzy numbers, have been solved in [25] by α-cut method and also in [29] using the superimposition by analytically. In [3, 5, 6, 30] researchers have solved the fuzzy equations numerically. The numerical solution of the fuzzy polynomial equations by the neural network was investigated in [4], as well as was investigated using the ranking method [33]. Abbasbandy and Amirfakhrian considered the application of fuzzy polynomial equations [2]. They proposed a method for acquiring the best fuzzy function approximation on a set of points. Amirfakhrian presented the numerical solution of the algebraic fuzzy equation of degree n with a crisp variable in [8] and so with crisp coefficients in [9]. Researchers have studied the solutions of fully fuzzy equations as $\tilde{a}\tilde{x}^2+\tilde{b}\tilde{x}+\tilde{c}=\tilde{d}$, based on the restricted variation method and based on the optimization theory [7, 31]. The primary purpose of this study is solving an nth degree algebraic fuzzy complex equation with fuzzy complex unknown and real crisp coefficients. The given method is a numerical method that transforms an equation into a system of nonlinear equations. The obtained system is solved by the Gauss-Newton iterative method. The organization of this article is as follows: We first define the parametric form as a complex fuzzy number, conjugate of it, and some basic properties of them, in section 2. Then, we introduce an nth degree algebraic fuzzy complex equation, with complex fuzzy unknown and crisp coefficients. section 3 contains the Gauss-Newton method to solve a system of nonlinear equations. In section 4, we introduce an algorithm to get the solutions of an nth degree algebraic fuzzy complex equation, for both cases nonnegative or negative root, by a numerical method. Finally, some examples are presented in section 5.

Received: 12 November 2019 ; Accepted: 18 January 2021.
* Corresponding author. Email: amirfakhrian@iauctb.ac.ir.

2. Preliminaries

Definition 2.1. A fuzzy number $\tilde{u}$ is a convex normalized fuzzy set of the real line $\mathbb{R}$ such that there exists exactly one $x_0 \in \mathbb{R}$ with $\mu_{\tilde{u}}(x_0) = 1$ ($\mu_{\tilde{u}}$ is called the membership function which is upper semi-continuous and compactly supported. We denote by $F(\mathbb{R})$ the set of all real fuzzy numbers [10].

Definition 2.2. A fuzzy number in the parametric form is presented by $\tilde{u} = (\underline{u}, \overline{u})$, where functions $\underline{u}$ and $\overline{u}$; $0 \le r \le 1$ apply the following conditions [9, 23, 24, 32, 34]:

(1) $\underline{u}$ is monotonically increasing left continuous function on $[0, 1]$.
(2) $\overline{u}$ is monotonically decreasing left continuous function on $[0, 1]$.
(3) $\underline{u}(r) \le \overline{u}(r)$, $0 \le r \le 1$.

Suppose $\tilde{u} = (\underline{u}, \overline{u})$ and $\tilde{v} = (\underline{v}, \overline{v})$ belong to $F(\mathbb{R})$. Some operations on fuzzy numbers are defined as the below forms [9]:

$$x > 0 : x\tilde{u} = (x\underline{u}, x\overline{u}), \tag{2.1}$$

$$x < 0 : x\tilde{u} = (x\overline{u}, x\underline{u}), \tag{2.2}$$

$$\tilde{u} + \tilde{v} = (\underline{u} + \underline{v}, \overline{u} + \overline{v}), \tag{2.3}$$

$$\tilde{u} - \tilde{v} = (\underline{u} - \overline{v}, \overline{u} - \underline{v}), \tag{2.4}$$

$$\tilde{u}.\tilde{v} = (\min\{\underline{u}.\underline{v}, \underline{u}.\overline{v}, \overline{u}.\underline{v}, \overline{u}.\overline{v}\}, \quad \max\{\underline{u}.\underline{v}, \underline{u}.\overline{v}, \overline{u}.\underline{v}, \overline{u}.\overline{v}\}). \tag{2.5}$$

A fuzzy number $\tilde{u}$ is nonnegative (non-positive) if for $x < 0$ ($x > 0$), we have $\mu_{\tilde{u}}(x) = 0$, equivalently if on $[0, 1]$ we have $\underline{u} \ge 0$ ($\overline{u} \le 0$). Also a fuzzy number $\tilde{u}$ is positive (negative) if for $x \le 0$ ($x \ge 0$), we have $\mu_{\tilde{u}}(x) = 0$, equivalently if on $[0, 1]$ we have $\underline{u} > 0$ ($\overline{u} < 0$). So we have

- If $\tilde{u}$ and $\tilde{v}$ are nonnegative fuzzy numbers, then $\tilde{u}.\tilde{v} = (\underline{u}.\underline{v}, \overline{u}.\overline{v})$.
- If $\tilde{u}$ and $\tilde{v}$ are non-positive fuzzy numbers, then $\tilde{u}.\tilde{v} = (\overline{u}.\overline{v}, \underline{u}.\underline{v})$.
- If $\tilde{u}$ be a non-negative fuzzy number and $\tilde{v}$ be a non-positive fuzzy number, then $\tilde{u}.\tilde{v} = (\underline{u}.\overline{v}, \overline{u}.\underline{v})$.

Definition 2.3. Inherit from the real complex numbers, a fuzzy complex number $\tilde{z}$, is defined in [22] in the form of $\tilde{z} = \tilde{x} + i\tilde{y}$, where $\tilde{x}$ and $\tilde{y}$ are two fuzzy real numbers. $\tilde{x}$ is the fuzzy real part of $\tilde{z}$ while $\tilde{y}$ represents the fuzzy imaginary part.

We denote by $F(\mathbb{C})$, the set of all fuzzy complex numbers.

Definition 2.4. Since the real and imaginary parts are both fuzzy numbers, we can define a parametric representation for fuzzy complex numbers as

$$\tilde{z} = \tilde{x} + i\tilde{y} = (\underline{x}, \overline{x}) + i(\underline{y}, \overline{y}), \tag{2.6}$$

in which $\underline{x}, \overline{x}, \underline{y}, \overline{y}$ satisfy in requirements of Definition 2.2.

Let $\tilde{z}_1 = \tilde{x}_1 + i\tilde{y}_1 = (\underline{x}_1, \overline{x}_1) + i(\underline{y}_1, \overline{y}_1)$ and $\tilde{z}_2 = \tilde{x}_2 + i\tilde{y}_1 = (\underline{x}_2, \overline{x}_2) + i(\underline{y}_2, \overline{y}_2)$ are belonging to $F(\mathbb{C})$. We defined some operations on fuzzy complex numbers as follows:

Addition:

$$\tilde{z}_1 + \tilde{z}_2 = (\underline{x}_1 + \underline{x}_2 \ , \ \overline{x}_1 + \overline{x}_2) + i(\underline{y}_1 + \underline{y}_2 \ , \ \overline{y}_1 + \overline{y}_2). \tag{2.7}$$

Subtraction:

$$\tilde{z}_1 - \tilde{z}_2 = (\underline{x}_1 - \overline{x}_2 \ , \ \overline{x}_1 - \underline{x}_2) + i(\underline{y}_1 - \overline{y}_2 \ , \ \overline{y}_1 - \underline{y}_2). \tag{2.8}$$

Multiplication:

$$\tilde{z}_1.\tilde{z}_2 = (\tilde{x}_1\tilde{x}_2 - \tilde{y}_1\tilde{y}_2) + i(\tilde{x}_1\tilde{y}_2 + \tilde{y}_1\tilde{x}_2). \tag{2.9}$$

Let $F(\mathbb{R})^+$ be the set of all non-negative fuzzy real numbers and $F(\mathbb{R})^-$ be the set of all non-positive fuzzy real numbers. If $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{y}_1$ and $\tilde{y}_2$ belong to $F(\mathbb{R})^+$, then

$$\tilde{z}_1.\tilde{z}_2 = (\underline{x}_1\underline{x}_2 - \overline{y}_1\overline{y}_2 \ , \ \overline{x}_1\overline{x}_2 - \underline{y}_1\underline{y}_2) + i(\underline{x}_1\underline{y}_2 + \underline{y}_1\underline{x}_2 \ , \ \overline{x}_1\overline{y}_2 + \overline{y}_1\overline{x}_2). \tag{2.10}$$

