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An exponential cubic B-spline algorithm for solving the nonlinear Coupled Burgers' equation

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Abstract The collocation method based on the exponential cubic B-splines (ECB-splines) together with the Crank-Nicolson formula is used to solve nonlinear coupled Burgers' equation (CBE). This method is tested by studying three different problems. The proposed scheme is compared with some existing methods. It produced accurate results with the suitable selection of the free parameter of the ECB-spline function. It produces accurate results. Stability of the fully discretized CBE is investigated by the Von Neumann analysis.

Keywords. Coupled Burgers' equation, Exponential cubic B-spline, Stability analysis, Collocation.

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1. Introduction

The collocation method which is a variant of finite element method (FEM) has been widely used for solving partial differential equations (PDEs). The collocation method is a numerical method for obtaining approximate solutions of PDEs. Development of high-speed computer allows to be improved collocation method to solve PDEs on both complex domain and complicated boundary conditions (BCs). It also provides higher order approximation which result from a vector of constants multiplied by a set of basis functions. The collocation method fulfill quite well in obtaining solution of a large class of PDEs. It is often used low order piece-wise continuous polynomials function to establish approximate solutions in the numerical techniques. It is also known that one of the success of the collocation method depends on the choice of the basis function on reducing error. The collocation method with the B-splines is efficient due both to giving smaller system of algebraic equations that has lower computational complexity and providing higher order continuous approximation depending on using the B-splines of high degree. McCartin [16, 17] has described the exponential cubic B-spline (ECB-spline) function as the generalization of the B-splines and given its

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properties. He has also showed a reliable algorithm by using the exponential spline functions to obtain solutions of the hyperbolic conservation laws [18]. The ECB-spline includes a free parameter which is determined by scanning the predetermined interval with small increment for the ECB. The best free parameter is determined for the ECB-spline functions for solving the differential equations over defined region. The use of the ECB-spline is not common for numerical methods to solve system of PDE. There exist few papers dealing with the solution of the differential equations using ECB-spline. Some variants of the collocation methods based on the exponential splines are established to find solutions of the singular perturbation problems in the studies [26, 28, 33]. According to results of paper [28], ECB-spline collocation method is seen to produce lower error than the cubic B-spline collocation in solving the singularly perturbation problem. The ECB-spline collocation algorithm is set up for solving for various PDEs [3, 4, 5, 7, 9, 35]. The convection-diffusion equation is solved by way of the ECB-spline collocation method in the study [23]. Then ECB-spline collocation algorithms designed to obtain numerical solutions of the Reaction Diffusion system [6]. In that study isothermal system and Brusselator system are studied and also two different approximates are proposed for the numerical solutions of Gray Scott Autocatalysis system in one dimension [12]. For this purpose the sinc differential quadrature and ECB-spline collocation methods are set up. In [20], R.C. Mittal studied on reaction-diffusion problem which includes linear problem, isothermal system, Brusselator system and Gray Scott system via differential quadrature method.

The purpose of this paper is to apply the ECB-spline collocation method to the coupled Burgers' equation (CBE) and see results whether accuracy increases or not. CBE in the following form:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + k_1 u \frac{\partial u}{\partial x} + k_2 (uv)_x = 0, \tag{1.1}$$

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} + k_1 v \frac{\partial v}{\partial x} + k_3 (uv)_x = 0 \tag{1.2}$$

where k_1, k_2 and k_3 are real constants and subscripts x and t indicate differentiation, x distance and t time, is considered. Boundary conditions for $t > 0$:

$$u(a,t) = f_1(a,t), \quad u(b,t) = f_2(b,t), \tag{1.3}$$

$$v(a,t) = g_1(a,t), \quad v(b,t) = g_2(b,t) \tag{1.4}$$

and initial conditions for $a \leq x \leq b$:

$$u(x,0) = f(x), \quad v(x,0) = g(x) \tag{1.5}$$

will be decided in the later sections according to test problem.

The mathematical model of the CBE describes some nonlinear phenomena in many areas of scientific field such as model of polydisperse sedimentation and approximate theory of flow through shock wave traveling in a viscous fluid. Thus, engineers and scientists are concerned on discovering new application of CBE and investigating its properties. Many researches have been performed for solution of the CBE numerically and analytically. It is known that there is no precise analytical solution of the CBE. Scientists have developed various numerical techniques to obtain numerical solutions of the CBE. The difficulty in the numerical techniques of CBE arises due to viscosity parameters and nonlinear terms. Many numerical methods have used to obtain the solution of the CBE given in the following list.

Fourth order accurate compact ADI scheme [27],
A chebyshev spectral collocation method [11],
A meshfree technique [1, 10],
The fourier pseudospectral method [29],
The generalized two-dimensional differential transform method [15],
Generalized differential quadrature method [24],
A robust technique for solving optimal control of CBE [31],
A differential quadrature method [21],
Galerkin quadratic B-spline finite element method [14],
Trigonometric B-spline collocation method [32],
A fully implicit finite-difference method [36],
A composite numerical scheme based on finite difference [13],
Logarithmic finite-difference method [37],
Modified cubic B-spline collocation method [22],
The cubic B-spline collocation method [19],
The trigonometric quintic B-spline collocation method [25]
The trigonometric cubic B-spline collocation method [30]
The septic B-spline collocation method [34].

The task in this paper is to set up collocation method accompanied with the combination of the ECB-spline basis as an approximation function to investigate solutions of the CBE. The paper is organized as follows. In section 2, some details about ECB-spline collocation method are provided. In section 3, the initial states are documented. In section 4 the stability of the fully discretized CBE is investigated by the Von Neumann analysis. In section 5, numerical results for three different problems and some related figures and tables are given in order to show the efficiency as well as the accuracy of the proposed method. Finally, conclusions are followed in section 6.

