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Fractional study on heat and mass transfer of MHD Oldroyd-B fluid with ramped velocity and temperature

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Abstract

This study explores the time-dependent flow of MHD Oldroyd-B fluid under the effect of ramped wall velocity and temperature. The flow is confined to an infinite vertical plate embedded in a permeable surface with the impact of heat generation and thermal radiation. Solutions of velocity, temperature, and concentration are derived symmetrically by applying non-dimensional parameters along with Laplace transformation (LT) and numerical inversion algorithm. Graphical results for different physical constraints are produced for the velocity, temperature, and concentration profiles. Velocity and temperature profile decrease by increasing the effective Prandtl number. The existence of an effective Prandtl number may reflect the control of the thickness of momentum and enlargement of thermal conductivity. Velocity is decreasing for κ, M, Pr_{reff}, and S_c while increasing for G_r and G_c. Temperature is an increasing function of the fractional parameter. Additionally, Atangana-Baleanu (ABC) model is good to explain the dynamics of fluid with better memory effect as compared to other fractional operators.

Keywords. Oldroyd-B fluid, Fractional differential operator, Ramped velocity and temperature.

2010 Mathematics Subject Classification. 65L05, 34K06, 34K28.

1. Introduction

The theory of Newtonian and non-Newtonian fluid describes the mechanical behavior of different real fluids. The motion of such real fluids has left a significant impact on the field of science, environmental engineering and industry. The attributes of fluid flow trace the diversity of physical structure for non-Newtonian fluid flow. In such fluid, stress and rate of strain have a non-linear relationship. Oldroyd-B fluids have become a significant model of rate type fluid. This model is the special extension of upper viscoelastic Maxwell fluid with retardation time. The procedure for the flow of rate type fluids was discussed by Oldroyd [24]. It describes the relaxation and retardation phenomena of viscoelastic fluid [44]. In present days, the significance of such fluids with mixed convection flows under influence of magnetohydrodynamic (MHD) force and thermal radiation have different applications in the power field, solar collection, polymer fabrication, aerodynamic heating and chemical industry [9, 23, 38, 45].

In the literature, there is an insufficient study which deals with flows under ramped wall temperature and ramped wall velocity conditions . Physically, the implementation of ramped wall velocity and temperature in real life problems has a significant role but mathematically it is difficult to handle such conditions. The diagnoses of prognosis, establishing treatments, analysis of heart functions and blood vessels system [6, 17, 22, 39] are major applications of ramp velocity. Firstly, authors [2] discussed the simultaneous use of ramped velocity and temperature. Seth et al. [35–37] investigated heat and mass transfer phenomena with ramp temperature conditions. Recently, Tiwana et al. [41] analyzed MHD Oldroyd-B fluid under threffect of ramped temperature and velocity.

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Table 1. Nomenclature

Symbol	Quantity
κ, γ	Fractional parameters
υ	Kinematic viscosity
g	Acceleration due to gravity
β_T	Thermal expansion coefficient
β_C	Volumetric expansion coefficient
ρ	Fluid density
σ	Electrical conductivity
C_p	Specific heat
D_m	Chemical molecular diffusivity
p	Laplace parameter
Q	Heat generation/absorption
R	Dimensionless chemical reaction
$\tilde{u}^*$	Dimensional velocity
V	Dimensionless velocity
$\tilde{T}^*$	Dimensional temperature
T	Dimensionless temperature
$\tilde{C}^*$	Dimensional fluid concentration
C	Dimensionless fluid concentration
K	Porosity parameter
G_c	Mass Grashof number
G_r	Thermal Grashof number
T_w	Temperature of the plate
T_∞	Temperature of fluid far away from the plate
T_∞^*	Temperature of fluid far away from the plate
C_w	Concentration on the plate
C_∞	Concentration of the fluid far away from the plate
λ	Relaxation time
λ_r	Retardation time
P_{reff}	Effective Prandtl number
S_c	Schmidt number
B_0	Magnetic field
M	Hartmann number
k	Thermal conductivity
q_r	Radiative heat flux
σ	Electrical conductivity of the fluid
σ_1	Stafan–Boltzman constant
K_1	Absorption coefficient

In 1967, Caputo was the first mathematician to develop the fractional operator by using Laplace convolutions production of power-law functions and fractional derivatives. This was the first fractional operator to fix the problem of the Riemann–Liouville fractional operator. However, the kernel of these fractional operators is singular at $t = \tau$, which leads to some erroneous results. Fractional calculus was further developed in 2015, Caputo and Fabrizio presented the Caputo-Fabrizio fractional operator with a nonsingular exponential kernel [7]. However, CF-operator has also been criticized because the solution of CF-operator is an exponential equation, not an exponential function; the kernel of CF-operator is nonsingular but local. To overcome these issues, a new approach of the fractional kernel has been discussed in fractional differential operators due to their significant ability in the application of biological sciences.

Atangana and Baleanu came up with a new approach by introducing the Atanagana-Baleanu fractional. Mittag Leffler nonsingular kernel is considered as a new fractional operator which provides bounded solution and stabilizing point [4]. As compared to the classical model, the memory effect is much stronger in fractional derivatives. Cancer treatment and blood flow through veins via MHD and ultraslow diffusion are suitable applications of these definitions in research [14, 16]. The convergence and its stability are controlled by a numerical approach. Recently, Riaz et al. [32] investigate the role of local and nonlocal kernels on MHD Oldroyd-B fluid flow with slip conditions. Convective flow with ramped wall temperature for non-singular kernel analyzed by Riaz et al. [27]. Furthermore, authors [29] discussed the comprehensive report on MHD Oldroyd-B fluid with slip effect and time boundary conditions with integer order, CF and ABC fractional derivative. In literature, many researcher worked on different problems regarding application of fractional order derivatives [1, 5, 8, 10, 15, 18–20, 25, 26, 28, 30, 31, 33, 34, 43].

Talha et al. [3] investigate the solution of MHD Oldroyd-B fluid under the impact of heat consumption/generation with ramped wall temperature and velocity. The main objective of this paper is to investigate MHD Oldroyd-B fluid with definition of non-integer order derivatives. The solution of fluid velocity, temperature and concentration are obtained by Caputo (C), Caputo-Fabrizio (CF) and Atangana-Baleanu (ABC) fractional derivative models under influence of ramped velocity and temperature. These non-integer order derivatives are good for handling mathematical calculations. This article is well organized. Section 2 helps to drive the governing partial differential equations. The solution of concentration, temperature and velocity gradient can be achieved through C, CF and ABC fractional models with help of Laplace transformation and inversion algorithm in sections 3, 4, and 5 respectively. In section 6, the influence of physical parameters discusses through graphically by MATHCAD-15 software. Finally, the conclusion of the present article is given at the end.

2. Modeling of the Problem

We discuss unsteady magnetohydrodynamic (MHD) fractional Oldroyd-B fluid flow under Boussinesq approximations over an infinite plate. The presence of the concentration effect with chemical reaction is taken into consideration. Figure (1) represents the flow geometry of magnetized Oldroyd-B fluid. Under these presumptions, the governing equation for Oldroyd-B fluid with appropriate conditions are defined below [3]:

$$\begin{aligned}\left(1+\tilde{\lambda}_1\frac{\partial}{\partial\tilde{t}^*}\right)\frac{\partial\tilde{u}^*(\tilde{y}^*,\tilde{t}^*)}{\partial\tilde{t}^*} &= \upsilon\left(1+\tilde{\lambda}_2\frac{\partial}{\partial\tilde{t}^*}\right)\frac{\partial^2\tilde{u}^*(\tilde{y}^*,\tilde{t}^*)}{\partial\tilde{y}^{2*}} - \left(1+\tilde{\lambda}_1\frac{\partial}{\partial\tilde{t}^*}\right)\frac{\sigma B_0^2}{\rho}\tilde{u}^*(\tilde{y}^*,\tilde{t}^*)\\ &\quad - \left(1+\tilde{\lambda}_2\frac{\partial}{\partial\tilde{t}^*}\right)\frac{\mu\phi}{\rho k^*}\tilde{u}^*(\tilde{y}^*,\tilde{t}^*) + g\beta_T\left(1+\tilde{\lambda}_1\frac{\partial}{\partial\tilde{t}^*}\right)(\tilde{T}^*-\tilde{T}^*_\infty)\\ &\quad + g\beta_C\left(1+\tilde{\lambda}_1\frac{\partial}{\partial\tilde{t}^*}\right)(\tilde{C}^*-\tilde{C}^*_\infty), \end{aligned} \tag{2.1}$$

$$\rho C_p\left(\frac{\partial\tilde{T}^*(\tilde{y}^*,\tilde{t}^*)}{\partial\tilde{t}^*}\right) = k\frac{\partial^2\tilde{T}^*(\tilde{y}^*,\tilde{t}^*)}{\partial\tilde{y}^{2*}} - \frac{\partial q_r}{\partial\tilde{y}} - Q_o(\tilde{T}^*-\tilde{T}^*_\infty), \tag{2.2}$$

$$\frac{\partial\tilde{C}^*(\tilde{y}^*,\tilde{t}^*)}{\partial\tilde{t}^*} = D_m\frac{\partial^2\tilde{C}^*(\tilde{y}^*,\tilde{t}^*)}{\partial\tilde{y}^{2*}} - R^*\ (\tilde{C}^*-\tilde{C}^*_\infty). \tag{2.3}$$

The appropriate conditions are given below:

$$\tilde{u}^*\left(\tilde{y}^*,0\right)=0,\quad \tilde{T}^*\left(\tilde{y}^*,0\right)=\tilde{T}^*_\infty,\quad \tilde{C}^*\left(\tilde{y}^*,0\right)=\tilde{C}^*_\infty,\quad \frac{\partial\tilde{u}^*}{\partial\tilde{t}}=\frac{\partial\tilde{u}^*}{\partial\tilde{y}}=0, \tag{2.4}$$

$$\tilde{T}^*(0,\tilde{t}^*)=\begin{cases}\tilde{T}^*_\infty+(\tilde{T}^*_w\frac{\tilde{t}^*}{t_0}-\tilde{T}^*_\infty\frac{\tilde{t}^*}{t_0}), & 0<\tilde{t}^*\le t_0;\\ \tilde{T}^*_w, & \tilde{t}^*>t_0\end{cases}, \tag{2.5}$$

$$\tilde{u}^*(0,\tilde{t}^*)=\begin{cases}U_0\frac{\tilde{t}^*}{t_0}, & 0<\tilde{t}^*\le t_0;\\ U_0, & \tilde{t}^*>t_0\end{cases},\qquad \tilde{C}^*(0,\tilde{t}^*)=\tilde{C}_w, \tag{2.6}$$