If $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{y}_1$ and $\tilde{y}_2$ belong to $F(\mathbb{R})^-$, then

$$\tilde{z}_1.\tilde{z}_2 = (\overline{x}_1\overline{x}_2 - \underline{y}_1\underline{y}_2 \ , \ \underline{x}_1\underline{x}_2 - \overline{y}_1\overline{y}_2) + i(\overline{x}_1\overline{y}_2 + \overline{y}_1\overline{x}_2 \ , \ \underline{x}_1\underline{y}_2 + \underline{y}_1\underline{x}_2). \tag{2.11}$$

Definition 2.5. Conjugate of a fuzzy complex number $\tilde{z} = \tilde{x} + i\tilde{y}$ is given by

$$\bar{\tilde{z}} = \tilde{x} - i\tilde{y} = (\underline{x}, \overline{x}) + i(-\overline{y}, -\underline{y}). \tag{2.12}$$

The relation between $\tilde{z}$ and $\bar{\tilde{z}}$ are as the below form:

(1) $\tilde{z} + \bar{\tilde{z}} = 2\tilde{x} + i\tilde{0}_y$,
(2) $\tilde{z} - \bar{\tilde{z}} = \tilde{0}_x + 2i\tilde{y}$,
(3) $\tilde{z}.\bar{\tilde{z}} = \tilde{x}^2 + \tilde{y}^2 + \tilde{0}_{xy}$,
where for a fuzzy number $\tilde{u}$, $\tilde{0}_u$ is a fuzzy zero with respect to $\tilde{u}$:$\tilde{0}_u = \tilde{u} - \tilde{u}$.

2.1. **Fuzzy polynomial over the set of fuzzy complex numbers.** In this section, we generalize the concept of the fuzzy algebraic equation to complex form based on [9].
Let $a_1, \cdots, a_n \in \mathbb{R}$, $a_n \neq 0$ and n be a natural number. A complex fuzzy polynomial of degree at most $n \geq 1$ with fuzzy complex variable is shown as

$$\tilde{P}_n(\tilde{z}) = \sum_{j=1}^{n} a_j \tilde{z}^j. \tag{2.13}$$

An nth degree algebraic fuzzy complex equation with fuzzy complex unknown and crisp coefficients, is

$$a_n\tilde{z}^n + \cdots + a_1\tilde{z} = \tilde{b}, \tag{2.14}$$

where $\tilde{b} \in F(\mathbb{C})$.

Definition 2.6. A fuzzy number $\tilde{u}$ ($\tilde{u} \in F(\mathbb{R})$) has "$m-$degree polynomial form" if there exist two polynomials of degree at most m, p_m and q_m, such that $\tilde{u} = (p_m \ , \ q_m)$[9].
Let $F_m(\mathbb{R})$ be the set of all $m-$ degree polynomial form fuzzy real numbers. By $F_m(\mathbb{C})$ we also denote the set of all fuzzy complex numbers with $m-$ degree polynomial form:

$$F_m(\mathbb{C}) = \left\{\tilde{z} = \tilde{x} + i\tilde{y} : \quad \tilde{x}, \ \tilde{y} \in F_m(\mathbb{R})\right\}. \tag{2.15}$$

The conjugate of a root of an algebraic fuzzy complex equation is n't a root of the equation, necessarily. Because $\bar{\tilde{z^n}}$ and $\bar{\tilde{z}}^n$ are not the same, even if real and imaginary parts of $\tilde{z}$ are non-negative or non-positive, and even for $n = 2$.

3. Solving the system of equations

The system of linear equations containing s variables and d equations ($d \geq s$) can be shown like $F(X) = (F_1(X), \cdots, F_d(X))^T = 0$, in which $X = [x_i]_{s\times 1}$. For the solution of the system, the iterative process of Gauss-Newton [20, 21], is as the following:
First, it is required to use the Jacobian matrix $J(X) = [\frac{\partial F_i(X)}{\partial X_j}]$, where $i = 1, \cdots, d$ and $j = 1, \cdots, s$, then we select an arbitrary starting vector $X^{(0)}$ as an approximated solution. To gain the desired solution one have to solve the system $J(X^k)H^k = -F(X^k)$, and in the next iterate is replaced $X^{(k)}$ by $X^{(k)} + H^{(k)}$ in other words $X^{(k+1)} = X^{(k)} + H^{(k)}$. This system is evaluated by the least squares method as the following form

$$J(X^{(k)})^T J(X^{(K)})H^{(k)} = -J(X^{(k)})^T F(X^{(k)}). \tag{3.1}$$

The conditions on convergence and uniqueness of the solution are studied in [17, 18, 26].

Theorem 3.1. *[9] Let N be a $d \times s$, matrix where $d > s$ and the linear system of equations $NX = C$ has a unique solution. With any initial vector, the sequence of the Gauss-Newton method will be converged to the exact solution of a linear system of equations at the first iterations.*

4. Solving an Algebraic Fuzzy Complex Equation

We consider the following nth degree algebraic complex fuzzy equation with the complex fuzzy variable as

$$\sum_{j=1}^{n} a_j \tilde{z}^j = \tilde{b} \tag{4.1}$$

where $a_n \neq 0$ and $\tilde{b} \in F(\mathbb{C})$. By inserting $\tilde{z} = \tilde{x} + i\tilde{y}$ and $\tilde{b} = \tilde{c} + i\tilde{d}$ in (4.1), we acquire

$$\sum_{j=1}^{n} a_j (\tilde{x} + i\tilde{y})^j = \tilde{c} + i\tilde{d}, \tag{4.2}$$

where $\tilde{x}$ and $\tilde{y}$ are single modal value fuzzy numbers in $F_m(\mathbb{R})$, and $\tilde{c}, \tilde{d}$ are known fuzzy numbers in parametric form belonging to $F_l(\mathbb{R})$, such that $l \leq nm$.
From (4.2), according to binomial expansion, we have

$$\sum_{j=1}^{n} a_j (\sum_{k=0}^{j} \binom{j}{k} \tilde{x}^k (i\tilde{y})^{j-k}) = (\sum_{q=0}^{l} \underline{c_q}\ r^q, \sum_{q=0}^{l} \overline{c_q}\ r^q) + i(\sum_{q=0}^{l} \underline{d_q}\ r^q, \sum_{q=0}^{l} \overline{d_q}\ r^q). \tag{4.3}$$

Left hand side of (4.3) is equal to

$$\sum_{j=1}^{n} a_j (\sum_{k=0}^{[j/2]} (-1)^k \binom{j}{j-2k} \tilde{x}^{j-2k} \tilde{y}^{2k} + i \sum_{k=0}^{j-[j/2]-1} (-1)^k \binom{j}{j-(2k+1)} \tilde{x}^{j-(2k+1)} \tilde{y}^{2k+1}). \tag{4.4}$$

Since the multiplication of fuzzy numbers is dependent on the sign of numbers, so we consider the "Eq. (4.4)" in two special cases, i.e first case is that $\tilde{x}$ and $\tilde{y}$ are nonnegative fuzzy numbers belonging to $F_m(\mathbb{R})$, and another case is that both of them are negative fuzzy numbers belonging to $F_m(\mathbb{R})$.
First case:
By taking $\tilde{x} = (\underline{x}, \overline{x})$, $\tilde{y} = (\underline{y}, \overline{y})$ as non-negative single modal value fuzzy numbers belonging to $F_m(\mathbb{R})$ and by considering (4.4), Eq. (4.3) is equivalent to following two equations:

$$\sum_{j=1}^{n} a_j \sum_{k=0}^{[j/2]} (-1)^k \binom{j}{j-2k} (\underline{x}^{j-2k} \underline{y}^{2k}, \overline{x}^{j-2k} \overline{y}^{2k}) = (\sum_{q=0}^{l} \underline{c_q}\ r^q, \sum_{q=0}^{l} \overline{c_q}\ r^q), \tag{4.5}$$

$$\sum_{j=1}^{n} a_j \sum_{k=0}^{j-[j/2]-1} (-1)^k \binom{j}{j-(2k+1)} (\underline{x}^{j-(2k+1)} \underline{y}^{2k+1}, \overline{x}^{j-(2k+1)} \overline{y}^{2k+1}) = (\sum_{q=0}^{l} \underline{d_q}\ r^q, \sum_{q=0}^{l} \overline{d_q}\ r^q). \tag{4.6}$$