2. Exponential cubic B-spline collocation method

Let π be partition of the problem domain $[a, b]$ defined at the knots $\pi : a = x_0 < x_1 < \ldots < x_N = b$ with mesh spacing $h = (b - a)/N$. The ECB-spline, $B_j(x)$, with knots at the points of π can be defined as

$$B_j(x) = \begin{cases} b_2\left((x_{j-2}-x) - \frac{1}{p}\left(\sinh(p\,(x_{j-2}-x))\right)\right) & [x_{j-2}, x_{j-1}]\,, \\ a_1 + b_1\,(x_j - x) + c_1 \exp\left(p\,(x_j - x)\right) + d_1 \exp\left(-p\,(x_j - x)\right) & [x_{j-1}, x_j]\,, \\ a_1 + b_1(x - x_j) + c_1 \exp\left(p\,(x - x_j)\right) + d_1 \exp\left(-p\,(x - x_j)\right) & [x_j, x_{j+1}]\,, \\ b_2\left((x - x_{j+2}) - \frac{1}{p}(\sinh\left(p\,(x - x_{j+2})\right))\right) & [x_{j+1}, x_{j+2}]\,, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

where

$$a_1 = \frac{phc}{phc - s},\; b_1 = \frac{p}{2}\left(\frac{c(c-1)+s^2}{(phc-s)(1-c)}\right),\; b_2 = \frac{p}{2(phc-s)},$$

$$c_1 = \frac{1}{4}\left(\frac{e^{-ph}(1-c)+s(e^{-ph}-1)}{(phc-s)(1-c)}\right),$$

$$d_1 = \frac{1}{4}\left(\frac{e^{ph}(c-1)+s(e^{ph}-1)}{(phc-s)(1-c)}\right).$$

and $c = \cosh(ph)$, $s = \sinh(ph)$, p is a free parameter. On the particular interval $[0, 1]$, the ECB-spline function is depicted for $p = 1$ in Figure 1.

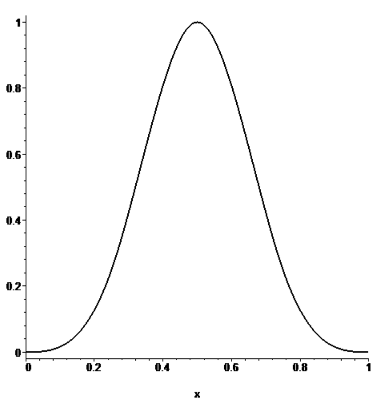


FIGURE 1. ECB-splines for $p = 1$ over the interval [0,1].

$\{B_{-1}(x),\ B_0(x),\cdots,B_{N+1}(x)\}$ forms a basis for the functions defined over the interval $[a,b]$. Each subinterval $[x_j,x_{j+1}]$ is covered by four consecutive ECB-spline. The ECB-spline and its first and second derivatives vanish outside its support interval $[x_{j-2},x_{j+2}]$. Each basis function $B_j(x)$ has got the continuous second derivatives. The values of $B_j(x)$, $B_j^{'}(x)$ and $B_j^{''}(x)$ at the knots x_j can be computed from Eq. (2.1) given Table 1.

Table 1. Values of $B_j(x)$, $B_j^{'}(x)$ and $B_j^{''}(x)$.

x	x_{j-2}	x_{j-1}	x_j	x_{j+1}	x_{j+2}
B_j	0	$\frac{s-ph}{2(phc-s)}$	1	$\frac{s-ph}{2(phc-s)}$	0
$B_j^{'}$	0	$\frac{p(1-c)}{2(phc-s)}$	0	$\frac{p(c-1)}{2(phc-s)}$	0
$B_j^{''}$	0	$\frac{p^2s}{2(phc-s)}$	$-\frac{p^2s}{phc-s}$	$\frac{p^2s}{2(phc-s)}$	0

Since the B-splines are basis functions, an approximate function can be constructed by the combination of the basis functions. Thus, approximate solutions $U(x,t)$ and $V(x,t)$ to the analytical solution $u(x,t)$ and $v(x,t)$ can be assumed in the forms:

$$U(x,t)=\sum_{j=-1}^{N+1}\delta_jB_j(x),\ V(x,t)=\sum_{j=-1}^{N+1}\phi_jB_j(x) \tag{2.2}$$

where δ_j and ϕ_j are time dependent parameters. The first and second derivatives also can be defined by:

$$U'(x,t)=\sum_{j=-1}^{N+1}\delta_jB_j'(x),\ V'(x,t)=\sum_{j=-1}^{N+1}\phi_jB_j^{'}(x) \tag{2.3}$$

$$U''(x,t)=\sum_{j=-1}^{N+1}\delta_jB_j''(x),\ V''(x,t)=\sum_{j=-1}^{N+1}\phi_jB_j''(x). \tag{2.4}$$

Using the Eq. (2.2)-(2.4) and Table 1, we see that the nodal values U_j, V_j, their first derivatives U_j', V_j' and second derivatives U_j'', V_j'' at the knots are given in terms of parameters by the following relations:

$$U_j=U(x_j,t)=\tfrac{s-ph}{2(phc-s)}\left(\delta_{j-1}+\tfrac{2(phc-s)}{s-ph}\delta_j+\delta_{j+1}\right) \tag{2.5}$$

$$U_j'=U'(x_j,t)=\tfrac{p(1-c)}{2(phc-s)}\left(\delta_{j-1}-\delta_{j+1}\right) \tag{2.6}$$

$$U_j''=U''(x_j,t)=\tfrac{p^2s}{2(phc-s)}\left(\delta_{j-1}-2\delta_j+\delta_{j+1}\right) \tag{2.7}$$

$$V_j=V(x_j,t)=\tfrac{s-ph}{2(phc-s)}\left(\phi_{j-1}+\tfrac{2(phc-s)}{s-ph}\phi_j+\phi_{j+1}\right) \tag{2.8}$$

$$V_j' = V'(x_j,t) = \frac{p(1-c)}{2(phc-s)}\left(\phi_{j-1} - \phi_{j+1}\right) \tag{2.9}$$

$$V_j'' = V''(x_j,t) = \frac{p^2 s}{2(phc-s)}\left(\phi_{j-1} - 2\phi_j + \phi_{j+1}\right). \tag{2.10}$$

Time discretization of unknown U and V is managed by way of the Crank–Nicolson scheme in the CBE to obtain following equation:

$$\frac{U^{n+1}-U^n}{\Delta t} - \frac{U_{xx}^{n+1}+U_{xx}^n}{2} + k_1\frac{(UU_x)^{n+1}+(UU_x^n)^n}{2} + k_2\frac{(UV)_x^{n+1}+(UV)_x^n}{2} = 0 \tag{2.11}$$

$$\frac{V^{n+1}-V^n}{\Delta t} - \frac{V_{xx}^{n+1}+V_{xx}^n}{2} + k_1\frac{(VV_x)^{n+1}+(VV_x^n)^n}{2} + k_3\frac{(UV)_x^{n+1}+(UV)_x^n}{2} = 0 \tag{2.12}$$

where $U^{n+1} = U(x,t_n+\Delta t)$ and $V^{n+1} = V(x,t_n+\Delta t)$. The nonlinear terms $(UU_x)^{n+1}$, $(VV_x)^{n+1}$ and $(UV)_x^{n+1}$in Eq. (2.11)-(2.12) are linearized with Taylor expansion:

$$(UU_x)^{n+1} \quad = U^{n+1}U_x^n + U^nU_x^{n+1} - U^nU_x^n \tag{2.13}$$

$$(VV_x)^{n+1} \quad = V^{n+1}V_x^n + V^nV_x^{n+1} - V^nV_x^n \tag{2.14}$$

$$(UV)_x^{n+1} \quad = U_x^{n+1}V^n + U_x^nV^{n+1} - U_x^nV^n + U^{n+1}V_x^n + U^nV_x^{n+1} - U^nV_x^n \tag{2.15}$$