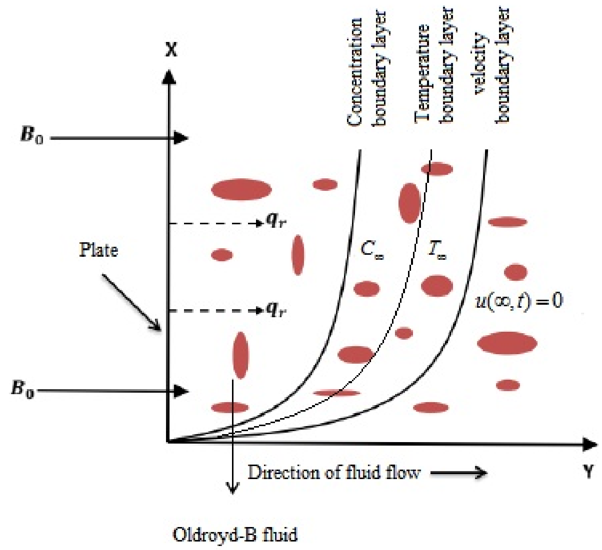

FIGURE 1. Geometrical presentation for Oldroyd-B Model

$$\tilde{u}^*\left(\tilde{y}^*,\tilde{t}^*\right)\to 0,\ \tilde{T}^*\left(\tilde{y}^*,\tilde{t}^*\right)\to \tilde{T}^*_\infty,\ \tilde{C}^*\left(\tilde{y}^*,\tilde{t}^*\right)\to \tilde{C}^*_\infty\ as\ \tilde{y}^*\to\infty. \tag{2.7}$$

We introduce the dimensionless function and parameters in eqs. (2.1) to (2.7) as mentioned below:

$$\tilde{u}^*=\frac{V}{U_0},\quad \tilde{y}^*=\frac{\zeta U_0}{\upsilon},\quad \tilde{t}^*=\frac{\tau U_0^2}{\upsilon},\quad t_0=\frac{\upsilon}{U_0^2},\quad \tilde{T}^*=\frac{T-\tilde{T}^*_\infty}{\tilde{T}^*_w-\tilde{T}^*_\infty},\quad P_r=\frac{\mu C_p}{k},$$
$$\tilde{\lambda}_1^*=\frac{\lambda U_0^2}{\upsilon},\quad \tilde{\lambda}_2=\frac{\lambda_r U_0^2}{\upsilon},\quad G_r=\frac{\upsilon g\beta_T(\tilde{T}^*_w-\tilde{T}^*_\infty)}{U_0^3},\quad G_c=\frac{\upsilon g\beta_C(\tilde{C}^*_w-\tilde{C}^*_\infty)}{U_0^3},$$
$$N_r=\frac{16\sigma_1 T^3_\infty}{3k^*K_1},\quad S_c=\frac{\upsilon}{D_m},\quad Q=\frac{\upsilon^2 Q_o}{\rho C_p U_0^2},\quad R^*=Rt_0,\quad \frac{1}{K}=\frac{\phi\upsilon^2}{k^*U_0^2},\ M=\frac{\sigma B_0^2\upsilon}{\rho U_0^2}. \tag{2.8}$$

Applying (2.8) into eqs. (2.1) - (2.7), required set of dimensionless governing equations in form of PDE's system have been present as:

$$\left(a_1+\lambda\frac{\partial}{\partial\tau}\right)\frac{\partial V(\zeta,\tau)}{\partial\tau}=\left(1+\lambda_r\frac{\partial}{\partial\tau}\right)\frac{\partial^2 V(\zeta,\tau)}{\partial\zeta^2}+G_r\left(1+\lambda\frac{\partial}{\partial\tau}\right)T(\zeta,\tau)+G_c\left(1+\lambda\frac{\partial}{\partial\tau}\right)C(\zeta,\tau)-a_2V(\zeta,\tau), \tag{2.9}$$

$$\frac{\partial T(\zeta,\tau)}{\partial\tau}=\frac{1}{P_{reff}}\left(\frac{\partial^2 T(\zeta,\tau)}{\partial\zeta^2}\right)-Q\ T(\zeta,\tau), \tag{2.10}$$

$$\frac{\partial C(\zeta,\tau)}{\partial\tau}=\frac{1}{S_c}\left(\frac{\partial^2 C(\zeta,\tau)}{\partial\zeta^2}\right)-R\ C(\zeta,\tau), \tag{2.11}$$

with corresponding conditions

$$V(\zeta,0)= T(\zeta,0)= C(\zeta,0)=0, \quad u_t(y,0)=u_y(y,0)=0, \; for \; \zeta\geq 0, \tag{2.12}$$

$$T(0,\tau)=V(0,\tau)=\begin{cases}\tau & 0<\tau\leq\tau_0;\\ 1, & \tau>\tau_0,\end{cases}, \quad C(\zeta,\tau)=1, \tag{2.13}$$

$$V(\zeta,\tau)\to 0, \qquad T(\zeta,\tau)\to 0, \qquad C(\zeta,\tau)\to 0, \quad as \qquad \zeta\to\infty. \tag{2.14}$$

2.1. **Development of Governing Equations in terms of Singular & Non-Singular Kernel.** Before developing the governing equations in terms of fractional differential operator, we define Caputo time derivative with its Laplace transform defined below [12].

$${}^{C}D_t^{\zeta}f(y,t)=\frac{1}{\Gamma(n-\zeta)}\int_a^t\left(\frac{f^{(n)}(\tau)}{(t-\tau)^{\zeta+1-n}}\right)d\tau, \tag{2.15}$$

$$\mathcal{L}\left({}^{C}D_t^{\zeta}f(y,t)\right)=s^{\zeta}\mathcal{L}\left(f(y,t)\right)-s^{\zeta-1}f(y,0). \tag{2.16}$$

The symbol of gamma function is $\Gamma(.)$.
The (CF) fractional derivative and its (LT) are defined as:

$${}^{CF}D_t^{\zeta}f(y,t)=\frac{1}{1-\zeta}\int_0^t exp\left(-\frac{\zeta(t-\tau)}{1-\zeta}\right)\frac{\partial f(y,\tau)}{\partial\tau}d\tau, \; 0<\zeta<1, \tag{2.17}$$

$$\mathcal{L}\left({}^{CF}D_t^{\zeta}f(y,t)\right)=\frac{s\mathcal{L}\left(f(y,t)\right)-f(y,0)}{(1-\zeta)s+\zeta}. \tag{2.18}$$

The (ABC) fractional derivative with its (LT) are defined as:

$${}^{ABC}D_t^{\zeta}f(y,t)=\frac{1}{1-\zeta}\int_0^t \mathbf{E}_{\zeta}\left(-\frac{\zeta(t-\tau)^{\tau}}{1-\zeta}\right)\frac{\partial f(y,\tau)}{\partial\tau}d\tau, \tag{2.19}$$

$$\mathcal{L}\left({}^{ABC}D_t^{\zeta}f(y,t)\right)=\frac{s^{\zeta}\mathcal{L}\left(f(y,t)\right)-s^{\zeta-1}f(y,0)}{(1-\zeta)s^{\zeta}+\zeta}. \tag{2.20}$$

3. Optimal Concentration Field via Fractional Operators

3.1. **Concentration Field via Integer order Approach.** By applying Laplace transformation to Eq. (2.11) with the help of initial condition on concentration results are given by Iftikhar et al. [11]

$$Sc\left(q+R\right)\overline{C}\left(\zeta,p\right)=\frac{\partial^2\overline{C}\left(\zeta,p\right)}{\partial\zeta^2}, \tag{3.1}$$

and solution is given as

$$\overline{C}\left(\zeta,p\right)=\frac{1}{p}e^{-\zeta\sqrt{Sc}\sqrt{p+R}}. \tag{3.2}$$

3.2. **Concentration Field Via Caputo Approach.** Solving Eq. (2.11) using C definition (2.15) and (2.16), we have results present in Riaz et al. [27].

$$\frac{\partial^2 \bar{C}_c(\zeta,p)}{\partial\zeta^2} - S_c\left(P^\kappa + R\right)\bar{C}_c(\zeta,p) = 0, \tag{3.3}$$

the solution of above second order differential equation is given by

$$\bar{C}_c(\zeta,p) = c_1 e^{\zeta\sqrt{S_c(p^\kappa+R)}} + c_2 e^{-\zeta\sqrt{S_c(p^\kappa+R)}}, \tag{3.4}$$

in order to find the values of constants c_1 and c_2, we use corresponding boundary conditions for concentration and after some simplifications, we have

$$\bar{C}_c(\zeta,p) = \frac{1}{p} e^{-\zeta\sqrt{S_c(p^\kappa+R)}}. \tag{3.5}$$

3.3. **Concentration Field Via Caputo-Fabrizio Approach.** Concentration field with Caputo-Fabrizio time-fractional is given as follow after using (2.17) and (2.18) with (2.11) Riaz et al. [27].

$$\frac{\partial^2 \bar{C}_{cf}(\zeta,p)}{\partial\zeta^2} - S_c\left[\frac{p}{(1-\kappa)p+\kappa} + R\right]\bar{C}_{cf}(\zeta,p) = 0, \tag{3.6}$$

$$\bar{C}_{cf}(\zeta,p) = c_1 e^{\zeta\sqrt{S_c\left[\frac{p}{(1-\kappa)p+\kappa}+R\right]}} + c_2 e^{-\zeta\sqrt{S_c\left[\frac{p}{(1-\kappa)p+\kappa}+R\right]}}, \tag{3.7}$$

and

$$\bar{C}_{cf}(\zeta,p) = \frac{1}{p} e^{-\zeta\sqrt{S_c\left(\frac{p}{(1-\kappa)p+\kappa}+R\right)}}, \tag{3.8}$$

where $c_1 = 0$ and $c_2 = \frac{1}{p}$.