By defining

$$\alpha_{j,k} = a_j (-1)^k \tbinom{j}{j-2k}, \qquad \beta_{j,k} = a_j (-1)^k \tbinom{j}{j-(2k+1)}, \tag{4.7}$$

and considering

$$\hat{\alpha}_{j,k} = \begin{cases} \alpha_{j,k}, & \alpha_{j,k} \geq 0, \\ 0, & \alpha_{j,k} < 0, \end{cases} \quad , \check{\alpha}_{j,k} = \begin{cases} 0, & \alpha_{j,k} \geq 0, \\ \alpha_{j,k}, & \alpha_{j,k} < 0, \end{cases} \tag{4.8}$$

Eq. (4.5) reduces to the following form:

$$\begin{cases} \sum_{j=1}^{n} \sum_{k=0}^{[j/2]} (\hat{\alpha}_{j,k} \underline{x}^{j-2k} \underline{y}^{2k} + \check{\alpha}_{j,k} \overline{x}^{j-2k} \overline{y}^{2k}) = \sum_{q=0}^{l} \underline{c_q}\ r^q, \\ \sum_{j=1}^{n} \sum_{k=0}^{[j/2]} (\hat{\alpha}_{j,k} \overline{x}^{j-2k} \overline{y}^{2k} + \check{\alpha}_{j,k} \underline{x}^{j-2k} \underline{y}^{2k}) = \sum_{q=0}^{l} \overline{c_q}\ r^q. \end{cases} \tag{4.9}$$

Similarly by considering

$$\hat{\beta}_{j,k} = \begin{cases} \beta_{j,k}, & \beta_{j,k} \geq 0, \\ 0, & \beta_{j,k} < 0, \end{cases} \quad , \check{\beta}_{j,k} = \begin{cases} 0, & \beta_{j,k} \geq 0, \\ \beta_{j,k}, & \beta_{j,k} < 0, \end{cases} \tag{4.10}$$

Eq.(4.6) will be of the following form:

$$\begin{cases} \sum_{j=1}^{n} \sum_{k=0}^{j-[j/2]-1} (\hat{\beta}_{j,k} \underline{x}^{j-(2k+1)} \underline{y}^{2k+1} + \check{\beta}_{j,k} \overline{x}^{j-(2k+1)} \overline{y}^{2k+1}) = \sum_{q=0}^{l} \underline{d_q}\ r^q, \\ \sum_{j=1}^{n} \sum_{k=0}^{[j-[j/2]-1]} (\hat{\beta}_{j,k} \overline{x}^{j-(2k+1)} \overline{y}^{2k+1} + \check{\beta}_{j,k} \underline{x}^{j-(2k+1)} \underline{y}^{2k+1}) = \sum_{q=0}^{l} \overline{d_q}\ r^q. \end{cases} \tag{4.11}$$

Let

$$\begin{cases} \underline{x}(r) = \sum_{s=0}^{m} \underline{x}_s\ r^s, & \overline{x}(r) = \sum_{s=0}^{m} \overline{x}_s\ r^s, \\ \underline{y}(r) = \sum_{s=0}^{m} \underline{y}_s\ r^s, & \overline{y}(r) = \sum_{s=0}^{m} \overline{y}_s r^s. \end{cases} \tag{4.12}$$

Eq.(4.9) and Eq.(4.11), are equivalence to the following equations

$$\begin{cases} \sum_{j=1}^{n} \sum_{k=0}^{[j/2]} (\hat{\alpha}_{j,k} (\sum_{s=0}^{m} \underline{x}_s\ r^s)^{j-2k} (\sum_{s=0}^{m} \underline{y}_s\ r^s)^{2k} \\ \qquad + \check{\alpha}_{j,k} (\sum_{s=0}^{m} \overline{x}_s\ r^s)^{j-2k} (\sum_{s=0}^{m} \overline{y}_s r^s)^{2k}) = \sum_{q=0}^{l} \underline{c_q}\ r^q, \\ \sum_{j=1}^{n} \sum_{k=0}^{[j/2]} (\hat{\alpha}_{j,k} (\sum_{s=0}^{m} \overline{x}_s\ r^s)^{j-2k} (\sum_{s=0}^{m} \overline{x}_s\ r^s)^{2k} \\ \qquad + \check{\alpha}_{j,k} (\sum_{s=0}^{m} \underline{x}_s\ r^s)^{j-2k} (\sum_{s=0}^{m} \overline{y}_s\ r^s)^{2k}) = \sum_{q=0}^{l} \overline{c_q}\ r^q, \end{cases} \tag{4.13}$$

and

$$\begin{cases} \sum_{j=1}^{n} \sum_{k=0}^{j-[j/2]-1} (\hat{\beta}_{j,k} (\sum_{s=0}^{m} \underline{x}_s\ r^s)^{j-(2k+1)} (\sum_{s=0}^{m} \underline{y}_s r^s)^{2k+1} \\ \qquad + \check{\beta}_{j,k} (\sum_{s=0}^{m} \overline{x}_s\ r^s)^{j-(2k+1)} (\sum_{s=0}^{m} \overline{y}_s\ r^s)^{2k+1}) = \sum_{q=0}^{l} \underline{d_q}\ r^q, \\ \qquad , \\ \sum_{j=1}^{n} \sum_{k=0}^{[j-[j/2]-1]} (\hat{\beta}_{j,k} (\sum_{s=0}^{m} \overline{x}_s\ r^s)^{j-(2k+1)} (\sum_{s=0}^{m} \overline{y}_s\ r^s)^{2k+1} \\ \qquad + \check{\beta}_{j,k} (\sum_{s=0}^{m} \underline{x}_s r^s)^{j-(2k+1)} (\sum_{s=0}^{m} \underline{y}_s r^s)^{2k+1}) = \sum_{q=0}^{l} \overline{d_q}\ r^q. \end{cases} \tag{4.14}$$

Let for $i = 0, 1, \cdots, nm; L_i, U_i, M_i$ and N_i be the coefficients of r^i in (4.13) and (4.14), respectively, thus

$$\begin{cases} \sum_{i=0}^{nm} L_i(\underline{x}_0, \cdots, \underline{x}_m, \overline{x}_0, \cdots, \overline{x}_m, \underline{y}_0, \cdots, \underline{y}_m, \overline{y}_0, \cdots, \overline{y}_m) r^i = 0, \\ \sum_{i=0}^{nm} U_i(\underline{x}_0, \cdots, \underline{x}_m, \overline{x}_0, \cdots, \overline{x}_m, \underline{y}_0, \cdots, \underline{y}_m, \overline{y}_0, \cdots, \overline{y}_m) r^i = 0, \\ \sum_{i=0}^{nm} M_i(\underline{x}_0, \cdots, \underline{x}_m, \overline{x}_0, \cdots, \overline{x}_m, \underline{y}_0, \cdots, \underline{y}_m, \overline{y}_0, \cdots, \overline{y}_m) r^i = 0, \\ \sum_{i=0}^{nm} N_i(\underline{x}_0, \cdots, \underline{x}_m, \overline{x}_0, \cdots, \overline{x}_m, \underline{y}_0, \cdots, \underline{y}_m, \overline{y}_0, \cdots, \overline{y}_m) r^i = 0. \end{cases} \tag{4.15}$$

Since the fuzzy numbers are one modal fuzzy numbers, we consider the following equations

$$Q_1(\underline{x}_0, \cdots, \underline{x}_m, \overline{x}_0, \cdots, \overline{x}_m) = \underline{x}(1) - \overline{x}(1) = 0 \tag{4.16}$$

and

$$Q_2(\underline{y}_0, \cdots, \underline{y}_m, \overline{y}_0, \cdots, \overline{y}_m) = \underline{y}(1) - \overline{y}(1) = 0 \tag{4.17}$$