Place Eq. (2.5)-(2.10) in (2.11)-(2.12) to have fully discretized system of equations:

$$\nu_{m1}\delta_{m-1}^{n+1} + \nu_{m2}\phi_{m-1}^{n+1} + \nu_{m3}\delta_m^{n+1} + \nu_{m4}\phi_m^{n+1} + \nu_{m5}\delta_{m+1}^{n+1} + \nu_{m6}\phi_{m+1}^{n+1}$$

$$= \nu_{m7}\delta_{m-1}^n + \nu_{m8}\delta_m^n + \nu_{m9}\delta_{m+1}^n \tag{2.16}$$

and

$$\nu_{m10}\delta_{m-1}^{n+1} + \nu_{m11}\phi_{m-1}^{n+1} + \nu_{m12}\delta_m^{n+1} + \nu_{m13}\phi_m^{n+1} + \nu_{m14}\delta_{m+1}^{n+1} + \nu_{m15}\phi_{m+1}^{n+1}$$

$$= \nu_{m7}\phi_{m-1}^n + \nu_{m8}\phi_m^n + \nu_{m9}\phi_{m+1}^n \tag{2.17}$$

where

$$\begin{aligned}
\nu_{m1} &= \left(1+\frac{\Delta t}{2}k_1K_2+\frac{\Delta t}{2}k_2L_2\right)\alpha_1+\frac{\Delta t}{2}\left(k_1K_1+k_2L_1\right)\beta_1-\frac{\Delta t}{2}\gamma_1\\
\nu_{m2} &= \frac{\Delta t}{2}\left(k_1K_2\right)\alpha_1+\frac{\Delta t}{2}\left(k_2K_1\right)\beta_1\\
\nu_{m3} &= \left(1+\frac{\Delta t}{2}k_1K_2+\frac{\Delta t}{2}k_2L_2\right)\alpha_2-\frac{\Delta t}{2}\gamma_2\\
\nu_{m4} &= \frac{\Delta t}{2}\left(k_1K_2\right)\alpha_2\\
\nu_{m5} &= \left(1+\frac{\Delta t}{2}k_1K_2+\frac{\Delta t}{2}k_2L_2\right)\alpha_1-\frac{\Delta t}{2}\left(k_1K_1+k_2L_1\right)\beta_1-\frac{\Delta t}{2}\gamma_1\\
\nu_{m6} &= \frac{\Delta t}{2}\left(k_1K_2\right)\alpha_1-\frac{\Delta t}{2}\left(k_2K_1\right)\beta_1\\
\nu_{m7} &= \alpha_1+\frac{\Delta t}{2}\gamma_1\\
\nu_{m8} &= \alpha_2+\frac{\Delta t}{2}\gamma_2\\
\nu_{m9} &= \alpha_1+\frac{\Delta t}{2}\gamma_1\\
\nu_{m10} &= \frac{\Delta t}{2}\left(k_3L_2\right)\alpha_1+\frac{\Delta t}{2}\left(k_3L_1\right)\beta_1\\
\nu_{m11} &= \left(1+\frac{\Delta t}{2}k_1L_2+\frac{\Delta t}{2}k_3K_2\right)\alpha_1+\frac{\Delta t}{2}\left(k_1L_1+k_3K_1\right)\beta_1-\frac{\Delta t}{2}\gamma_1\\
\nu_{m12} &= \frac{\Delta t}{2}\left(k_3L_2\right)\alpha_2\\
\nu_{m12} &= \left(1+\frac{\Delta t}{2}k_1L_2+\frac{\Delta t}{2}k_3K_2\right)\alpha_2-\frac{\Delta t}{2}\gamma_2\\
\nu_{m14} &= \frac{\Delta t}{2}\left(k_3L_2\right)\alpha_1-\frac{\Delta t}{2}\left(k_3L_1\right)\beta_1\\
\nu_{m15} &= \left(1+\frac{\Delta t}{2}k_1L_2+\frac{\Delta t}{2}k_3K_2\right)\alpha_1-\frac{\Delta t}{2}\left(k_1L_1+k_3K_1\right)\beta_1-\frac{\Delta t}{2}\gamma_1
\end{aligned}$$

$$\begin{aligned}
&K_1=\alpha_1\delta^n_{m-1}+\alpha_2\delta^n_m+\alpha_1\delta^n_{m+1},\quad L_1=\alpha_1\phi^n_{m-1}+\alpha_2\phi^n_m+\alpha_1\phi^n_{m+1}\\
&K_2=\beta_1\delta^n_{m-1}-\beta_1\delta^n_{m+1},\quad L_2=\beta_1\phi^n_{m-1}-\beta_1\phi^n_{m+1}
\end{aligned}$$

$$\begin{aligned}
\alpha_1 &= \frac{s-ph}{2(phc-s)},\ \alpha_2=1,\ \beta_1=\frac{p(1-c)}{2(phc-s)}\\
\gamma_1 &= \frac{p^2s}{2(phc-s)},\ \gamma_2=-\frac{p^2s}{phc-s}.
\end{aligned}$$

The system with (2.16) and (2.17) can be converted into the following matrices system;

$$\mathbf{A}\mathbf{d}^{n+1}=\mathbf{B}\mathbf{d}^n \tag{2.18}$$

where

$$\mathbf{A} = \begin{bmatrix} \nu_{m1} & \nu_{m2} & \nu_{m3} & \nu_{m4} & \nu_{m5} & \nu_{m6} & & & & \\ \nu_{m10} & \nu_{m11} & \nu_{m12} & \nu_{m13} & \nu_{m14} & \nu_{m15} & & & & \\ & & \nu_{m1} & \nu_{m2} & \nu_{m3} & \nu_{m4} & \nu_{m5} & \nu_{m6} & & \\ & & \nu_{m10} & \nu_{m11} & \nu_{m12} & \nu_{m13} & \nu_{m14} & \nu_{m15} & & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & & \nu_{m1} & \nu_{m2} & \nu_{m3} & \nu_{m4} & \nu_{m5} & \nu_{m6} \\ & & & & \nu_{m10} & \nu_{m11} & \nu_{m12} & \nu_{m13} & \nu_{m14} & \nu_{m15} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \nu_{m7} & 0 & \nu_{m8} & 0 & \nu_{m9} & 0 & & & & \\ 0 & \nu_{m7} & 0 & \nu_{m8} & 0 & \nu_{m9} & & & & \\ & & \nu_{m7} & 0 & \nu_{m8} & 0 & \nu_{m9} & 0 & & \\ & & 0 & \nu_{m7} & 0 & \nu_{m8} & 0 & \nu_{m9} & & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & & \nu_{m7} & 0 & \nu_{m8} & 0 & \nu_{m9} & 0 \\ & & & & 0 & \nu_{m7} & 0 & \nu_{m8} & 0 & \nu_{m9} \end{bmatrix}$$