3.4. **Concentration Field Via Atangana-Baleanu Approach.** Concentration field with Atangana-Baleanu time-fractional derivative is given by applying (2.19) and (2.20) on (2.11), we obtained results present in Riaz et al. [27].

$$\frac{\partial^2 \bar{C}_{abc}(\zeta,p)}{\partial\zeta^2} - S_c\left(\frac{p^\kappa}{(1-\kappa)p^\kappa+\kappa} + R\right)\bar{C}_{abc}(\zeta,p) = 0, \tag{3.9}$$

$$\bar{C}_{abc}(\zeta,p) = c_1 e^{\zeta\sqrt{S_c\left(\frac{p^\kappa}{(1-\kappa)p^\kappa+\kappa}+R\right)}} + c_2 e^{-\zeta\sqrt{S_c\left(\frac{p^\kappa}{(1-\kappa)p^\kappa+\kappa}+R\right)}}, \tag{3.10}$$

by using boundary conditions on concentration, we have following simplified form

$$\bar{C}_{abc}(\zeta,p) = \frac{1}{p} e^{-\zeta\sqrt{S_c\left(\frac{p^\kappa}{(1-\alpha)p^\kappa+\kappa}+R\right)}}. \tag{3.11}$$

4. Optimal Temperature Field via Fractional Operators

4.1. **Temperature Field via Caputo Approach.** We utilize integral transformation to acquire the solutions of temperature given by Eq. (2.10) using (2.13). We exchange the partial derivative with fractional derivative of order κ, and Eq. (2.10) can be written as:

$${}^{C}D_\tau^\kappa T\left(\zeta,\tau\right) = \frac{1}{P_{reff}}\frac{\partial^2 T(\zeta,\tau)}{\partial\zeta^2} - QT(\zeta,\tau). \tag{4.1}$$

We prefer to apply Laplace transform given in (2.16) on Eq. (4.1). The resultant form of above expression is,

$$\frac{\partial^2 \bar{T}(\zeta,p)}{\partial\zeta^2} = P_{reff}\left(p^\kappa + Q\right)\bar{T}\left(\zeta,p\right). \tag{4.2}$$

The solution of homogenous part of second order partial differential equation say (4.2) is,

$$\bar{T}(\zeta,p) = c_1 e^{-\zeta\sqrt{P_{reff}(p^\kappa+Q)}} + c_2 e^{\zeta\sqrt{P_{reff}(p^\kappa+Q)}}. \tag{4.3}$$

With the help of (2.12)-(2.14), we find out the values of constants c_1 and c_2, we have

$$\bar{T}(\zeta,p)=\left(\frac{1-e^{-p}}{p^2}\right)\times e^{-\zeta\sqrt{P_{reff}(p^{\kappa}+Q)}}. \tag{4.4}$$

4.2. **Temperature Field via Caputo-Fabrizio Approach.** We utilize integral transformation to acquire the solutions of temperature given by Eq. (2.10) using (2.13). We prefer to apply Laplace transform given in (2.18) on Eq. (2.10). The resultant form of above expression is,

$$\frac{\partial^2\bar{T}(\zeta,p)}{\partial\zeta^2}=P_{reff}\left(\frac{b_1p+b_2}{b_3p+b_4}\right)\bar{T}(\zeta,p). \tag{4.5}$$

The solution of homogenous part of second order partial differential equation say (4.5) is,

$$\bar{T}(\zeta,p)=c_1e^{-\zeta\sqrt{P_{reff}\left(\frac{b_1p+b_2}{b_3p+b_4}\right)}}+c_2e^{\zeta\sqrt{P_{reff}\left(\frac{b_1p+b_2}{b_3p+b_4}\right)}}. \tag{4.6}$$

With the help of (2.12)-(2.14), we find out the values of constants c_1 and c_2, we have

$$\bar{T}(\zeta,p)=\left(\frac{1-e^{-p}}{p^2}\right)\times e^{-\zeta\sqrt{P_{reff}\left(\frac{b_1p+b_2}{b_3p+b_4}\right)}}. \tag{4.7}$$

4.3. **Temperature Field via Atangana-Baleanu Approach.** We utilize integral transformation to acquire the solutions of temperature given by Eq. (2.10) using (2.13). We prefer to apply Laplace transform given in (2.20) on Eq. (2.10). The resultant form of above expression is,

$$\frac{\partial^2\bar{T}(\zeta,p)}{\partial\zeta^2}=P_{reff}\left(\frac{b_1p^{\kappa}+b_2}{b_3p^{\kappa}+b_4}\right)\bar{T}(\zeta,p). \tag{4.8}$$

The solution of homogenous part of second order partial differential equation say (4.8) is,

$$\bar{T}(\zeta,p)=c_1e^{-\zeta\sqrt{P_{reff}\left(\frac{b_1p^{\kappa}+b_2}{b_3p^{\kappa}+b_4}\right)}}+c_2e^{\zeta\sqrt{P_{reff}\left(\frac{b_1p^{\kappa}+b_2}{b_3p^{\kappa}+b_4}\right)}}. \tag{4.9}$$

With the help of (2.12)-(2.14), we find out the values of constants c_1 and c_2, we have

$$\bar{T}(\zeta,p)=\left(\frac{1-e^{-p}}{p^2}\right)\times e^{-\zeta\sqrt{P_{reff}\left(\frac{b_1p^{\kappa}+b_2}{b_3p^{\kappa}+b_4}\right)}}, \tag{4.10}$$

where $\quad b_1=1-Q+\kappa Q,\qquad b_2=\kappa Q,\qquad b_3=1-\kappa,\qquad b_4=\kappa.$

5. Optimal Velocity Field via Fractional Operators

5.1. **Velocity Field via Caputo Approach.** We utilize integral transformation to acquire the solutions of temperature given by Eq. (2.9). We exchange the partial derivative with fractional derivative of order α, and Eq. (2.9) can be written as:

$$\left(a_1+\lambda^C D_\tau^\kappa\right)\frac{\partial V}{\partial\tau}=\left(1+\lambda_r^C D_\tau^\gamma\right)\frac{\partial^2 V}{\partial\zeta^2}+G_r\left(1+\lambda^C D_\tau^\kappa\right)T+G_c\left(1+\lambda^C D_\tau^\kappa\right)C-a_2V. \tag{5.1}$$

We prefer to apply Laplace transform given in (2.16) on equation (2.9). The resultant form of above expression is

$$\left(a_1+\lambda p^\kappa\right)p\,\bar{V}(\zeta,p)=\left(1+\lambda_r p^\gamma\right)\frac{\partial^2\bar{V}(\zeta,p)}{\partial\zeta^2}+G_r\left(1+\lambda p^\kappa\right)\bar{T}(\zeta,p)+G_c\left(1+\lambda p^\kappa\right)\bar{C}(\zeta,p)-a_2\bar{V}(\zeta,p). \tag{5.2}$$

The solution of homogeneous part of (5.2) is,

$$\bar{V}(\zeta,p)=c_1e^{-\zeta\sqrt{\left(\frac{(a_1+\lambda\ p^\kappa)p+a_2}{1+\lambda_r\ p^\gamma}\right)}}+c_2e^{\zeta\sqrt{\left(\frac{(a_1+\lambda\ p^\kappa)p+a_2}{1+\lambda_r\ p^\gamma}\right)}}. \tag{5.3}$$

The general solution can be give as follow after making use of $\bar{T}(\zeta,\tau)$ and $\bar{C}(\zeta,\tau)$,

$$\begin{aligned}\bar{V}(\zeta,p) = c_1 e^{-\zeta\sqrt{\left(\frac{(a_1+\lambda\ p^{\kappa})p+a_2}{1+\lambda_r\ p^{\gamma}}\right)}} + c_2 e^{\zeta\sqrt{\left(\frac{(a_1+\lambda\ p^{\kappa})p+a_2}{1+\lambda_r\ p^{\gamma}}\right)}} \\ - \frac{G_r(1+\lambda p^{\kappa})(1-e^{-p})}{p^2\left[\left(P_{reff}(p^{\kappa}+Q)\right)(1+\lambda_r p^{\gamma}) - \left((a_1+\lambda p^{\kappa})p+a_2\right)\right]} e^{-\zeta\sqrt{P_{reff}(p^{\kappa}+Q)}} \\ - \frac{G_c(1+\lambda p^{\kappa})}{p\left[(S_c(p^{\kappa}+R))(1+\lambda_r p^{\gamma}) - ((a_1+\lambda p^{\kappa})p+a_2)\right]} e^{-\zeta\sqrt{S_c(p^{\kappa}+R)}},\end{aligned} \tag{5.4}$$

with the help of Eqs. (2.12)-(2.14), we find out the values of constants c_1 and c_2 for velocity equation:

$$\begin{aligned}\bar{V}(\zeta,p) = \left(\frac{1-e^{-p}}{p^2}\right) e^{-\zeta\sqrt{\left(\frac{(a_1+\lambda\ p^{\kappa})p+a_2}{1+\lambda_r\ p^{\gamma}}\right)}} \\ + \frac{G_r(1+\lambda p^{\kappa})(1-e^{-p})}{p^2\left[\left(P_{reff}(p^{\kappa}+Q)\right)(1+\lambda_r p^{\gamma}) - \left((a_1+\lambda p^{\kappa})p+a_2\right)\right]} \times \left\{ e^{-\zeta\sqrt{\left(\frac{(a_1+\lambda\ p^{\kappa})p+a_2}{1+\lambda_r\ p^{\gamma}}\right)}} - e^{-\zeta\sqrt{P_{reff}(p^{\kappa}+Q)}} \right\} \\ + \frac{G_c(1+\lambda p^{\kappa})}{p\left[(S_c(p^{\kappa}+R))(1+\lambda_r p^{\gamma}) - ((a_1+\lambda p^{\kappa})p+a_2)\right]} \times \left\{ e^{-\zeta\sqrt{\left(\frac{(a_1+\lambda\ p^{\kappa})p+a_2}{1+\lambda_r\ p^{\gamma}}\right)}} - e^{-\zeta\sqrt{S_c(p^{\kappa}+R)}} \right\}.\end{aligned} \tag{5.5}$$