Eqs.(4.15), (4.16), and (4.17) yield that

$$\begin{cases} L_i(\underline{x}_0, \cdots, \underline{x}_m, \overline{x}_0, \cdots, \overline{x}_m, \underline{y}_0, \cdots, \underline{y}_m, \overline{y}_0, \cdots, \overline{y}_m) = 0, & i = 0, \cdots, nm \\ U_i(\underline{x}_0, \cdots, \underline{x}_m, \overline{x}_0, \cdots, \overline{x}_m, \underline{y}_0, \cdots, \underline{y}_m, \overline{y}_0, \cdots, \overline{y}_m) = 0, & i = 0, \cdots, nm \\ M_i(\underline{x}_0, \cdots, \underline{x}_m, \overline{x}_0, \cdots, \overline{x}_m, \underline{y}_0, \cdots, \underline{y}_m, \overline{y}_0, \cdots, \overline{y}_m) = 0, & i = 0, \cdots, nm \\ N_i(\underline{x}_0, \cdots, \underline{x}_m, \overline{x}_0, \cdots, \overline{x}_m, \underline{y}_0, \cdots, \underline{y}_m, \overline{y}_0, \cdots, \overline{y}_m) = 0, & i = 0, \cdots, nm \\ Q_1(\underline{x}_0, \cdots, \underline{x}_m, \overline{x}_0, \cdots, \overline{x}_m) = 0, & \\ Q_2(\underline{y}_0, \cdots, \underline{y}_m, \overline{y}_0, \cdots, \overline{y}_m) = 0, & \end{cases} \tag{4.18}$$

which is a system of the nonlinear equations with $4nm + 6$ equations and $4m + 4$ unknowns.

We solve the system (4.18) by an iterative Gauss-Newton method that introduced in section 3. Defining

$$J = \begin{pmatrix} \frac{\partial L_i}{\partial \underline{x}_j} & \frac{\partial L_i}{\partial \overline{x}_j} & \frac{\partial L_i}{\partial \underline{y}_j} & \frac{\partial L_i}{\partial \overline{y}_j} \\ \frac{\partial U_i}{\partial \underline{x}_j} & \frac{\partial U_i}{\partial \overline{x}_j} & \frac{\partial U_i}{\partial \underline{y}_j} & \frac{\partial U_i}{\partial \overline{y}_j} \\ \frac{\partial M_i}{\partial \underline{x}_j} & \frac{\partial M_i}{\partial \overline{x}_j} & \frac{\partial M_i}{\partial \underline{y}_j} & \frac{\partial M_i}{\partial \overline{y}_j} \\ \frac{\partial N_i}{\partial \underline{x}_j} & \frac{\partial N_i}{\partial \overline{x}_j} & \frac{\partial N_i}{\partial \underline{y}_j} & \frac{\partial N_i}{\partial \overline{y}_j} \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \tag{4.19}$$

and

$$B = -\begin{pmatrix} L_i \\ U_i \\ M_i \\ N_i \\ Q_1 \\ Q_2 \end{pmatrix}, X = \begin{pmatrix} \underline{x}_j \\ \overline{x}_j \\ \underline{y}_j \\ \overline{y}_j \end{pmatrix}, H = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{4m+4} \end{pmatrix}, \tag{4.20}$$

where $i = 0, 1, \cdots, mn$; $j = 0, 1, \cdots, m$, we have $JH = B$ $(J = J(X), B = B(X), H = H(X))$. We solve this system with an initial vector $X^{(0)}$, by the least squares method and then we improve this guess. We find the sequence $Y^{(k)}$ as follows: With an initial vector $X^{(0)}$, in iteration k, we compute $J^{(k)}$ and $B^{(k)}$ and solve $J^{(k)}H^{(k)} = B^{(k)}$ by the least squares method as the following: $(J^{(k)} = J(X^k)$, $B^{(k)} = B(X^k)$ $)$

$${J^{(k)}}^T J^{(k)} H^{(k)} = {J^{(k)}}^T B^{(k)}. \tag{4.21}$$

The solution will be improved by

$$X^{(k+1)} = X^{(k)} + H^{(k)}. \tag{4.22}$$

If $X^{(k)}$ converges to a vector X^*; then $\tilde{x}^*$ in the form of

$$\tilde{x}^* = \Big(\sum_{s=0}^{m} {\underline{x}_s}^* \, r^s \; , \; \sum_{s=0}^{m} {\overline{x}_s}^* \, r^s \Big) \tag{4.23}$$

and $\tilde{y}^*$ in the form of

$$\tilde{y}^* = \Big(\sum_{s=0}^{m} {\underline{y}_s}^* \, r^s \; , \; \sum_{s=0}^{m} {\overline{y}_s}^* \, r^s \Big) \tag{4.24}$$

are the parts of the Gauss-Newton solution. If $\underline{x}^*$ and $\underline{y}^*$ are both increasing, $\overline{x}^*$ and $\overline{y}^*$ are both decreasing, then $\tilde{z}^* = \tilde{x}^* + i\tilde{y}^*$ is a solution of (2.14).

Second case:

For a negative fuzzy number such as $\tilde{x} = (\underline{x}, \overline{x})$, we have

$$\tilde{x}^{2j} = (\overline{x}^{2j}, \underline{x}^{2j}),\ \tilde{x}^{2j+1} = (\underline{x}^{2j+1}, \overline{x}^{2j+1}), \tag{4.25}$$

therefore, if $\tilde{x}$ and $\tilde{y}$ be two negative fuzzy numbers, then real and imaginary parts of (4.3) will be rewritten to the following two relations :

Real part of (4.3) is:

$$\begin{aligned}\sum_{j=1}^{n} a_j \sum_{k=0}^{[j/2]} (-1)^k \binom{j}{j-2k} \tilde{x}^{j-2k}\tilde{y}^{2k} = &\sum_{j=1}^{l_1} a_{2j} \sum_{k=0}^{j} (-1)^k \binom{2j}{2j-2k} (\overline{x}^{2j-2k}, \underline{x}^{2j-2k})(\overline{y}^{2k}, \underline{y}^{2k}) \\ &+ \sum_{j=0}^{l_2} a_{2j+1} \sum_{k=0}^{j} (-1)^k \binom{2j+1}{2j+1-2k} (\underline{x}^{2j+1-2k}, \overline{x}^{2j+1-2k})(\overline{y}^{2k}, \underline{y}^{2k}).\end{aligned} \tag{4.26}$$

Imaginary part of (4.3) is:

$$\begin{aligned}\sum_{j=1}^{n} a_j \sum_{k=0}^{j-[j/2]-1} (-1)^k \binom{j}{j-(2k+1)} \tilde{x}^{j-(2k+1)}\tilde{y}^{2k+1} = &\sum_{j=1}^{l_1} a_{2j} \sum_{k=0}^{j-1} (-1)^k \binom{2j}{2j-(2k+1)} (\underline{x}^{2j-2k-1}, \overline{x}^{2j-2k-1})(\underline{y}^{2k+1}, \overline{y}^{2k+1}) \\ &+ \sum_{j=0}^{l_2} a_{2j+1} \sum_{k=0}^{j} \binom{2j+1}{2j-2k} (\overline{x}^{2j-2k}, \underline{x}^{2j-2k})(\underline{y}^{2k+1}, \overline{y}^{2k+1}).\end{aligned} \tag{4.27}$$

In (4.26) and (4.27) l_1 and l_2 are as follows:

$$l_1 = [\tfrac{n}{2}] \quad , l_2 = \begin{cases} [\frac{n}{2}] & \text{n is odd} \\ [\frac{n}{2}] - 1 & \text{n is even} \end{cases} \tag{4.28}$$