The system (2.18) consists of $2N+2$ linear equation in $2N+6$ unknown parameters $\mathbf{d}^{n+1} = (\delta_{-1}^{n+1}, \phi_{-1}^{n+1}, \delta_0^{n+1}, \phi_0^{n+1}, \ldots, \delta_{n+1}^{n+1}, \phi_{n+1}^{n+1})$. To obtain a unique solution additional four constraints are needed. By imposing the Dirichlet boundary conditions this will lead us to the following relations:

$$\delta_{-1}^{n+1} = (f_1(a,t) - \alpha_2\delta_0^{n+1} - \alpha_1\delta_1^{n+1})/\alpha_1 \tag{2.19}$$

$$\phi_{-1}^{n+1} = (g_1(a,t) - \alpha_2\phi_0^{n+1} - \alpha_1\phi_1^{n+1})/\alpha_1 \tag{2.20}$$

$$\delta_{N+1}^{n+1} = (f_2(b,t) - \alpha_1\delta_{N-1}^{n+1} - \alpha_2\delta_N^{n+1})/\alpha_1 \tag{2.21}$$

$$\phi_{N+1}^{n+1} = (g_2(b,t) - \alpha_1\phi_{N-1}^{n+1} - \alpha_2\phi_N^{n+1})/\alpha_1 \tag{2.22}$$

Elimination of the parameters δ_{-1}^{n+1}, ϕ_{-1}^{n+1}, δ_{N+1}^{n+1}, $\phi_{N+1}^{n+1}$from the system (2.18) gives us a solvable system. Once the initial parameters are determined, parameters δ_j^n and ϕ_j^n , $j = -1, \ldots, N+1$ can be found from the iterative system (2.18), so that using the equations (2.5) and (2.8) solutions at the defined time steps can be found iteratively.

3. The initial state

Initial parameters δ_{-1}^0, ϕ_{-1}^0, δ_0^0, $\phi_0^0, \ldots, \delta_{N+1}^0$, ϕ_{N+1}^0 must be determined to start the iterative equations (2.16) and (2.17). Initial conditions provides following equations:

$$U^0(a,0) = \frac{s-ph}{2(phc-s)}\delta_{-1}^0 + \delta_0^0 + \frac{s-ph}{2(phc-s)}\delta_1^0 \tag{3.1}$$

$$U^0(x_j,0) = \frac{s-ph}{2(phc-s)}\delta^0_{j-1} + \delta^0_j + \frac{s-ph}{2(phc-s)}\delta^0_{j+1}, \quad j=1,2,...,N-1 \tag{3.2}$$

$$U^0(b,0) = U^0_N = \frac{s-ph}{2(phc-s)}\delta^0_{N-1} + \delta^0_N + \frac{s-ph}{2(phc-s)}\delta^0_{N+1} \tag{3.3}$$

and

$$V^0(a,0) = \frac{s-ph}{2(phc-s)}\phi^0_{-1} + \phi^0_0 + \frac{s-ph}{2(phc-s)}\phi^0_1 \tag{3.4}$$

$$V^0(x_j,0) = \frac{s-ph}{2(phc-s)}\phi^0_{j-1} + \phi^0_j + \frac{s-ph}{2(phc-s)}\phi^0_{j+1}, \quad j=1,2,...,N-1 \tag{3.5}$$

$$V^0(b,0) = \frac{s-ph}{2(phc-s)}\phi^0_{N-1} + \phi^0_N + \frac{s-ph}{2(phc-s)}\phi^0_{N+1} \tag{3.6}$$

The system (3.1)-(3.3) which is consist of $N+1$ equations and $N+3$ unknowns, δ^0_{-1} and δ^0_{N+1} are eliminated from the system (3.1)-(3.3) using the derivative boundary conditions:

$$\delta^0_{-1} = \delta^0_1 + \frac{2(phc-s)}{p(1-c)}U'_0, \; \delta^0_{N+1} = \delta^0_{N-1} - \frac{2(phc-s)}{p(1-c)}U'_N, \tag{3.7}$$

and if the system of equations is rearranged for the above conditions, then following form is obtained

$$\begin{bmatrix} 1 & \frac{s-ph}{phc-s} & & & & \\ \frac{s-ph}{2(phc-s)} & 1 & \frac{s-ph}{2(phc-s)} & & & \\ & & & \ddots & & \\ & & & \frac{s-ph}{2(phc-s)} & 1 & \frac{s-ph}{2(phc-s)} \\ & & & & \frac{s-ph}{phc-s} & 1 \end{bmatrix} \begin{bmatrix} \delta^0_0 \\ \delta^0_1 \\ \vdots \\ \delta^0_{N-1} \\ \delta^0_N \end{bmatrix}$$

$$= \begin{bmatrix} U'_0 - \frac{s-ph}{p(1-c)}U'_0 \\ U'_1 \\ \vdots \\ U'_{N-1} \\ U'_N - \frac{s-ph}{p(c-1)}U'_N \end{bmatrix}$$

which can also be solved using Thomas algorithm [8]. As the same way, from the system (3.4)-(3.6), ϕ_{-1}^0 and ϕ_{N+1}^0 can be eliminated using derivative conditions which gives the equations:

$$\phi_{-1}^0 = \phi_1^0 + \frac{2(phc-s)}{p(1-c)}V_0', \; \phi_{N+1}^0 = \phi_{N-1}^0 - \frac{2(phc-s)}{p(1-c)}V_N', \tag{3.8}$$

the following three bounded matrix is obtained.

$$\begin{bmatrix} 1 & \frac{s-ph}{phc-s} & & & & \\ \frac{s-ph}{2(phc-s)} & 1 & \frac{s-ph}{2(phc-s)} & & & \\ & & \ddots & & & \\ & & & \frac{s-ph}{2(phc-s)} & 1 & \frac{s-ph}{2(phc-s)} \\ & & & & \frac{s-ph}{phc-s} & 1 \end{bmatrix} \begin{bmatrix} \phi_0^0 \\ \phi_1^0 \\ \vdots \\ \phi_{N-1}^0 \\ \phi_N^0 \end{bmatrix}$$

$$= \begin{bmatrix} V_0' - \frac{s-ph}{p(1-c)}V_0' \\ V_1' \\ \vdots \\ V_{N-1}' \\ V_N' - \frac{s-ph}{p(c-1)}V_N' \end{bmatrix}$$