5.2. **Velocity Field via Caputo-Fabrizio Approach.** We utilize integral transformation to acquire the solutions of temperature given by Eq. (2.9) using (2.13). We prefer to apply Laplace transform given in (2.18) on Eq. (2.9). The resultant form of the above expression is,

$$\begin{aligned}\left(a_1 + \lambda\left(\frac{p}{(1-\kappa)p+\kappa}\right)\right) p\ \bar{V} = \left(1+\lambda_r\left(\frac{p}{(1-\gamma)p+\gamma}\right)\right)\frac{\partial^2 \bar{V}}{\partial\zeta^2} + G_r\left(1+\lambda\left(\frac{p}{(1-\kappa)p+\kappa}\right)\right)\bar{T} \\ + G_c\left(1+\lambda\left(\frac{p}{(1-\kappa)p+\kappa}\right)\right)\bar{C} - a_2\bar{V}.\end{aligned} \tag{5.6}$$

The general solution can be given as follow after making use of $\bar{T}(\zeta,p)$ and $\bar{C}(\zeta,p)$,

$$\bar{V}(\zeta,p) = c_1 e^{-\zeta\sqrt{\frac{(p+a_6)\left((a_1p+a_8)p+a_2\right)}{\left((p+a_4)(a_7p+a_6)\right)}}} + c_2 e^{\zeta\sqrt{\frac{(p+a_6)\left((a_1p+a_8)p+a_2\right)}{\left((p+a_4)(a_7p+a_6)\right)}}}$$
$$- \frac{G_r(a_9p+a_4)(p+a_6)(1-e^{-p})}{p^2\left[P_{reff}\left((a_3+Q)p+Qa_4\right)\left(a_7p+a_6\right)-(p+a_6)\left((a_1p+a_8)p+a_2\right)\right]} \times e^{-\zeta\sqrt{\frac{P_{reff}\left((a_3+Q)p+Qa_4\right)}{\left(p+a_4\right)}}}$$
$$- \frac{G_c(a_9p+a_4)(p+a_6)}{p\left[S_c\left((a_3+R)p+Ra_4\right)\left(a_7p+a_6\right)-(p+a_6)\left((a_1p+a_8)p+a_2\right)\right]} \times e^{-\zeta\sqrt{\frac{S_c\left((a_3+R)p+Ra_4\right)}{\left(p+a_4\right)}}}, \quad (5.7)$$

with the help of Eqs. (2.12)-(2.14), we find out the values of constants c_1 and c_2 for velocity equation:

$$\bar{V}(\zeta,p) = \left(\frac{1-e^{p}}{p^2}\right) \times e^{-\zeta\sqrt{\frac{(p+a_6)\left((a_1p+a_8)p+a_2\right)}{\left((p+a_4)(a_7p+a_6)\right)}}}$$
$$+ \frac{G_r(a_9p+a_4)(p+a_6)(1-e^{-p})}{p^2\left[P_{reff}\left((a_3+Q)p+Qa_4\right)\left(a_7p+a_6\right)-(p+a_6)\left((a_1p+a_8)p+a_2\right)\right]}$$
$$\times\left\{e^{-\zeta\sqrt{\frac{(p+a_6)\left((a_1p+a_8)p+a_2\right)}{\left((p+a_4)(a_7p+a_6)\right)}}} - e^{-\zeta\sqrt{\frac{P_{reff}\left((a_3+Q)p+Qa_4\right)}{\left(p+a_4\right)}}}\right\}$$
$$+ \frac{G_c(a_9p+a_4)(p+a_6)}{p\left[S_c\left((a_3+R)p+Ra_4\right)\left(a_7p+a_6\right)-(p+a_6)\left((a_1p+a_8)p+a_2\right)\right]}$$
$$\times\left\{e^{-\zeta\sqrt{\frac{(p+a_6)\left((a_1p+a_8)p+a_2\right)}{\left((p+a_4)(a_7p+a_6)\right)}}} - e^{-\zeta\sqrt{\frac{S_c\left((a_3+R)p+Ra_4\right)}{\left(p+a_4\right)}}}\right\}, \quad (5.8)$$

where $a_1 = 1+\lambda M+\frac{\lambda_r}{K}$, $a_2 = 1+\frac{M}{K}$, $a_3 = \frac{1}{1-\kappa}$, $a_4 = \frac{\kappa}{1-\kappa}$, $a_5 = \frac{1}{1-\gamma}$, $a_6 = \frac{\gamma}{1-\gamma}$, $a_7 = 1+\lambda_r a_5$, $a_8 = a_1a_4+\lambda a_3$, $a_9 = 1+\lambda a_3$.

5.3. **Velocity Field via Atangana-Baleanu Approach.** We utilize integral transformation to acquire the solutions of temperature given by Eq. (2.9) using (2.13). We prefer to apply Laplace transform given in (2.20) on Eq. (2.9). The resultant form of the above expression is,

$$\left(a_1+\lambda\left(\frac{p^{\kappa}}{(1-\kappa)p^{\kappa}+\kappa}\right)\right)p\,\bar{V}=\left(1+\lambda_r\left(\frac{p^{\gamma}}{(1-\gamma)p^{\gamma}+\gamma}\right)\right)\frac{\partial^2\bar{V}}{\partial\zeta^2}+G_r\left(1+\lambda\left(\frac{p^{\kappa}}{(1-\kappa)p^{\kappa}+\kappa}\right)\right)\bar{T}$$
$$+G_c\left(1+\lambda\left(\frac{p^{\kappa}}{(1-\kappa)p^{\kappa}+\kappa}\right)\right)\bar{C}-a_2\bar{V}. \tag{5.9}$$

The general solution can be give as follow after making use of $\bar{T}(\zeta,p)$ and $\bar{C}(\zeta,p)$,

$$\bar{V}(\zeta,\tau)=c_1 e^{-\zeta\sqrt{\frac{(p^{\gamma}+a_6)\left((a_1p^{\kappa}+a_8)p+a_2\right)}{\left((p^{\kappa}+a_4)(a_7p^{\gamma}+a_6)\right)}}}+c_2 e^{\zeta\sqrt{\frac{(p^{\gamma}+a_6)\left((a_1p^{\kappa}+a_8)p+a_2\right)}{\left((p^{\kappa}+a_4)(a_7p^{\gamma}+a_6)\right)}}}$$
$$-\frac{G_r(a_9p^{\kappa}+a_4)(p^{\gamma}+a_6)(1-e^{-p})}{p^2\left[P_{reff}\left((a_3+Q)p^{\kappa}+Qa_4\right)\left(a_7p^{\gamma}+a_6\right)-(p^{\gamma}+a_6)\left((a_1p^{\kappa}+a_8)p+a_2\right)\right]}$$
$$\times e^{-\zeta\sqrt{\frac{P_{reff}\left((a_3+Q)p^{\kappa}+Qa_4\right)}{\left(p^{\kappa}+a_4\right)}}}$$
$$-\frac{G_c(a_9p^{\kappa}+a_4)(p^{\gamma}+a_6)}{p\left[S_c\left((a_3+R)p^{\kappa}+Ra_4\right)\left(a_7p^{\gamma}+a_6\right)-(p^{\gamma}+a_6)\left((a_1p^{\kappa}+a_8)p+a_2\right)\right]}$$
$$\times e^{-\zeta\sqrt{\frac{S_c\left((a_3+R)p^{\kappa}+Ra_4\right)}{\left(p^{\kappa}+a_4\right)}}}, \tag{5.10}$$

with the help of Eqs. (2.12)-(2.14), we find out the values of constants c_1 and c_2 for velocity equation:

$$\begin{aligned}\bar{V}(\zeta,p) &= \left(\frac{1-e^{p}}{p^2}\right)e^{-\zeta\sqrt{\frac{(p^\gamma+a_6)\left((a_1p^\kappa+a_8)p+a_2\right)}{\left((p^\kappa+a_4)(a_7p^\gamma+a_6)\right)}}}\\
&-\frac{G_r(a_9p^\kappa+a_4)(p^\gamma+a_6)(1-e^{-p})}{p^2\left[P_{reff}\left((a_3+Q)p^\kappa+Qa_4\right)\left(a_7p^\gamma+a_6\right)-(p^\gamma+a_6)\left((a_1p^\kappa+a_8)p+a_2\right)\right]}\\
&\times\left\{e^{-\zeta\sqrt{\frac{(p^\gamma+a_6)\left((a_1p^\kappa+a_8)p+a_2\right)}{\left((p^\kappa+a_4)(a_7p^\gamma+a_6)\right)}}}-e^{-\zeta\sqrt{\frac{P_{reff}\left((a_3+Q)p^\kappa+Qa_4\right)}{\left(p^\kappa+a_4\right)}}}\right\}\\
&-\frac{G_c(a_9p^\kappa+a_4)(p^\gamma+a_6)}{p\left[S_c\left((a_3+R)p^\kappa+Ra_4\right)\left(a_7p^\gamma+a_6\right)-(p^\gamma+a_6)\left((a_1p^\kappa+a_8)p+a_2\right)\right]}\\
&\times\left\{e^{-\zeta\sqrt{\frac{(p^\gamma+a_6)\left((a_1p^\kappa+a_8)p+a_2\right)}{\left((p^\kappa+a_4)(a_7p^\gamma+a_6)\right)}}}-e^{-\zeta\sqrt{\frac{S_c\left((a_3+R)p^\kappa+Ra_4\right)}{\left(p^\kappa+a_4\right)}}}\right\}.\end{aligned} \tag{5.11}$$

As $\kappa \to 1$, the non-integer fractional models are reduced into the classical model. Further, if we neglect mass Grahsof number ($G_c = 0$) in Eq. (2.9), the results are identical which was obtained by Anwar et al. [3].
In our fluid models, we use the classical computational technique to solve the fluid models using definitions of fractional derivatives. There are many numerical algorithms which are used to calculate their inverses like Stehfest's and Tzou's algorithms for semi-analytical solutions [40, 42]. Recently, Riaz et al. [13] and Madeeha et al. [21] analyzed the numerical Laplace method to show the accuracy of inversion algorithms by solving fractional differential equations effectively and reliably way. Tzou's calculation for our numerical inverse Laplace

$$v(r,t)=\frac{e^{4.7}}{t}\left[\frac{1}{2}\bar{v}\left(r,\frac{4.7}{t}\right)+Re\left\{\sum_{k=1}^{N_1}(-1)^k\bar{v}\left(r,\frac{4.7+k\pi i}{t}\right)\right\}\right],$$

where Re(.) is the real part, i is the imaginary unit and N_1 is a natural number [42].