Thus Eqs. (4.5) and (4.6) reduce to following:

$$\begin{aligned}&\sum_{j=1}^{l_1} a_{2j} \sum_{k=0}^{j} (-1)^k \binom{2j}{2j-2k} (\overline{x}^{2j-2k}, \underline{x}^{2j-2k})(\overline{y}^{2k}, \underline{y}^{2k}) + \sum_{j=0}^{l_2} a_{2j+1} \sum_{k=0}^{j} (-1)^k \binom{2j+1}{2j+1-2k} (\underline{x}^{2j+1-2k}, \overline{x}^{2j+1-2k})(\overline{y}^{2k}, \underline{y}^{2k}) \\ &= (\sum_{q=0}^{l} \underline{c_q} r^q \ , \ \sum_{q=0}^{l} \overline{c_q} r^q),\end{aligned} \tag{4.29}$$

and

$$\sum_{j=1}^{l_1} a_{2j} \sum_{k=0}^{j-1} (-1)^k \binom{2j}{2j-2k-1} (\underline{x}^{2j-2k-1}, \overline{x}^{2j-2k-1})(\underline{y}^{2k+1}, \overline{y}^{2k+1}) + \sum_{j=0}^{l_2} a_{2j+1} \sum_{k=0}^{j} \binom{2j+1}{2j-2k} (\overline{x}^{2j-2k}, \underline{x}^{2j-2k})(\underline{y}^{2k+1}, \overline{y}^{2k+1}) \tag{4.30}$$

$$= (\sum_{q=0}^{l} \underline{d_q} r^q, \sum_{q=0}^{l} \overline{d_q} r^q). \tag{4.31}$$

By using

$$\alpha_{j,k} = (-1)^k a_{2j} \tbinom{2j}{2j-2k}, \qquad \beta_{j,k} = (-1)^k a_{2j+1} \tbinom{2j+1}{2j+1-2k}, \tag{4.32}$$

and considering

$$\hat{\alpha}_{j,k} = \begin{cases} \alpha_{j,k}, & \alpha_{j,k} \geq 0, \\ 0, & \alpha_{j,k} < 0, \end{cases} \qquad \check{\alpha}_{j,k} = \begin{cases} 0, & \alpha_{j,k} \geq 0, \\ \alpha_{j,k}, & \alpha_{j,k} < 0, \end{cases} \quad , \tag{4.33}$$

and

$$\hat{\beta}_{j,k} = \begin{cases} \beta_{j,k}, & \beta_{j,k} \geq 0, \\ 0, & \beta_{j,k} < 0, \end{cases} \quad \check{\beta}_{j,k} = \begin{cases} 0, & \beta_{j,k} \geq 0, \\ \beta_{j,k}, & \beta_{j,k} < 0, \end{cases} , \tag{4.34}$$

Eq. (4.29) is changed to the following equations

$$\begin{cases} \sum_{j=1}^{l_1} \sum_{k=0}^{j} (\hat{\alpha}_{j,k} \overline{x}^{(2j-2k)} \overline{y}^{2k} + \check{\alpha}_{j,k} \underline{x}^{(2j-2k)} \underline{y}^{2k}) + \\ \sum_{j=0}^{l_2} \sum_{k=0}^{j} (\hat{\beta}_{j,k} \underline{x}^{2j-2k+1} \underline{y}^{2k} + \check{\beta}_{j,k} \overline{x}^{2j-2k+1} \overline{y}^{2k}) = \sum_{q=0}^{l} \underline{c_q} r^q, \\ \\ \sum_{j=1}^{l_1} \sum_{k=0}^{j} (\hat{\alpha}_{j,k} \underline{x}^{(2j-2k)} \underline{y}^{2k} + \check{\alpha}_{j,k} \overline{x}^{2j-2k} \overline{y}^{2k}) + \\ \sum_{j=0}^{l_2} \sum_{k=0}^{j} (\hat{\beta}_{j,k} \overline{x}^{2j-2k+1} \overline{y}^{2k} + \check{\beta}_{j,k} \underline{x}^{(2j-2k+1)} \underline{y}^{2k}) = \sum_{q=0}^{l} \overline{c_q} r^q. \end{cases} \tag{4.35}$$

Also, by using

$$\hat{\gamma}_{j,k} = \begin{cases} \gamma_{j,k}, & \gamma_{j,k} \geq 0 \\ 0, & \gamma_{j,k} < 0, \end{cases} , \check{\gamma}_{j,k} = \begin{cases} 0, & \gamma_{j,k} \geq 0 \\ \gamma_{j,k}, & \gamma_{j,k} < 0, \end{cases} \tag{4.36}$$

and

$$\hat{\eta}_{j,k} = \begin{cases} \eta_{j,k}, & \eta_{j,k} \geq 0 \\ 0, & \eta_{j,k} < 0 \end{cases} , \check{\eta}_{j,k} = \begin{cases} 0, & \eta_{j,k} \geq 0 \\ \eta_{j,k}, & \eta_{j,k} < 0 \end{cases} \tag{4.37}$$

where

$$\gamma_{j,k} = (-1)^k a_{2j} \binom{2j}{2j-2k-1} \quad , \eta_{j,k} = (-1)^k a_{2j+1} \binom{2j+1}{2j-2k}, \tag{4.38}$$

Eq. (4.30) changes to the following system

$$\begin{cases} \sum_{j=1}^{l_1} \sum_{k=0}^{j-1} (\hat{\gamma}_{j,k} \overline{x}^{(2j-2k-1)} \overline{y}^{(2k+1)} + \check{\gamma}_{j,k} \underline{x}^{(2j-2k-1)} \underline{y}^{(2k+1)}) + \\ \sum_{j=0}^{l_2} \sum_{k=0}^{j} (\hat{\eta}_{j,k} \underline{x}^{(2j-2k)} \underline{y}^{(2k+1)} + \check{\eta}_{j,k} \overline{x}^{(2j-2k)} \overline{y}^{(2k+1)}) = \sum_{q=0}^{l} \underline{d_q} r^q, \\ \\ \sum_{j=1}^{l_1} \sum_{k=0}^{j-1} (\hat{\gamma}_{j,k} \underline{x}^{(2j-2k-1)} \underline{y}^{(2k+1)} + \check{\gamma}_{j,k} \overline{x}^{(2j-2k-1)} \overline{y}^{(2k+1)}) + \\ \sum_{j=0}^{l_2} \sum_{k=0}^{j} \hat{\eta}_{j,k} \overline{x}^{(2j-2k)} \overline{y}^{(2k+1)} + \check{\eta}_{j,k} \underline{x}^{(2j-2k)} \underline{y}^{(2k+1)} = \sum_{q=0}^{l} \overline{d_q} r^q. \end{cases} \tag{4.39}$$

Eqs.(4.35) and (4.39) yield that for $i = 0, \cdots, mn$; L_i, U_i, M_i and N_i are the coefficients of r^i, respectively, also $Q1$, $Q2$ are the same as (4.17), (4.18). Thus again we have $4nm + 6$ equations.

Algorithm

(1) Input an initial solution $X^{(0)}$.
(2) Evaluate $J^{(k)}$ and $B^{(k)}$ from (4.19) and (4.20).
(3) Solve $H^{(k)}$ from the system of linear equations ${J^{(k)}}^T J^{(k)} H^{(k)} = {J^{(k)}}^T B^{(k)}$.
(4) $X^{(k+1)} = X^{(k)} + H^{(k)}$.
(5) If the convergence condition is yield, then end.
(6) $k = k + 1$, and go to step 3.

Theorem and lemmas of this part are extended from theorem and lemmas related to [9].

Lemma 4.1. *Let $n = 1$ (i.e. $a\tilde{z} = \tilde{b}$), where real and imaginary parts of $\tilde{z}$ and $\tilde{b}$ belong to $F_m(\mathbb{R})$. If $a \neq 0$, then in each iteration, we have*

$$J^T J = a^2 I_{4m+4} + V, \tag{4.40}$$

and

$$(J^T J)^{-1} = \frac{1}{a^2} I_{4m+4} - \frac{1}{a^2(a^2 + 2m + 2)} V \tag{4.41}$$

such that

$$V = \begin{pmatrix} \psi & -\psi & O & O \\ -\psi & \psi & O & O \\ O & O & \psi & -\psi \\ O & O & -\psi & \psi \end{pmatrix}, \tag{4.42}$$

where $\psi_{ij} = 1$, for $i, j = 1, 2, \cdots, m+1$ [9].