4. Stability analysis

For the stability of the system (2.16) and (2.17) Von-Neumann method is applied. To do this in the nonlinear terms $(UV)_x$ in the equation system (1.1-1.2) one consider U and V as a local constants μ_1 and μ_2 respectively. The equation (1.1) can be written as

$$\begin{aligned} a_1\delta_{j-1}^{n+1} + a_2\delta_j^{n+1} + a_3\delta_{j+1}^{n+1} + a_4\phi_{j-1}^{n+1} + a_5\phi_{j+1}^{n+1} \\ = a_6\delta_{j-1}^n + a_7\delta_j^n + a_8\delta_{j+1}^n + a_9\phi_{j-1}^n + a_{10}\phi_{j+1}^n \end{aligned} \tag{4.1}$$

Now on substituting $\delta^n = A\xi^n \exp(ij\varphi)$ and $\phi^n = B\xi^n \exp(ij\varphi)$ in the equation (**??**) where A and B are amplitude, $\varphi = kh$, k is a mode number h is a step size and $i = \sqrt{-1}$, we obtain:

$$\begin{aligned} &\left\{ \begin{array}{l} a_1A\xi^{n+1}\exp(i(j-1)\varphi) + a_2A\xi^{n+1}\exp(ij\varphi) + a_3A\xi^{n+1}\exp(i(j+1)\varphi) \\ +a_4B\xi^{n+1}\exp(i(j-1)\varphi) + a_5B\xi^{n+1}\exp(i(j+1)\varphi) \end{array} \right. \\ = &\left\{ \begin{array}{l} a_6A\xi^n\exp(i(j-1)\varphi) + a_7A\xi^n\exp(ij\varphi) + a_8A\xi^n\exp(i(j+1)\varphi) \\ +a_9B\xi^n\exp(i(j-1)\varphi) + a_{10}B\xi^n\exp(i(j+1)\varphi) \end{array} \right. \end{aligned} \tag{4.2}$$

where

$$a_1 = \alpha_1 + \tfrac{\Delta t}{2}\left(k_1\mu_1 + k_2\mu_2\right)\beta_1 - \tfrac{\Delta t}{2}\gamma_1$$
$$a_2 = \alpha_2 - \tfrac{\Delta t}{2}\gamma_2$$
$$a_3 = \alpha_1 - \tfrac{\Delta t}{2}\left(k_1\mu_1 + k_2\mu_2\right)\beta_1 - \tfrac{\Delta t}{2}\gamma_1$$
$$a_4 = \tfrac{\Delta t}{2}k_2\mu_1\beta_1$$
$$a_5 = -\tfrac{\Delta t}{2}k_2\mu_1\beta_1$$
$$a_6 = \alpha_1 - \tfrac{\Delta t}{2}\left(k_1\mu_1 + k_2\mu_2\right)\beta_1 + \tfrac{\Delta t}{2}\gamma_1$$
$$a_7 = \alpha_2 + \tfrac{\Delta t}{2}\gamma_2$$
$$a_8 = \alpha_1 + \tfrac{\Delta t}{2}\left(k_1\mu_1 + k_2\mu_2\right)\beta_1 + \tfrac{\Delta t}{2}\gamma_1$$
$$a_9 = -\tfrac{\Delta t}{2}k_2\mu_1\beta_1$$
$$a_{10} = \tfrac{\Delta t}{2}k_1\mu_1\beta_1$$

Now dividing the both side of (4.2) with $\exp(ij\varphi)$ yield

$$|X_1 + iY|\,\xi^{n+1} = |X_2 + iY|\,\xi^n \tag{4.3}$$

where

$$X_1 = A\left[(2\alpha_1\cos(\varphi) + \alpha_2) - \Delta t\left(\tfrac{1}{2}\gamma_2 + \gamma_1\cos(\varphi)\right)\right] \tag{4.4}$$
$$X_2 = A\left[(2\alpha_1\cos(\varphi) + \alpha_2) + \Delta t\left(\tfrac{1}{2}\gamma_2 + \gamma_1\cos(\varphi)\right)\right] \tag{4.5}$$
$$Y = -\left[A\Delta t(k_1\mu_1 + k_2\mu_2) + B\Delta t k_2\mu_1\beta_1\sin(\varphi)\right] \tag{4.6}$$

$$|\xi| = \frac{|X_2 + iY|}{|X_1 + iY|} = \frac{\sqrt{X_2^2 + Y^2}}{\sqrt{X_1^2 + Y^2}} \leq 1 \Rightarrow 0 \leq X_1^2 - X_2^2 \tag{4.7}$$

From (4.7) the CBE is unconditionally stable since the modulus of the eigenvalues is less than or equal to one. This means that there is no restriction on space step h and time step Δt and step size in time level but we should choose those values of h and Δt, for which we get the best accuracy of the scheme. Since the CBE (1.1-1.2) is symmetric, the similar process can be applied to the Eq. (1.2), so that it also unconditionally stable.

5. Numerical tests

In this section, computational examples of the three test problems will be presented. For the CBE, the error is measured with error norm $L_\infty = \max_j \left|U_j - (U_N)_j^n\right|$. The obtained computational results will be compared with [2, 14, 19, 21, 22]. Order of convergence of the method is obtained by using the formula [35]:

$$\log\left(\frac{eh_1}{eh_2}\right) / \log\left(\frac{h_1}{h_2}\right) \tag{5.1}$$

where eh_1 and eh_2 are L_∞ errors for grid sizes h_1 and h_2, respectively.