6. Results and discussion

This section is dedicated to present physical interpretation of the obtained results via CF and AB differential operators under heat generation, ramp velocity and ramp temperature on MHD Oldroyd-B fluid. Results are investigated via Laplace transformation with inversion algorithm for velocity, energy and shear stress based on singular verses non-singular and local versus non-local kernels. The graphical representations are depicted for showing the influences of different physical parameters such as fractional parameters κ, magnetic effect M, Grashof number G_r and G_c, effective Prandtl number P_{reff}, Schmidth number S_c, relaxation time λ and retardation time λ_r on velocity and energy profile using the package of MATHCAD-15.

6.1. **Velocity profile for κ.** The influence of fraction parameter κ on velocity can be seen in Figure 2. By increasing the value of κ, fluid velocity decreases for variation of time. Further, it can be easily seen that ABC is good to explain the memory effect of velocity profile as compared to C and CF. As the kernel of ABC, possess the property of non-singularity and non-local due to which it is more effective as compared to C and CF.

6.2. **Velocity profile for M.** The influence of M is shown in Figure 3. By increasing the value of M, the resultant velocity the decrease. Physically, the reason behind decrease in velocity is Lorentz force produced by the magnetic field whose direction is opposite to the direction of flow which slows down the velocity. The behavior of non-integer order models in velocity for M is the same at different time.

6.3. **Velocity profile for G_r.** Figure 4 shows the impact on G_r for velocity field versus time. It is observed that as G_r increases, velocity becomes high because of enhancement in buoyancy force due to temperature gradient. Physically, there is a relation between thermal buoyancy forces and viscous hydrodynamic force which is related to G_r. As G_r increases, there is an increase in temperature gradient due to which the buoyancy effect dominates and hence rise in velocity is observed

6.4. **Velocity profile for Pr_{reff}.** Figure 5 depicted the influence of Pr_{reff} on velocity. Enhancement in Pr_{reff} reduces velocity. As we increase Pr_{reff}, the thermal conductivity decreases due to which viscosity increases which causes a decrease in velocity. Furthermore, with the increase in Pr_{reff} the boundary layer thickness decreases.

6.5. **Velocity profile for S_c.** Figure 6 analyzed the impact on S_c for velocity field versus time. It is defined as the ratio between viscous diffusion rate and mass diffusion rate. Physically, Schmidt number is used to characterize fluid motion and it relates the thickness of hydro-dynamic layers and mass transfer boundary layers. It is observed that with larger value of S_c, the velocity becomes decreases.

6.6. **Velocity profile for G_c.** Figure 7 discusses the behavior of velocity curves for G_c. Clearly, with the increase in G_c, velocity rises due to an increase in buoyancy force and buoyancy force increases due to concentration gradient. Physically, there is a relationship between concentration buoyancy forces and the viscous hydrodynamic force affected by G_c. Buoyancy effect is enhanced because of an increase in concentration gradient which increases as G_c increases; as a result, velocity decreases. The impact of G_c on fluid velocity is similar to G_r. The velocity for the ABC model is good as compared to other fractional models.

6.7. **Velocity profile for λ_r.** Figure 8 shows the behavior of velocity curves for λ_r. It is observed that velocity enhance with the increase in λ_r for all fractional models. The velocity behavior is also observed for variation of time. Clearly, ABC model achieved maximum velocity as compared to other fractional models.

6.8. **Velocity profile for λ.** Effect of velocity profile for retardation time parameter is illustrated in Figure 9. With the increase in λ, velocity decreases which is quite opposite behavior as compare to λ_r.

6.9. **Temperature profile for κ.** Figures 10 highlights the effect of fractional parameter on temperature profile for fractional models. Increase in κ, the resultant temperature decreases. Temperature for CF and ABC is more as compared to C in all cases. Moreover, as κ tends to 1, temperature curves for non-integer order approach integer order.

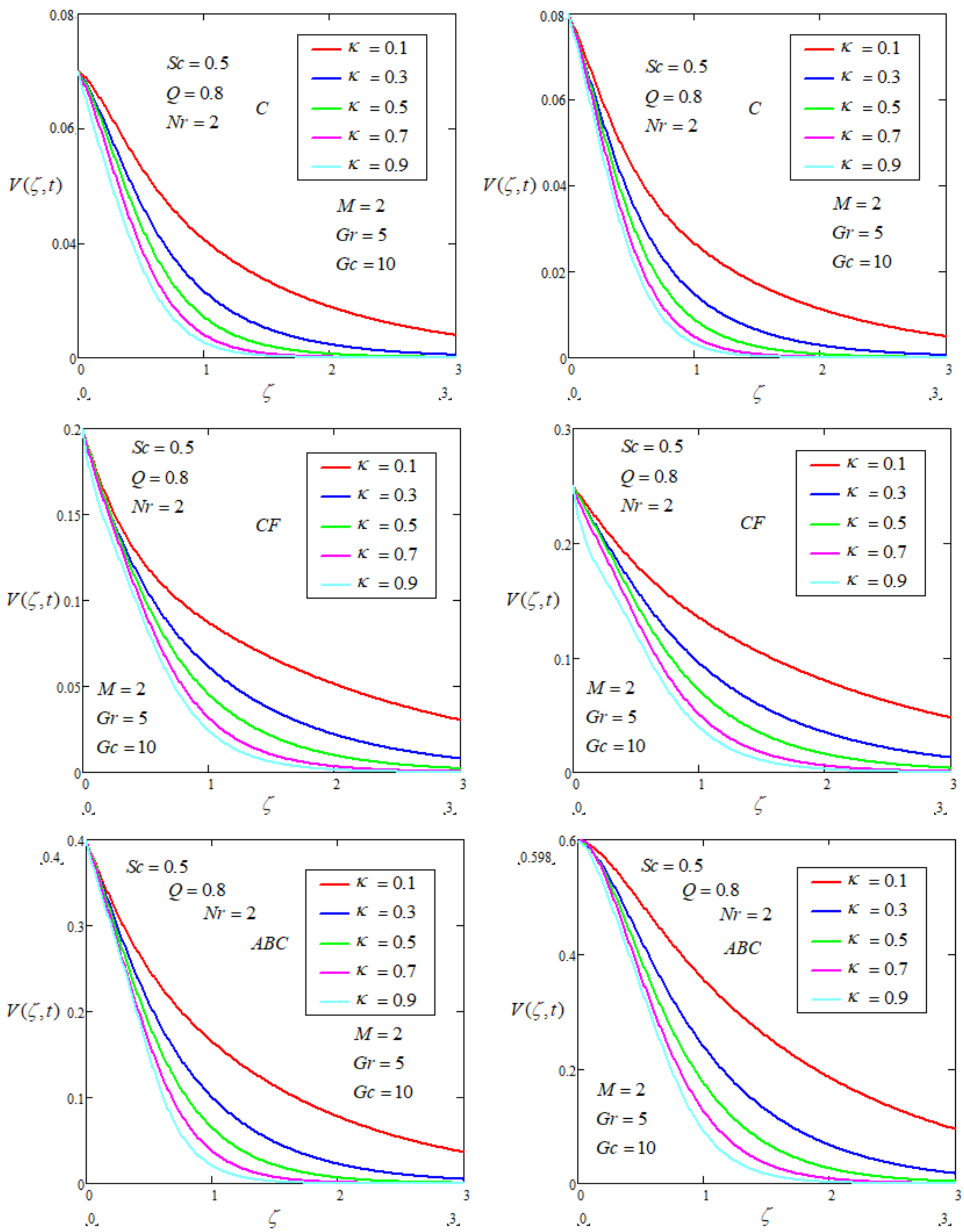

FIGURE 2. Plot via C, CF and AB-approaches for velocity with different values of κ.