Proof. For equation of degree 1, i.e. $a(\underline{x}, \overline{x}) + ia(\underline{y}, \overline{y}) = (\underline{b}, \overline{b})$, we have

$$\begin{cases} \sum\limits_{k=0}^{k=m} (\hat{a}\underline{x}_k + \check{a}\overline{x}_k - \underline{b}_k) r^k = 0 \\ \\ \sum\limits_{k=0}^{k=m} (\hat{a}\overline{x}_k + \check{a}\underline{x}_k - \overline{b}_k) r^k = 0 \\ \\ \quad \sum\limits_{k=0}^{k=m} (\hat{a}\underline{y}_k + \check{a}\overline{y}_k) r^k = 0 \\ \\ \quad \sum\limits_{k=0}^{k=m} (\hat{a}\overline{y}_k + \check{a}\underline{y}_k) r^k = 0 \end{cases} \tag{4.43}$$

where

$$\hat{a} = \begin{cases} a, & a \geq 0 \\ 0, & a < 0 \end{cases} \quad , \check{a} = \begin{cases} 0, & a \geq 0 \\ a, & a < 0. \end{cases} \tag{4.44}$$

Thus for $k = 0, \cdots, m+1$ we have

$$\begin{cases} L_k = \hat{a}\underline{x}_k + \check{a}\overline{x}_k - \underline{b}_k = 0 \\ U_k = \hat{a}\overline{x}_k + \check{a}\underline{x}_k - \overline{b}_k = 0 \\ M_k = \hat{a}\underline{y}_k + \check{a}\overline{y}_k = 0 \\ N_k = \hat{a}\overline{y}_k + \check{a}\underline{y}_k = 0 \\ \;Q_1 = \sum\limits_{k=0}^{k=m} (\underline{x}_k - \overline{x}_k) = 0 \\ \;Q_2 = \sum\limits_{k=0}^{k=m} (\underline{y}_k - \overline{y}_k) = 0 \end{cases} \tag{4.45}$$

thus Jacobian matrix has the following form

$$J = \begin{pmatrix} T_1 & T_2 & O & O \\ T_2 & T_1 & O & O \\ O & O & T_1 & T_2 \\ O & O & T_2 & T_1 \\ S^T & -S^T & O & O \\ O & O & S^T & -S^T \end{pmatrix} \tag{4.46}$$

where $T_1 = \hat{a}I_{m+1}, T_2 = \check{a}I_{m+1}$ and S^T is an $1 \times (m+1)$ matrix such that $S_i = 1$ for $i = 1, 2, \cdots, m+1$. This yields that

$$J^T J = \begin{pmatrix} a^2 I_{m+1} + \psi & -\psi & O & O \\ -\psi & a^2 I_{m+1} + \psi & O & O \\ O & O & a^2 I_{m+1} + \psi & -\psi \\ O & O & -\psi & a^2 I_{m+1} + \psi \end{pmatrix} = a^2 I_{4m+4} + V. \tag{4.47}$$

Let

$$B = \frac{1}{a^2} I_{4m+4} - \frac{V}{a^2(a^2 + 2m + 2)}. \tag{4.48}$$

we show that $BJ^T J = J^T JB = I$. Consequently

$$\begin{aligned} J^T JB &= (a^2 I_{4m+4} + V)(\frac{1}{a^2} I_{4m+4} - \frac{V}{a^2(a^2 + 2m + 2)}) \\ &= I_{4m+4} + \frac{1}{a^2(a^2 + 2m + 2)}((2m + 2)V - V^2) = I_{4m+4} \end{aligned} \tag{4.49}$$

because

$$V^2 = \begin{pmatrix} 2\psi^2 & -2\psi^2 & O & O \\ -2\psi^2 & 2\psi^2 & O & O \\ O & O & 2\psi^2 & -2\psi^2 \\ O & O & -2\psi^2 & 2\psi^2 \end{pmatrix} = 2(m + 1)V. \tag{4.50}$$

Similarly we have $BA^T A = I_{4m+4}$.
It can be proved for the matrix defined above we have

$$|J^T J| = a^{4(2m+1)}((m + 1)(4m + 4) + (4m + 4)a^2 + a^4). \tag{4.51}$$

□

Lemma 4.2. *For n=1 i.e.* $a\tilde{z} = \tilde{b}$, $a \neq 0$ *the algorithm is convergent to the exact solution of equation, at the first step, with every initial vector.*
proof: *According to Lemma 4.1,* $F(Y) = 0$ *is a linear system of equations, and by using Theorem 3.1, the proof is completed.*

Theorem 4.3. *If the equation is crisp then the sequence is given by the following recurrence equation:*

$$z_{k+1} = z_k - \frac{a_n z_k^n + \ldots + a_1 z_k - b}{n a_n z_k^{n-1} + \ldots + a_1} \tag{4.52}$$

Proof. We consider Eq. (4.1) at the following forms

$$F(x, y) = f_1(x, y) + i f_2(x, y) = 0. \tag{4.53}$$

and

$$P(z) - b = 0; \qquad z = x + iy. \tag{4.54}$$

By using Eq. (4.53), the system of $A^{(k)} H^k = B^{(k)}$, in iteration k is as the following form

$$\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = - \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{4.55}$$

the solution of the system (4.55) is

$$\begin{pmatrix} h_1 \\ \\ h_2 \end{pmatrix} = \frac{\begin{pmatrix} f_1.\frac{\partial f_2}{\partial y} - f_2.\frac{\partial f_1}{\partial y} \\ \\ -f_1.\frac{\partial f_2}{\partial x} + f_2.\frac{\partial f_1}{\partial x} \end{pmatrix}}{\frac{\partial f_1}{\partial x}.\frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y}.\frac{\partial f_2}{\partial x}}. \tag{4.56}$$

Therefore

$$z_{k+1} = z_k - \frac{f_1(\frac{\partial f_2}{\partial y} - i\frac{\partial f_2}{\partial x}) + f_2(-\frac{\partial f_1}{\partial y} + i\frac{\partial f_1}{\partial x})}{\frac{\partial f_1}{\partial x}.\frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y}.\frac{\partial f_2}{\partial x}}. \tag{4.57}$$

From (4.53), we have

$$\frac{\partial F}{\partial x} = \frac{\partial f_1}{\partial x} + i\frac{\partial f_2}{\partial x}, \tag{4.58}$$

$$\frac{\partial F}{\partial y} = \frac{\partial f_1}{\partial y} + i\frac{\partial f_2}{\partial y}, \tag{4.59}$$

From Eqs. (4.53) and (4.54) we have

$$F(x,y) = P(z) - b, \quad z = x + iy. \tag{4.60}$$

Now, from Eq. (4.60) we can write

$$\tfrac{\partial F}{\partial x} = P^{'}(z), \quad \tfrac{\partial F}{\partial y} = iP^{'}(z), \tag{4.61}$$

From (4.58), (4.59), and (4.61), we have

$$\frac{\partial f_2}{\partial x} + i\frac{\partial f_2}{\partial y} = -\frac{\partial f_1}{\partial y} + i\frac{\partial f_1}{\partial x} \tag{4.62}$$

by substituting (4.62) in (4.57), we have

$$z_{k+1} = z_k - \frac{(P(z) - b)(\frac{\partial f_2}{\partial y} - i\frac{\partial f_2}{\partial x})}{\frac{\partial f_1}{\partial x}.\frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y}.\frac{\partial f_2}{\partial x}}. \tag{4.63}$$

Now by multiplying $P^{'}(z) = \frac{\partial f_1}{\partial x} + i\frac{\partial f_2}{\partial x}$ by $\frac{\partial f_2}{\partial y}$ and by multiplying $iP^{'}(z) = \frac{\partial f_1}{\partial y} + i\frac{\partial f_2}{\partial y}$ by $\frac{\partial f_2}{\partial x}$, yield

$$\frac{\partial f_1}{\partial x}.\frac{\partial f_2}{\partial y} = \frac{\partial f_2}{\partial y}(P^{'}(z) - i\frac{\partial f_2}{\partial x}) \tag{4.64}$$

$$\frac{\partial f_2}{\partial x}.\frac{\partial f_1}{\partial y} = i\frac{\partial f_2}{\partial x}(P^{'}(z) - \frac{\partial f_2}{\partial y}). \tag{4.65}$$

By substituting (4.64) and (4.65) in (4.63), we have $z_{k+1} = z_k - \frac{P(z_k)}{P^{'}(z_k)}$. □

5. Numerical Examples

This section contains some numerical results about our investigations that have been solved by software Matlab. The numerical solutions of the following examples have not been provided up to now. They are not given by any specific references. A sequence of iterative solutions is given for Example 5.3. The process for all examples is the same.