Problem 1) Consider the CBE (1.1-1.2) with the following initial and boundary conditions:

$$u(x, 0) = v(x, 0) = \sin(x) \tag{5.2}$$

and

$$u(-\pi,t) = u(\pi,t) = v(-\pi,t) = v(\pi,t) = 0. \tag{5.3}$$

The exact solution is:

$$u(x,t) = v(x,t) = e^{-t}\sin(x) \tag{5.4}$$

with the parameters $k_1 = -2$, $k_2 = 1$ and $k_3 = 1$. We compute numerical solutions with various time-space steps over the interval $-\pi \le x \le \pi$. Table 2 illustrates the comparison of the error norms L_∞ obtained by the present method with those found with extended cubic B-spline Taylor-collocation method referenced [2] for time step $\Delta t = 0.001$ and space steps $h = \pi/100$ and $h = \pi/200$. Best solution is searched for the suitable free parameters chosen from the randomly determined interval (0,1] with very small increments. In the suggested method when the space step is reduced from $h = \pi/100$ to $h = \pi/200$, error norm L_∞ decreases by one decimal digit, seeing in Table 2. Suggested algorithm gives same accuracy with the Taylor-collocation method based on an extended B-spline functions. The B-spline Taylor-collocation method obtained with the choice of the free parameter $\lambda = 0$ in the extended cubic B-spline and suggested method with the free parameters $p = 1$ gives the same error seeing Table 3. Results of our method are also compared with that of the differential quadrature, the quadratic B-spline Galerkin and modified cubic B-spline collocation method. It is seen that accuracy of the presented method is higher than the methods listed in Table 4. Space pointwise rate of convergence is determined approximately as 2 achieved in the studies [14, 22] as well when time steps $\Delta t = 0.01, 0.0001$ are used given in Table 5. Crank-Nicolson method of order 2 is verified by calculating discrete rate of convergence as approximately 2 given in Table 5 for the time steps $\Delta t = 0.01, 0.0001$ and $p = 1$.

Table 2. $L_\infty \times 10^5$ for $t = 0.1$, $\Delta t = 0.001$, $u(x,t) = v(x,t)$ for Problem 1.

N	Present ($p = 1$)	Present ($p = 3.4103 \times 10^{-5}$)	[2] ($\lambda = 0$)	[2] (free parameter λ)
200	1.48916	0.00006	0.74326	0.00079
				($\lambda = -1.640 \times 10^{-4}$)
400	0.37285	0.00003	0.18534	0.00006
				($\lambda = -4.087 \times 10^{-5}$)

Table 3. $L_\infty \times 10^5$ for $t = 1$, $N = 400$, $u(x,t) = v(x,t)$ for Problem 1.

Δt	Present ($p = 1$)	Present (free parameter p)	[2] ($\lambda = 0$)	[2] (free parameter λ)
0.01	1.8194	0.00029	1.08691	0.00131
		($p = 4.6807 \times 10^{-4}$)		($\lambda = -5.896 \times 10^{-5}$)
0.001	1.5159	0.00017	1.10393	0.00036
		($p = 2.1092 \times 10^{-4}$)		($\lambda = -5.992 \times 10^{-5}$)

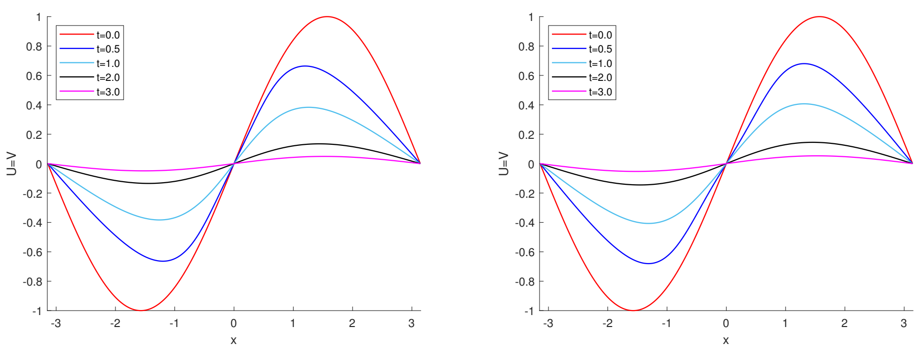
Graphical solutions are depicted in Figures 2-3 for various combinations of the parameters k_1, k_2, k_3. Simulations resembles pattern obtained by Mittal and Triphati [22].

Table 4. L_∞ for $\Delta t = 0.01$, $N = 50$, $(x,t) = v(x,t)$ for Problem 1.

t	Present ($p = 1$)	Present ($p = 3.14 \times 10^{-6}$)	[21]	[14]	[22]
0.5	7.9881×10^{-4}	6.3483×10^{-6}	1.51688×10^{-4}	2.26627×10^{-5}	1.10308×10^{-4}
1.0	9.6837×10^{-4}	7.7011×10^{-6}	1.83970×10^{-4}	1.46179×10^{-5}	1.33688×10^{-4}
2.0	7.1154×10^{-4}	5.6661×10^{-6}	1.35250×10^{-4}	0.73805×10^{-5}	9.81825×10^{-5}
3.0	3.9213×10^{-4}	3.1266×10^{-6}	7.46014×10^{-4}	0.40272×10^{-5}	1.02987×10^{-5}

Table 5. The rate of convergence for Problem 1.

	$\Delta t = 0.01$		$\Delta t = 0.0001$	
N	Present ($p = 1$)	order	Present ($p = 1$)	order
50	3.9213×10^{-4}		3.9213×10^{-4}	
100	9.9430×10^{-5}	1.9796	9.8185×10^{-5}	1.9932
150	4.4895×10^{-5}	1.9610	4.3651×10^{-5}	1.9993
200	2.5808×10^{-5}	1.9245	2.4563×10^{-5}	1.9986
250	1.6965×10^{-5}	1.8801	1.5721×10^{-5}	2.0000



(A) $k_1 = -2, k_2 = 1, k_3 = -8$ (B) $k_1 = -2, k_2 = 1, k_3 = -4$

Figure 2. Numerical solutions of $u = v$ for Problem 1 (k_1, k_2 fixed).

Problem 2) One of the exact solution of the CBE is:

$$u(x,t) = a_0 - 2A(\tfrac{2k_2-1}{4k_2k_3-1})\tanh(A(x-2At)) \tag{5.5}$$

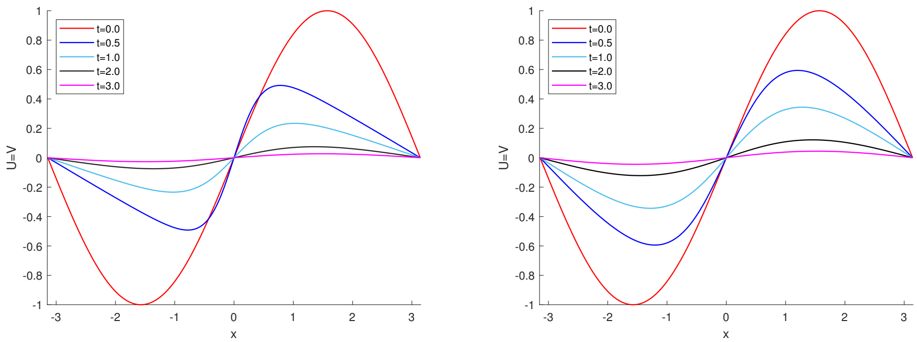
and

$$v(x,t) = a_0(\tfrac{2k_3-1}{2k_2-1}) - 2A(\tfrac{2k_2-1}{4k_2k_3-1})\tanh(A(x-2At)). \tag{5.6}$$

The initial and boundary conditions can be obtained via the exact solution

$$u(x,0) = a_0 - 2A(\tfrac{2k_2-1}{4k_2k_3-1})\tanh(Ax) \tag{5.7}$$

$$v(x,0) = a_0(\tfrac{2k_3-1}{2k_2-1}) - 2A(\tfrac{2k_2-1}{4k_2k_3-1})\tanh(Ax) \tag{5.8}$$



(A) $k_1 = -8, k_2 = 1, k_3 = 1$ (B) $k_1 = -4, k_2 = 1, k_3 = 1$

FIGURE 3. Numerical solutions of $u = v$ for Problem 1 (k_2, k_3) fixed.