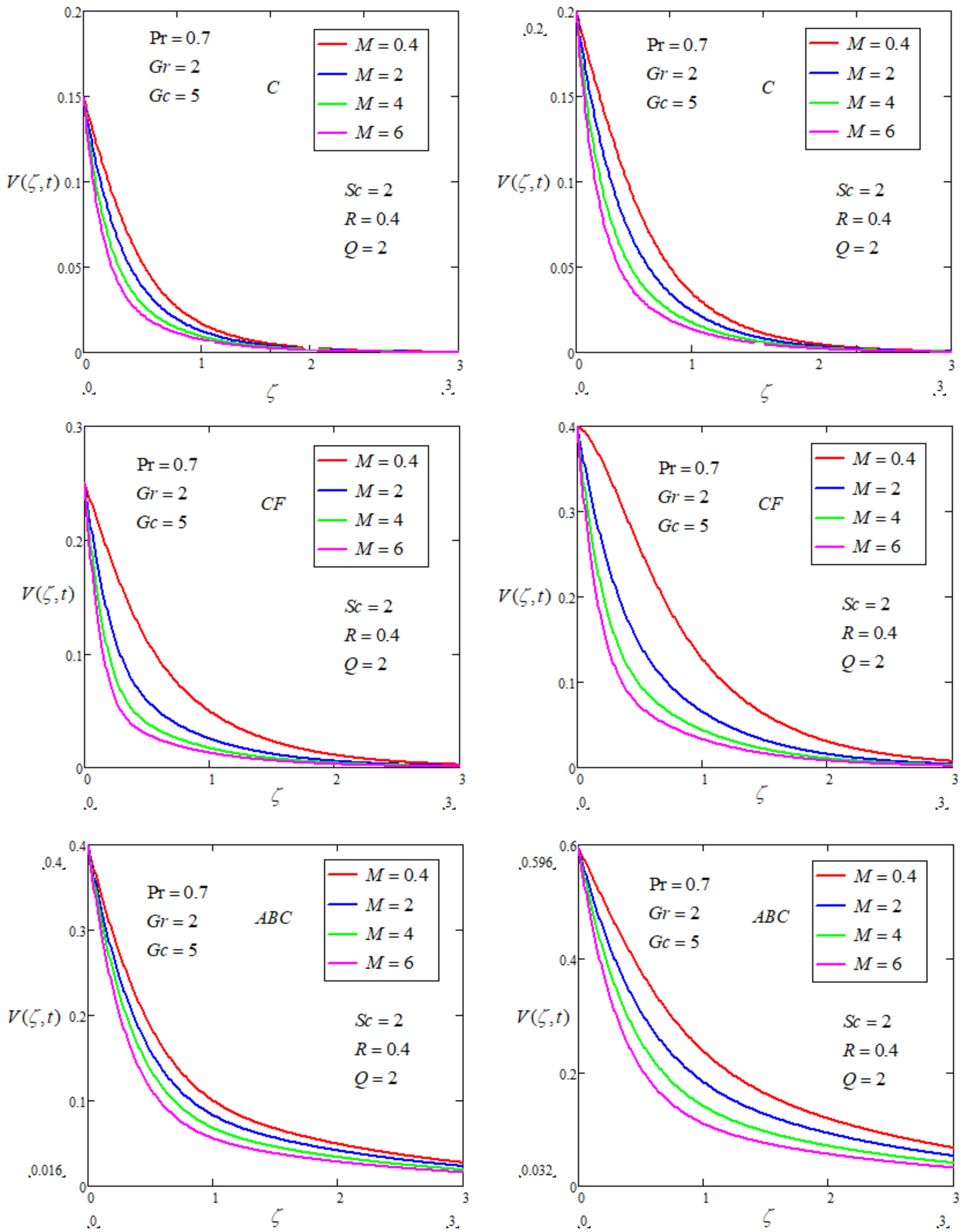

FIGURE 3. Plot via C, CF and AB-approaches for velocity with different values of M.

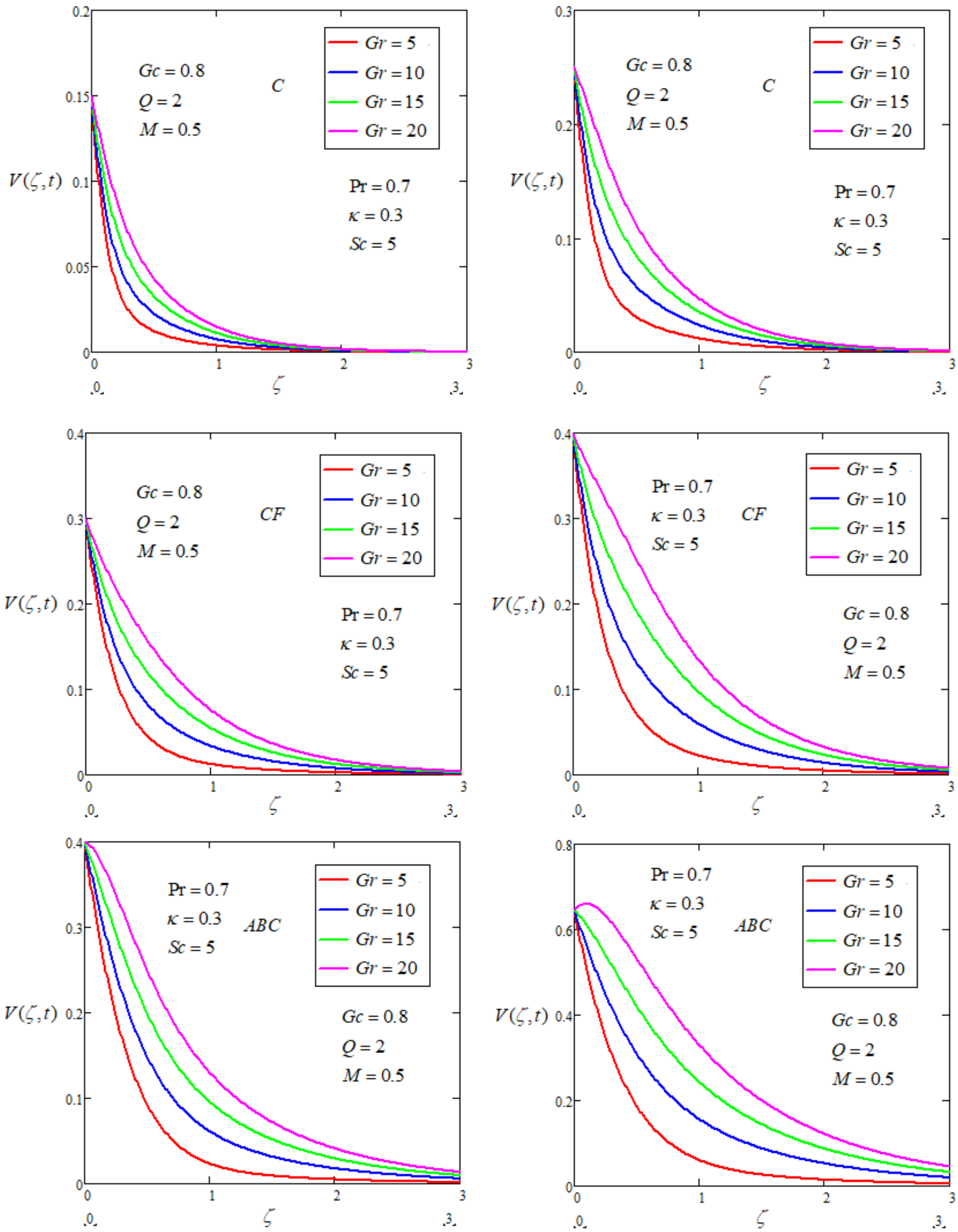

FIGURE 4. Plot via C, CF and AB-approaches for velocity with different values of G_r.

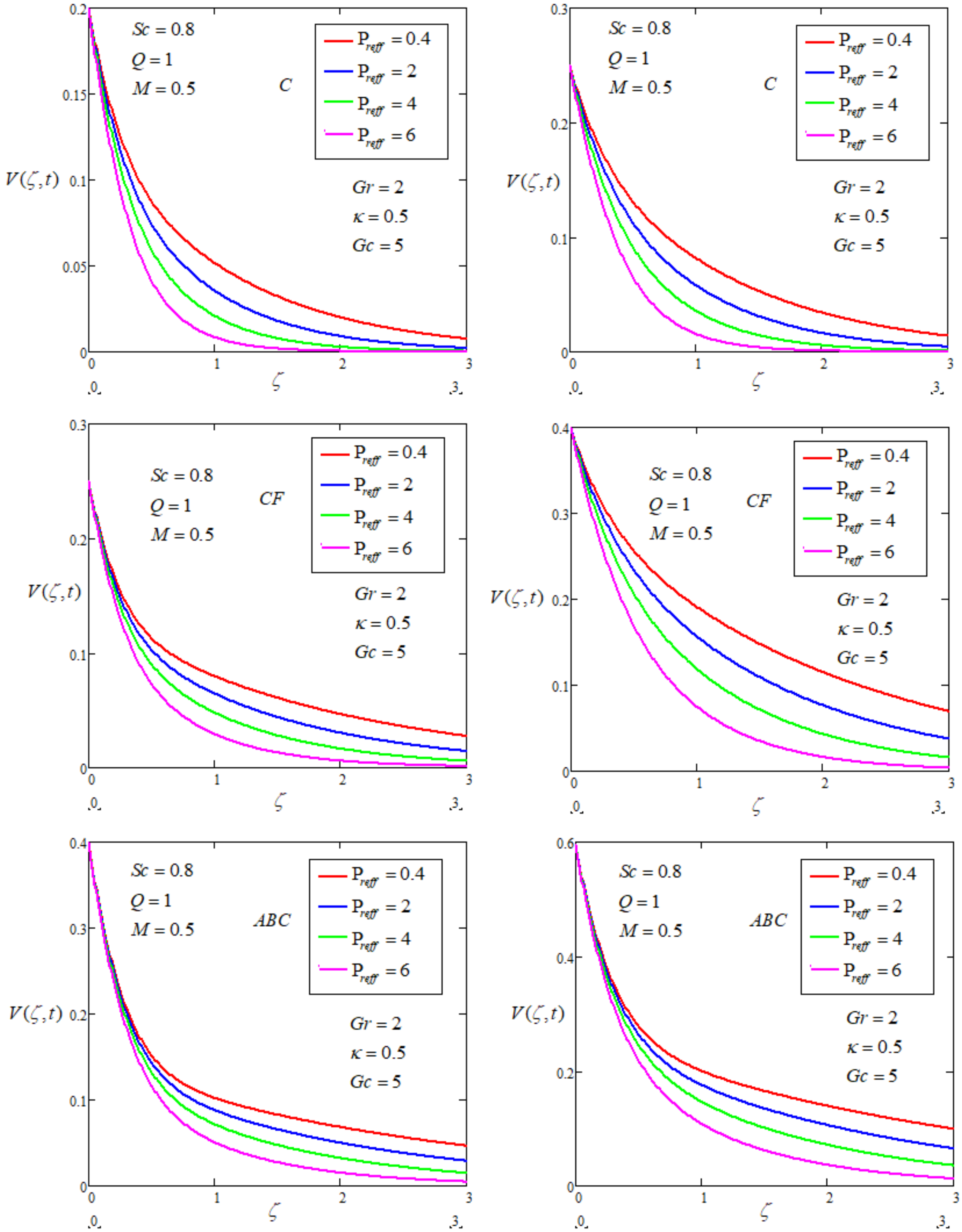

FIGURE 5. Plot via C, CF and AB-approaches for velocity with different values of P_{reff}.