Example 5.1. *Suppose we want to solve the equation*

$$5\tilde{z} = (-10 + 10r + 2r^2, 29 - 35r + 8r^2). \tag{5.1}$$

For this equation we consider $n = 1, m = 2$ and $l = 2$. Taking $\tilde{z} = (1 + r^2, 4 - r) + i(2, 2)$, after one iteration we have

$$\tilde{z}^* = (-2 + 2r + \frac{2}{5}r^2, \frac{29}{5} - 7r + \frac{8}{5}r^2). \tag{5.2}$$

Example 5.2. *Let we want to solve the equation $\tilde{z}^2 = -1$. For this equation we have $n = 2, l = 0$. We test our method for this equation with different values of m. Table 1 presents the results.*

TABLE 1. The solved results of the equation in Example 5.2.

m	$\tilde{z}^0$	*Number of iterations*	$\tilde{z}^*$
0	$(3,3) + i(-2,-2)$, $(4,4) + i(3,3)$	6 , 6	$i(-1,-1) = -i$, $i(1,1) = i$
1	$(4+r, 6-r) + i(3,3)$, $(1+r, 3-r) + i(-2,-2)$	7 , 5	$i(-1,-1) = -i$, $i(1,1) = i$
2	$(-4+r^2, -1-2r) + i(-4,-4)$, $(1,1) + i(1+r^2, 3-r)$	5 , 6	$i(-1,-1) = -i$, $i(1,1) = i$

Example 5.3. *We want to solve*

$$\tilde{z}^2 = (-4 + 4r - r^2, -r^2). \tag{5.3}$$

For this equation we have $n = 2, l = 2$.
We examine the method for this equation with values of $m = 1$ and $m = 2$. More details are given in Tables 2 and 3.

TABLE 2. The iterative solutions of the equation in Example 5.3 for $m = 1$.

Initial guess	$(2+r, 4-r) + i(1+r, 3-r)$
Iterative solutions	$(0.2308 + 1.0327r, 2.0327 - 0.2728r) + i$ $(0.8927 + 0.4450r, 1.5466 - 0.8054r)$
	$(0.5273 + 0.4499r, 0.4814 + 0.0369) + i$ $(-0.1012 + 0.7001r, 1.5506 - 0.7971r)$
	$(-0.0697 + 0.1009r, -0.0971 + 0.0361r) + i$ $(-0.0417 + 0.8638r, 1.9719 - 1.1541r)$
	$(0.0084 - 0.0136r, -0.0002 + 0.0061r) + i$ $(0.0061 + 1.0063r, 1.9981 - 0.9966r)$
	$(-0.0000 - 0.0001r, -0.0000 - 0.0000r) + i$ $(0.0000 + 1.0000r, 2.0000 - 0.9999r)$
	$(-0.0000 + 0.0000r, -0.0000 - 0.0000r) + i$ $(-0.0000 + 1.0000r, 2.0000 - 1.0000r)$
Number of iterations	7
Final solution	$i(r, 2-r)$

TABLE 3. The iterative solutions of the equation in Example 5.3 for $m = 2$.

Initial guess of solution	$(1+r^2, 4-2r) + i(2+r, 4-r^2)$
Iterative solutions	$(0.0622 + 0.2809r + 0.3379r^2, 1.9903 - 0.7550r - 0.0463r^2)+$ $i(1.2007 + 0.7334r - 0.1418r^2, 2.1673 - 0.2783r - 0.5398r^2)$
	$(0.0430 + 0.2506r + 0.0133r^2, 0.5355 + 0.0999r - 0.1264r^2)+$ $i(0.0293 + 1.5130r - 0.5707r^2, 1.7655 - 0.6945r - 0.1632r^2)$
	$(0.0172 + 0.0644r - 0.0298r^2, -0.0414 + 0.0802r + 0.0148r^2)+$ $i(-0.0036 + 1.1290r - 0.1573r^2, 2.0051 - 1.0394r + 0.0266r^2)$
	$(0.0018 + 0.0055r - 0.0103r^2, -0.0003 + 0.0007r - 0.0018r^2)+$ $i(0.0036 + 1.0071r - 0.0184r^2, 1.9995 - 1.0012r - 0.0010r^2)$
	$(0.0001 - 0.0000r - 0.0002r^2, -0.0000 - 0.0000r - 0.0000r^2)+$ $i(0.0000 + 1.0000r - 0.0001r^2, 2.0000 - 1.0000r - 0.0000r^2)$
	$(-0.0000 - 0.0000r + 0.0000r^2, -0.0000 - 0.0000r - 0.0000r^2)+$ $i(-0.0000 + 1.0000r + 0.0000r^2, 2.0000 - 1.0000r + 0.0000r^2)$
Number of iterations	7
Final solution	$i(r, 2-r)$

Example 5.4. *Let we want to solve the equation*

$$\tilde{z}^2 = (r^2 - 1, r^2 - 4r + 3) + 2i(r, 2-r). \tag{5.4}$$

For this equation we have $n = 2$ and $l = 2$. We solve this equation with values of $m = 1, 3$. Table 4 shows the results.

TABLE 4. The solved results of the equation in Example 5.4.

m	$\tilde{z}^0$	*Number of iterations*	$\tilde{z}^*$
1	$(2+r, 4-r) + i(1+r, 3-r)$	6	$(r, 2-r) + i(1,1)$
3	$(r^2+r^3, 3-r) + i(r^2+r^3, 3-r)$	6	$(r, 2-r) + i(1,1)$

Example 5.5. *We consider the equation*

$$\tilde{z}^2 - \tilde{z} = (7r - 6, -r^4 + r^2 - 7r + 8) + i(2r^3 + 2r^2 + r - 2, r^2 - 10r + 12) \tag{5.5}$$

for $n = 2$ and $l = 4$. We solve this equation for $m = 2,\ 3$. The results of solving the equation are given in table 5.

TABLE 5. The solved results of the equation in Example 5.5.

m	$\tilde{z}^0$	*Number of iterations*	$\tilde{z}^*$
2	$(1,1) + i(0,0)$	5	$(1+r, 3-r) + i(r^2, 2-r)$
3	$(1,1) + i(0,0)$	7	$(1+r, 3-r) + i(r^2, 2-r)$

Example 5.6. *We solve the equation*

$$\tilde{z}^3 = ((r+1)^3, (4-2r)^3); \quad n = 3, \ \ l = 3. \tag{5.6}$$

We examine the method for this equation with two values of m. Table 6 presents the results.

TABLE 6. The solved results of the equation in Example 5.6.

m	$\tilde{z}^0$	*Number of iterations*	$\tilde{z}^*$
1	$(1+r, 3-r) + i(3r, 5-2r)$	22	$(1+r, 4-2r)$
2	$(1+r+r^2, 4-r^2) + i(3r, 5-2r)$	18	$(1+r, 4-2r)$

Example 5.7. *Suppose that we want to solve the following equation for $n = 3$ and $l = 3$.*

$$\tilde{z}^3 + \tilde{z}^2 - \tilde{z} = (-r^3 + 3r^2 + 4r - 9, -r^3 + 7r^2 - 18r + 9) +$$

$$i(3r^2 + 2r - 2, 3r^2 - 14r + 6). \tag{5.7}$$

Table 7 presents the results of solving the equation by the method with the different values of m.