TABLE 6. L_∞ for $t = 1$, $\Delta t = 0.001$, $u(x,t)$ for Problem 2.

N	Present $(p = 1)$	[2] $(\lambda = 0)$	[2] (free parameter λ)
10	0.37323×10^{-5}	3.73505×10^{-5}	$0.00077 \times 10^{-5} (\lambda = 6 \times 10^{-5})$
100	0.37350×10^{-5}	3.73503×10^{-5}	$0.00078 \times 10^{-5} (\lambda = -4.087 \times 10^{-5})$

TABLE 7. L_∞ for $t = 1$, $\Delta t = 0.001$, $v(x,t)$ for Problem 2.

N	Present $(p = 1)$	[2] $(\lambda = 0)$	[2] (free parameter λ)
10	0.12569×10^{-5}	1.29030×10^{-5}	$0.00079 \times 10^{-5} (\lambda = -6 \times 10^{-5})$
100	0.12871×10^{-5}	1.29038×10^{-5}	$0.00079 \times 10^{-5} (\lambda = -4.087 \times 10^{-4})$

The second problem is run with the values of $a_0 = 0.05$, $k_1 = 2$, $k_2 = 1$, $k_3 = 0.3$ and $A = \frac{1}{2}\left[\frac{a_0(4k_2k_3 - 1)}{2k_2 - 1}\right]$, time step $\Delta t = 0.001$, $n = 10,\ 100$, $0 \leq x \leq 1$. Comparison of results of the presented method with those of [2] is given at time $t = 1$ in Tables 6-7 for $u(x,t)$ and $v(x,t)$ respectively. Although we get the fairly good results, extended Taylor-collocation method gives higher accuracy with the appropriate free parameters. To make further comparison with the parameters $N = 21$, $\Delta t = 0.01$, and $k_1 = 2$, results of the differential quadrature and modified cubic B-spline collocation method are documented in Tables 8-9 together with the our results when $p = 1$ for $u(x,t)$ and $v(x,t)$ respectively. And also numerical solutions for $u(x,t)$ and $v(x,t)$ $N = 21$, $\Delta t = 0.001$, $t = 1$, $0 \leq x \leq 1$, $k_2 = 0.1$ and $k_3 = 0.3$ are depicted in Figure 4. Suggested algorithm provides partially better accuracy

Problem 3) Consider the CBE (1.1-1.2) with the following initial conditions

$$u(x,0) = \begin{cases} \sin(2\pi x), & x \in [0, 0.5] \\ 0, & x \in (0.5, 1] \end{cases} \tag{5.9}$$

TABLE 8. L_∞ for $u(x,t)$, $N = 21$, $\Delta t = 0.01$ for Problem 2.

t	k_2	k_3	Present ($p = 1$)	[21]	[22]
0.5	0.1	0.3	8.8160×10^{-6}	4.173×10^{-5}	4.18921×10^{-5}
	0.3	0.03	9.2556×10^{-6}	4.585×10^{-5}	4.58483×10^{-5}
1.0	0.1	0.3	8.8878×10^{-6}	8.275×10^{-5}	8.26964×10^{-5}
	0.3	0.03	9.3324×10^{-6}	9.167×10^{-5}	9.14733×10^{-5}
3.0	0.1	0.3	8.9174×10^{-6}	2.408×10^{-4}	2.40120×10^{-4}
	0.1	0.03	9.3691×10^{-6}	2.747×10^{-4}	2.70420×10^{-4}

TABLE 9. L_∞ for $v(x,t)$, $N = 21$, $\Delta t = 0.01$ for Problem 2.

t	k_2	k_3	Present ($p = 1$)	[21]	[22]
0.5	0.1	0.3	2.8380×10^{-6}	5.418×10^{-5}	9.09474×10^{-6}
	0.3	0.03	1.1179×10^{-5}	2.826×10^{-5}	2.48218×10^{-5}
1.0	0.1	0.3	2.8686×10^{-6}	1.074×10^{-4}	1.69628×10^{-5}
	0.3	0.03	1.1269×10^{-5}	5.673×10^{-5}	4.96532×10^{-5}
3.0	0.1	0.3	2.9081×10^{-6}	3.119×10^{-4}	4.50548×10^{-5}
	0.1	0.03	1.1301×10^{-5}	1.663×10^{-4}	1.49831×10^{-5}

TABLE 10. L_∞ for $u(x,t)$, $N = 16$, $\Delta t = 0.01$ for Problem 2.

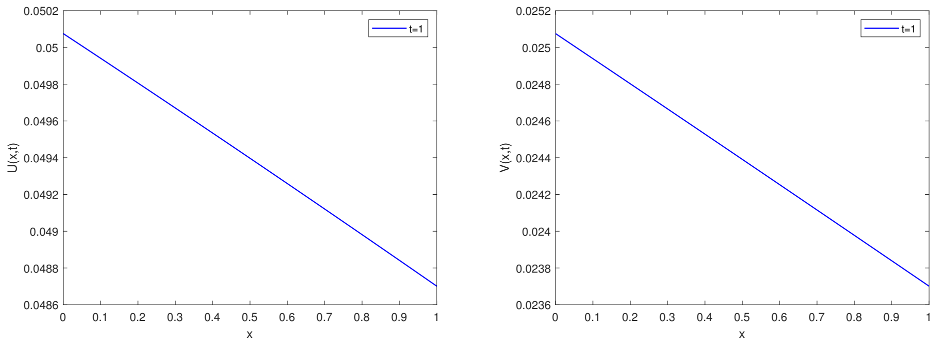
t	k_2	k_3	Suggested	[11]	[29]	[19]	[10]	[14]
0.5	0.1	0.3	8.830×10^{-6}	4.38×10^{-5}	9.618×10^{-4}	4.167×10^{-5}	4.108×10^{-5}	4.208×10^{-5}
0.5	0.3	0.03	9.270×10^{-6}	4.58×10^{-5}	4.310×10^{-4}	4.590×10^{-5}	4.285×10^{-5}	4.703×10^{-5}
1	0.1	0.3	8.901×10^{-6}	8.66×10^{-5}	1.152×10^{-3}	8.258×10^{-5}	8.201×10^{-5}	8.320×10^{-5}
1	0.3	0.03	9.346×10^{-6}	9.16×10^{-5}	1.268×10^{-3}	9.182×10^{-5}	8.873×10^{-5}	9.409×10^{-5}