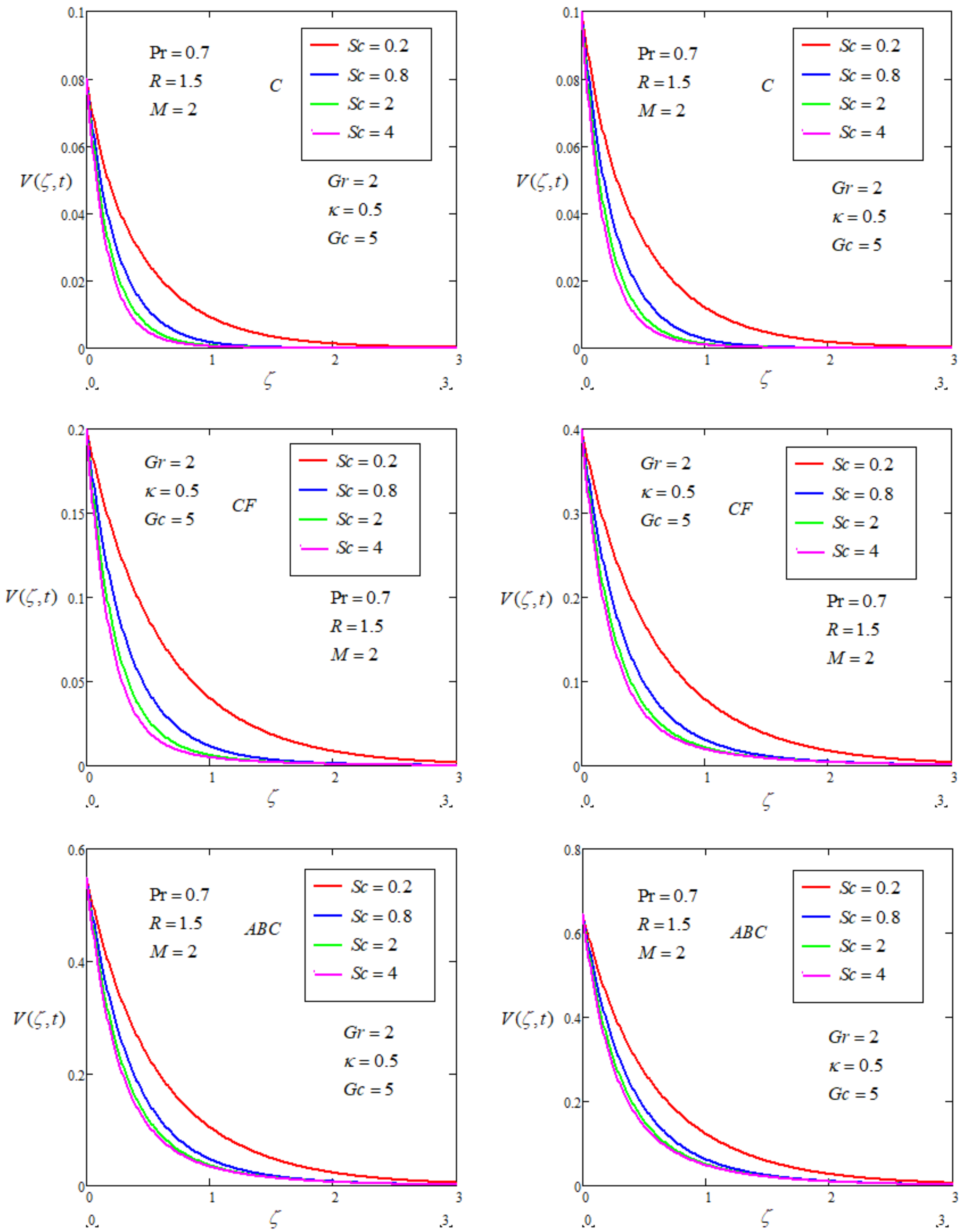

FIGURE 6. Plot via C, CF and AB-approaches for velocity with different values of Sc.

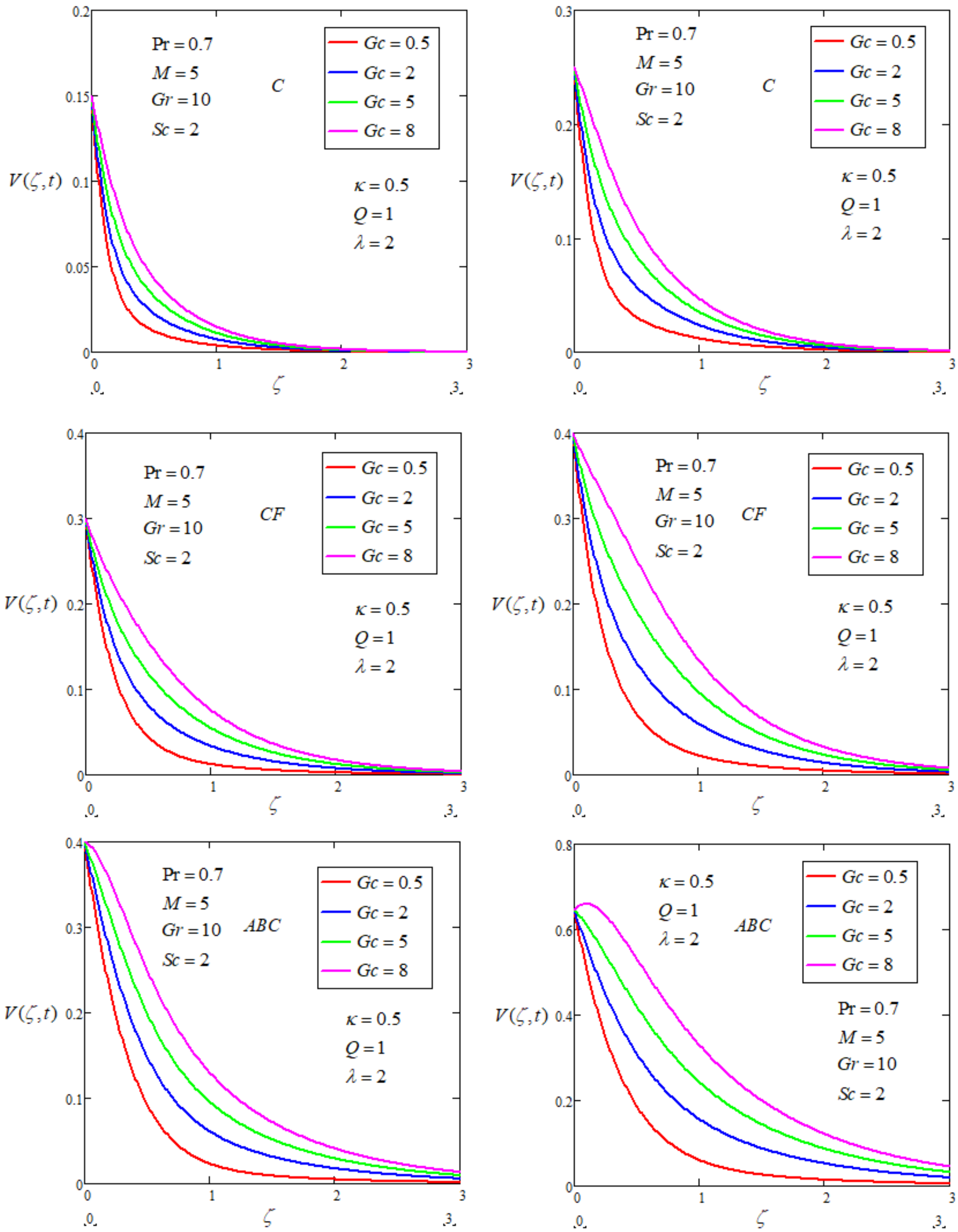

FIGURE 7. Plot via C, CF and AB-approaches for velocity with different values of G_c.

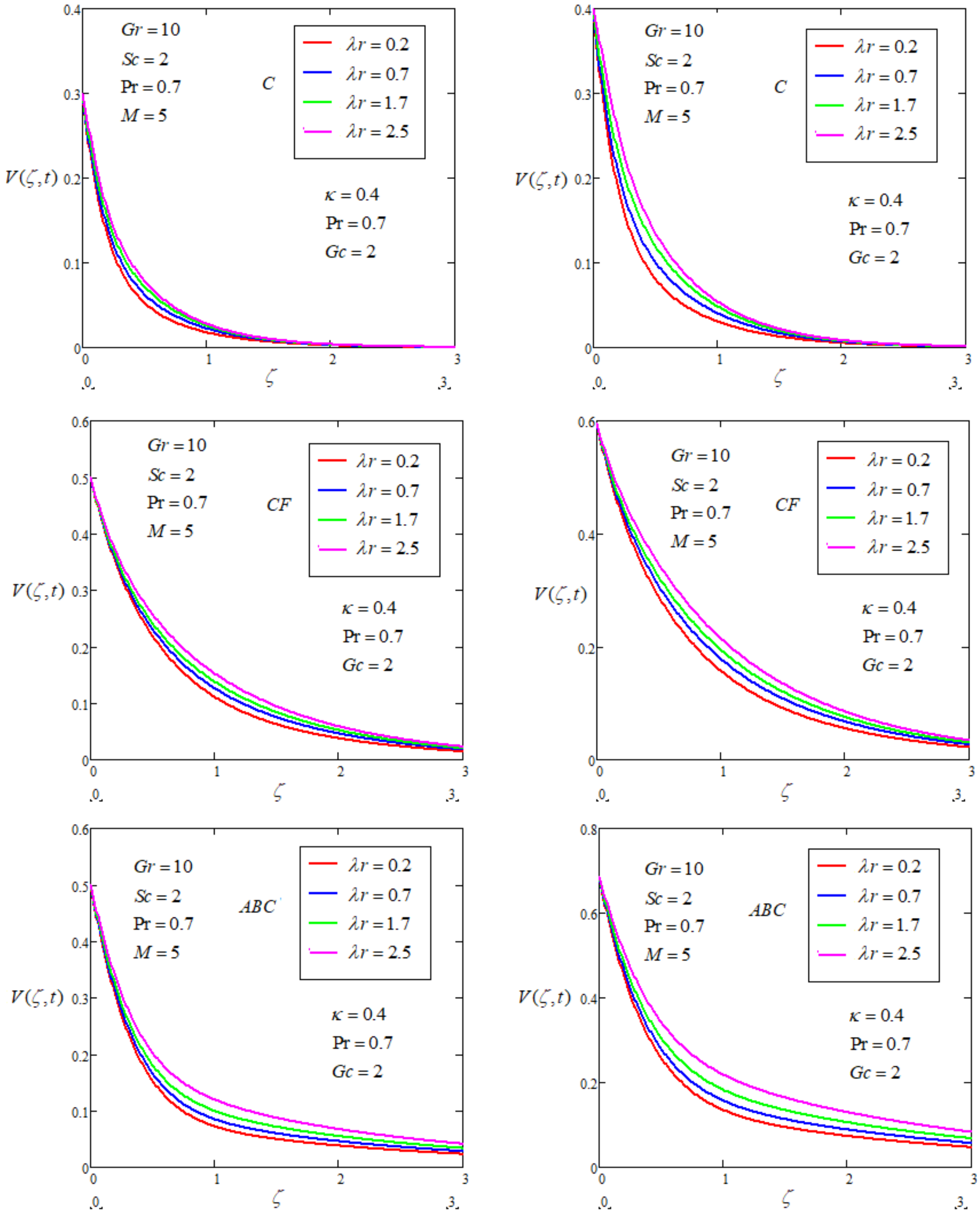

FIGURE 8. Plot via C, CF and AB-approaches for velocity with different values of λ_r.