TABLE 7. The solved results of the equation in Example 5.7.

m	$\tilde{z}^0$	*Number of iterations*	$\tilde{z}^*$
1	$(1,1) + i(2,2)$	6	$(r, 2-r) + i(1,1)$
2	$(4+r, 6-r) + i(3,3)$	8	$(r, 2-r) + i(1,1)$

Example 5.8. *Suppose that we want to solve the following equation*

$$\tilde{z}^2 + \tilde{z} = (e^{2r} + e^r - 1,\ \ e^{4-2r} + e^{2-r} - 1) + i(-2e^{2-r} - 1,\ \ -2e^r - 1), \tag{5.8}$$

by taking $\tilde{x}$ as nonnegative fuzzy number and $\tilde{y}$ as non-positive fuzzy number. Considering the nearest 18-degree polynomial form for the real and imaginary parts of the right hand side [1], $\tilde{z}^0 = (r, 2-r)$ and $m = 9$, after 7 iterations we have $\tilde{z}^* = (\underline{x}, \overline{x}) + i(\underline{y}, \overline{y})$ where

$$\underline{x} = 1.0000 + 1.0000r + 0.5000r^2 + 0.1667r^3 + 0.0417r^4 + 0.0083r^5 + 0.0014r^6 + 0.0002r^7 + 0.0000r^8 + 0.0000r^9, \tag{5.9}$$

$$\overline{x} = 7.3891 - 7.3891r + 3.6945r^2 - 1.2315r^3 + 0.3079r^4 - 0.0616r^5 + 0.0103r^6 - 0.0015r^7 + 0.0002r^8 - 0.0000r^9, \tag{5.10}$$

$$\underline{y} = -1.0000, \qquad \overline{y} = -1.0000 \tag{5.11}$$

For this solution the infinity norm of the error functions for left and right sides of real part are 2.135×10^{-5}, 4.366×10^{-5}. The infinity norm of both the left and right sides of the imaginary part is 0 and 0, respectively.

6. CONCLUSION

Up to now, there is not any analytically and numerically method for solving the algebraic fuzzy complex equation of degree n. In this article, the research has been done related to gain the roots of a particular algebraic fuzzy complex equation of degree n, by a numerical method. We plan to consider the solutions of another algebraic fuzzy complex equation, shortly.

ACKNOWLEDGMENT

The authors thank anonymous reviewers for their useful comments.

References

[1] S. Abbasband and M. Amirfakhrian, *The nearest approximation of a fuzzy quantity in parametric form*, App. Math. Comput., *172* (2006), 624–632.

[2] S. Abbasband and M. Amirfakhrian, *Numerical approximation of fuzzy functions by fuzzy polynomials*, App. Math. Comput., *174* (2006), 669–675.

[3] S. Abbasbandy and B. Asady, *Newton's method for solving fuzzy nonlinear equations*, App. Math. Comput., *159* (2004), 349–356.

[4] S. Abbasbandy and M. Otadi, *Numerical solution of fuzzy polynomials by fuzzy neural network*, App. Math. Comput., *181* (2006), 1084–1089.

[5] T. Allahviranloo, M. Otadi, and M. Moslem, *Iterative method for fuzzy equations*, Soft Comput., *12* (2007) 935–939.

[6] T. Allahviranloo and S. Asari, *Numerical solution of fuzzy polynomials by Newton-Rephson method*, J. App. Math., *7* (2011), 17–24.

[7] T. Allahviranloo and L. G. Moazam, *The solution of fuzzy quadratic equation based on optimization theory*, Sci. World J., *2014* (2014) 1–6.

[8] M. Amirfahkrian, *Numerical solution of algebraic fuzzy equations with crisp variable by Gauss- Newton method*, Appl. Math. Model., *32* (2008), 1859–1868.

[9] M. Amirfahkrian, *An iterative Gauss- Newton method to solve an algebraic fuzzy equation with crisp coefficients*, J. Intell Fuzzy Syst., *22* (2011) 207– 216.

[10] M. Amirfahkrian, *Some approximation methods in fuzzy logic*, Lambert Academic Publishing, (2012).

[11] J. J. Buckley, *The fuzzy mathematics of finance*, Fuzzy Sets Syst., *21* (1987), 257–273.

[12] J. J. Buckley, *Solving fuzzy equations in economics and finance*, Fuzzy Sets Syst., *48* (1992), 289–296.

[13] J. J. Buckley and Y. Qu, *Solving systems of linear fuzzy equations*, Fuzzy Sets Syst., *43* (1991), 33–43.

[14] J. J. Buckley and Y. Qu, *On using α-cuts to evaluate fuzzy equations*, Fuzzy Sets Syst., *38* (1990) 309–312.

[15] J. J. Buckley and Y. Qu, *Solving fuzzy equations: A new Solution Concept*, Fuzzy Sets Syst., *39* (1991) 291–301.

[16] J. J. Buckley and Y. Qu, *Solving linear and quadratic fuzzy equations*, Fuzzy Sets Syst., *38* (1990), 43–59.

[17] J. Chen and W. Li, *Local convergence results of Gauss-Newton's like method in weak conditions*, J. Math. Anal. Appl., *2* (2006), 1381–1394.

[18] J. Chen and W. Li, *Convergence of Gauss-Newton's method and uniqueness of the solution*, Appl. Math. Comput., *1* (2005), 686–705.

[19] S. H. Chen and X. W. Yang, *Interval finite element method for beam structhures*, Finite Elem. Anal. Des., *34* (2000), 75–88.

[20] P. Deuflhard, *Newton methods for nonlinear problems*, Springer Seris in Computational Mathematics, Springer-Verlag Berlin Heidelberg, Berline, 2004. .

[21] P. Deuflhard and A. Hohmann, *Numerical analysis in modern scientific computing*, Texts in Applied Mathematics, 2nd edition, Springer-Verlag New york, New York, 2003. .

[22] X. Fu and Q. Shen, *Fuzzy complex numbers and their application for classifires performance evaluation*, Pattern Recogni., *7* (2011), 1403–1417.

[23] M. Friedman, M. Ming, and A. Kndel, *Fuzzy linear systems*, Fuzzy Set. Syst., *2* (1998), 201–209.

[24] R. Goestschel and W. Voxman, *Elementary fuzzy calculus*, Fuzzy Set. Syst., *1* (1989), 31–43.

[25] G. J. Klir and B. Yuan, *Fuzzy sets and fuzzy logic theory and applications*, Prentice Hall of India Pvt. Ltd., Delhi, 2002.

[26] C. Li, W. H. Zhang, and X. Q. Jin, *Convergence and uniqueness properties of Gauss-Newton's method*, Comput. Math. Appl., *47* (2004), 1057–1067.

[27] M. Li Calzi, *Towards a general setting for the fuzzy mathematics of finance*, Fuzzy Set. Syst, *35* (1990), 265–280..

[28] Q. X. Li and S. F. Liu, *The foundation of the grey matrix and the grey input-output analysis*, Appl. Math. Model., *32* (2008), 267– 291.

[29] F. A. Mazarbhuiya, A. K. Mahanta, and H. K. Baruah, *Solution of fuzzy equation $A + X = B$ using the method of superimposition*, Appl. Math., *2* (2011), 1039–1045.

[30] M. Otadi and M. Mosleh, *Solution of fuzzy polynomial equations by modified Adomian Decomposition method*, Soft Comput., *15* (2011), 187–192. .

[31] L. G. Moazam, *The solution of fuzzy quadratic equations based on restricted varation*, Int. J. Ind. Math., *8* (2016), 395–400.

[32] M. Ma, M. Friedman, and A. Kandel, *A new fuzzy arithmetic*, fuzzy set. Syst., *108* (1999), 83–90.

[33] H. Rouhparvar,*Solving fuzzyp polynomial equation by ranking method*, First Joint Congress On Fuzzy and Intelligent Systems Ferdowsi University of Mashhad, (2007), 1–5.

[34] L. Stefanini, L. Sorini, and M. L. Guerra, *Parametric representation of fuzzy numbers and applications to calculus*, Fuzzy set. syst., *157* (2006), 2423–2455.

[35] C. C. Wu and N. B. Chang, *Grey input-output analysis and its application for environmental cost allocation*, Eur. J. of Oper. Res., *145* (2003), 175–201.