TABLE 11. L_∞ for $v(x,t)$, $N = 16$, $\Delta t = 0.01$ for Problem 2.

t	k_2	k_3	Suggested	[11]	[29]	[19]	[10]	[14]
0.5	0.1	0.3	2.849×10^{-6}	4.99×10^{-5}	3.331×10^{-4}	1.480×10^{-4}	3.731×10^{-5}	0.221×10^{-4}
0.5	0.3	0.03	1.120×10^{-5}	1.81×10^{-4}	1.148×10^{-3}	5.729×10^{-4}	7.680×10^{-5}	1.818×10^{-4}
1	0.1	0.3	2.880×10^{-6}	9.92×10^{-5}	1.162×10^{-3}	4.770×10^{-5}	7.394×10^{-5}	4.255×10^{-5}
1	0.3	0.03	1.128×10^{-5}	3.62×10^{-4}	1.638×10^{-3}	3.617×10^{-4}	1.572×10^{-4}	6.636×10^{-4}

$$v(x,0) = \begin{cases} 0, & x \in [0, 0.5] \\ -\sin(2\pi x), & x \in (0.5, 1] \end{cases} \tag{5.10}$$

and zero boundary conditions. We deal with the problem, which it has no analytical solution found yet. The running of the program has been carried out on using the parameters $k_2 = k_3 = 10$ time step $\Delta t = 0.001$, number of partitions as 50 over the interval $0 \leq x \leq 1$. Maximum values of $u(x,t)$ and $v(x,t)$, and their positions x are tabulated at some times for comparison purpose in Tables 10-11 for $u(x,t)$ and $v(x,t)$ respectively. Our maximum values of solutions and their positions x are very close to maximum values found with [19, 22].



(A) (B)

FIGURE 4. Numerical solutions of $u(x,t)$ and $v(x,t)$ $N = 21$, $\Delta t = 0.001$, $k_2 = 0.1$ and $k_3 = 0.3$ for Problem 2.

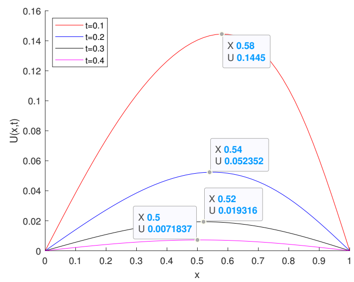
TABLE 12. Maximum values of $U(x,t)$ for $k_2 = 2, k_2 = k_3 = 10$ for Problem 3.

t	Present $(p = 1)$	[19]	[22]	at point
0.1	0.144501	0.14456	0.14449	0.58
0.2	0.052352	0.05237	0.05235	0.54
0.3	0.019316	0.01932	0.01931	0.52
0.4	0.007183	0.00718	0.00718	0.50

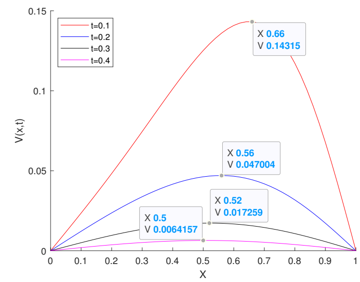
TABLE 13. Maximum values of $V(x,t)$ for $k_2 = 2, k_2 = k_3 = 10$ for Problem 3.

t	Present $(p = 1)$	[19]	[22]	at point
0.1	0.143155	0.14306	0.14314	0.66
0.2	0.047004	0.04697	0.04700	0.56
0.3	0.017259	0.01725	0.01726	0.52
0.4	0.006415	0.00641	0.00641	0.50

Visual solutions $u(x,t)$ and $v(x,t)$ are given at some times $t = 0.1, 0.2, 0.3$ in Figures 5-6 by using the parameters time step $\Delta t = 0.001$, $k_1 = 2$, $k_2 = k_3 = 10$ in the interval $[0,1]$ with 50 partitions. Decay of the solutions to zero can be observed as time increases. Computation has been repeated with parameters $k_1 = 2$, $k_2 = k_3 = 100$. Values are chosen to coincide with those used in the studies [19, 22] to compare the graphical solutions. Similar solution patterns are obtained with the studies.

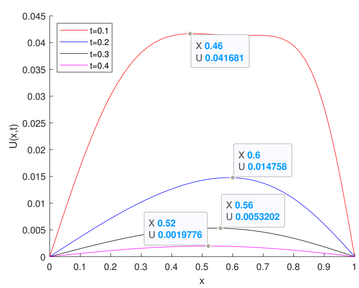


(A)

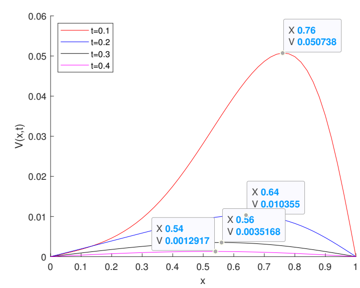


(B)

FIGURE 5. Numerical solutions of $u(x,t)$ and $v(x,t)$ at different time steps for $k_2 = k_3 = 10$ while $k_1 = 2$.



(A)



(B)

FIGURE 6. Numerical solutions of $u(x,t)$ and $v(x,t)$ at different time steps for $k_2 = k_3 = 100$ while $k_1 = 2$.

6. Conclusion

The ECB-spline collocation method has presented to get the numerical solutions of the CBE. The free parameter p in the ECB-spline is searched experimentally to get the best numerical solution. It is selected from the predetermined interval by taking small increments. The proposed method has produced less error than the methods listed in the tables for some test problems. The results are satisfactory and competent with some available solutions in the related literature. Another advantage is that the method can be used without the complex calculations to solve the system of differential equations reliably. Although the Galerkin method gives high accurate results than the collocation method to get solutions of the differential equations, it is shown that the present ECB-spline collocation method produces better results that the Galerkin methods with suitable selection of the free parameters. ECB-spline collocation method provides 7 banded algebraic equations which is easier to be solved, so that efficient algorithms are obtained to find the solution of the PDE.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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