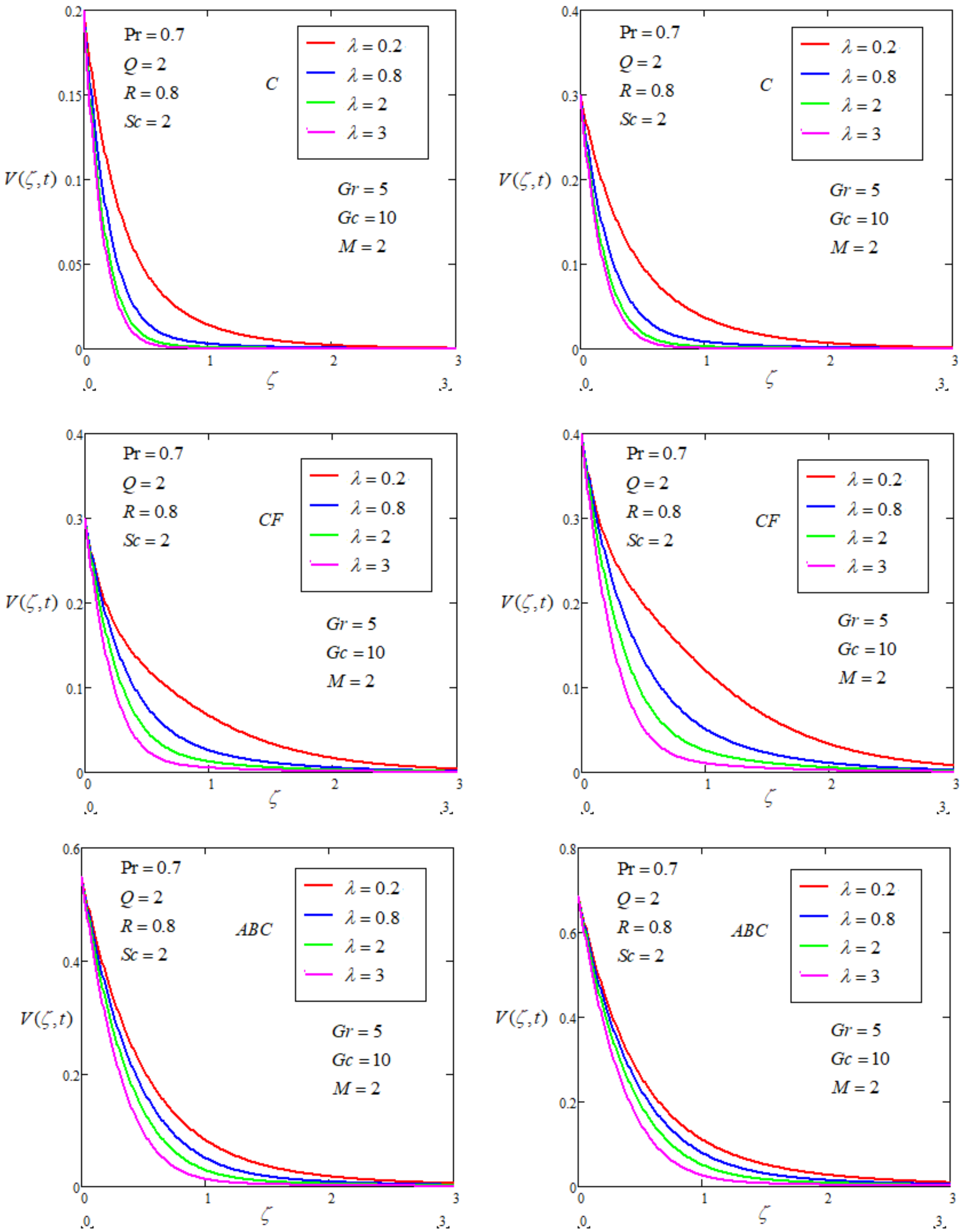

FIGURE 9. Plot via C, CF and AB-approaches for velocity with different values of λ.

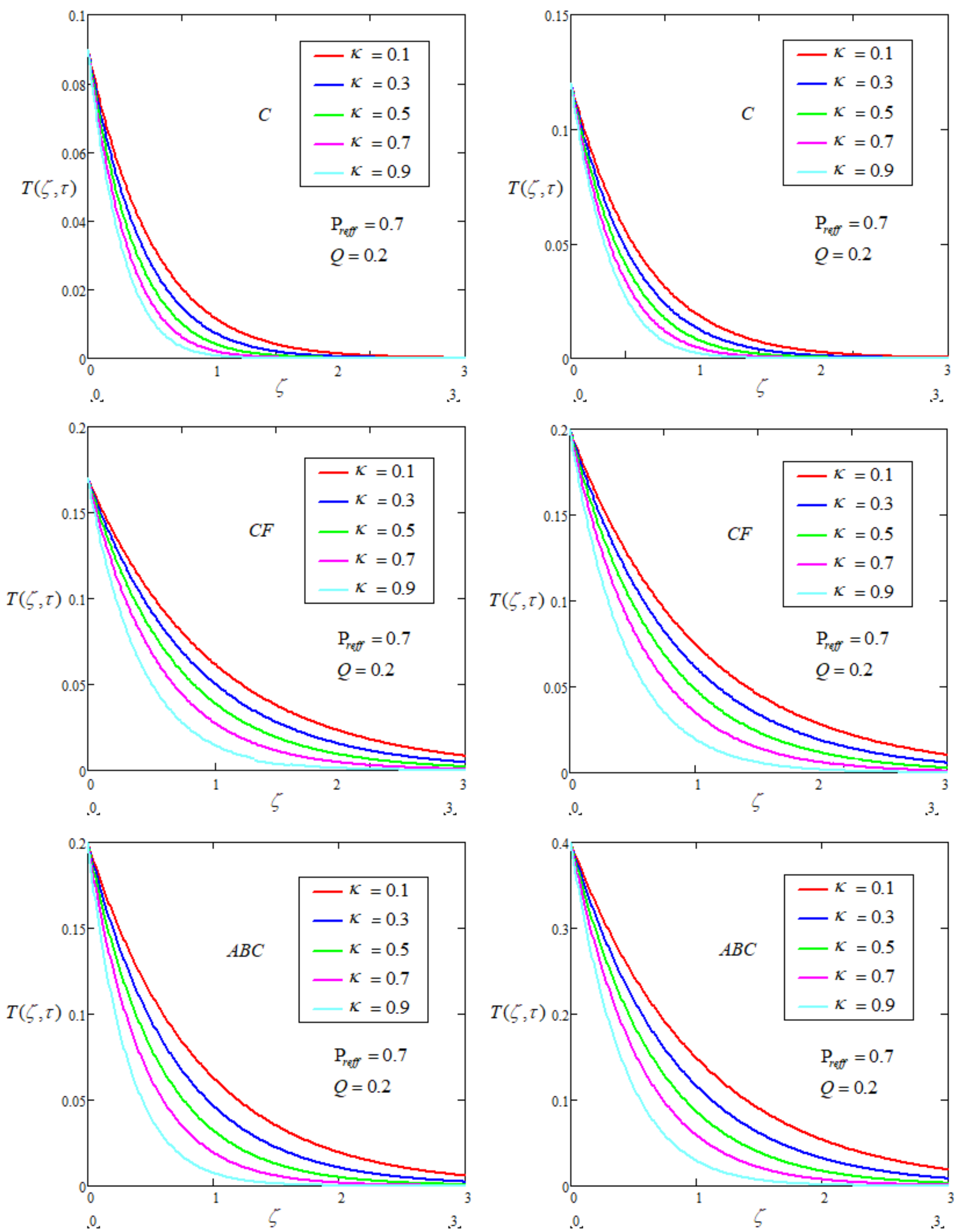

FIGURE 10. Plot via C, CF and AB-approaches for temperature profile with different values of κ.

7. Conclusion

The study of MHD Oldroyd-B fluid flow with ramped wall temperature and velocity under the influence thermal radiation in a porous medium has been discussed via non-integer models. The inversion algorithm and Laplace transform are used to find the velocity, temperature, and concentration. A comparison is made for all models. Some results from the literature can be recovered from our general results. Some significant remarks for this problem are:

- Velocity curves are showing decreasing behavior for fractional parameter κ and M.
- Velocity increases as G_r and G_c increase for C, CF and ABC.
- Velocity is an decreasing function of Pr_{reff} and Sc for all fractional models.
- Velocity showing opposite behavior for λ and λ_r for C, CF and ABC.
- Temperature decreases by magnify the value of fractional parameter.

Our present study includes analysis of Oldroyd-B fluid with ramped velocity and ramped temperature using fractional order derivatives. This work can be extend with different boundary conditions. In future we considering a more complex models like Jeffery fluid and some rotational models with different conditions. Moreover, new 3D graphical analysis for possible emerging parameters can also be taken into account.

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