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A new numerical fractional differentiation formula to approximate the Caputo-Fabrizio fractional derivative: error analysis and stability

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Abstract

In the present work, first of all, a new numerical fractional differentiation formula (called the CF2 formula) to approximate the Caputo-Fabrizio fractional derivative of order α, $(0<\alpha<1)$ is developed. It is established by means of the quadratic interpolation approximation using three points $(t_{j-2}, y(t_{j-2})), (t_{j-1}, y(t_{j-1}))$, and $(t_j, y(t_j))$ on each interval $[t_{j-1}, t_j]$ for $(j \geq 2)$, while the linear interpolation approximation are applied on the first interval $[t_0, t_1]$. As a result, the new formula can be formally viewed as a modification of the classical CF1 formula, which is obtained by the piecewise linear approximation for $y(t)$. Both the computational efficiency and numerical accuracy of the new formula is superior to that of the CF1 formula. The coefficients and truncation errors of this formula are discussed in detail. Two test examples show the numerical accuracy of the CF2 formula. The CF1 formula demonstrates that the new CF2 is much more effective and more accurate than the CF1 when solving fractional differential equations. Detailed stability analysis and region stability of the CF2 are also carefully investigated.

Keywords. Fractional differential equation, Stability, Caputo-Fabrizio fractional derivative, Numerical methods, Error analysis .

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1. Introduction

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order. The term fractional is a misnomer, but it is retained following the prevailing use. The fractional calculus may be considered an old and yet novel topic. It is an old topic since, starting from some speculations of Leibniz (1695, 1697) and Euler (1730). It has been developed up to nowadays [5]. Ross organized the first conference on fractional calculus and its applications at the University of New Haven in June 1974 and edited the proceedings, see [12]. Gorenflo and Mainardi [6], introduced the linear operators of fractional integration and fractional differentiation in the framework of the Riemann-Liouville fractional calculus. Particular attention is devoted to the technique of Laplace transforms for treating these operators in a way accessible to applied scientists, avoiding unproductive generalities and excessive mathematical rigor.

The authors [1] investigate the existence and uniqueness of solutions for a fractional boundary value problem involving four-point nonlocal Riemann-Liouville integral boundary conditions of a different order. The authors of [11] proposed a new fractional derivative without a singular kernel. Caputo and Fabrizio [4], proposed a new definition of the fractional derivative with a smooth kernel which takes on two different representations for the temporal and spatial variables. In [8], recently a new fractional differentiation was introduced to get rid of the singularity in the Riemann-Liouville and Caputo fractional derivative. The new fractional derivative has then generated a new class of ordinary differential equations. This class of fractional ordinary differential equations cannot be solved using a conventional Adams-Bashforth numerical scheme. In this paper, a new three-step fractional Adams-Bashforth scheme with the Caputo-Fabrizio derivative is formulated for the solution linear and nonlinear fractional differential equations.

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In [14], stability analysis of fractional-order nonlinear systems with delay is studied. The authors proposed the definition of Mittag-Leffler stability of the time-delay system and introduced the fractional Lyapunov direct method by using the properties of Mittag-Leffler function and Laplace transform. Then some new sufficient conditions ensuring asymptotical stability of the fractional-order nonlinear systems with delay is proposed firstly. The application of Riemann-Liouville fractional-order systems is extended by the fractional comparison principle and the Caputo fractional-order systems. The stability of linear fractional differential systems with commensurate order $1 < \alpha < 2$ and the corresponding perturbed systems is investigated. By using the Laplace transform, the asymptotic expansion of the Mittag-Leffler function, and the Gronwall inequality, some conditions on stability and asymptotic stability are given in [14]. The authors of [3] investigate the chaotic behavior and stability of fractional differential equations within a new generalized Caputo derivative. A semi-analytical method is proposed based on Adomian polynomials and a fractional Taylor series. Furthermore, the chaotic behavior of a fractional Lorenz system is numerically discussed. Since the fractional derivative includes two fractional parameters, chaos becomes more complicated than the one in Caputo fractional differential equations. Finally, Lyapunov stability is defined for the generalized fractional system. A sufficient condition of asymptotic stability is given and numerical results support the theoretical analysis.

The plan for the remainder of the paper is as follows. In section 2, a new numerical fractional differentiation formula(CF2-formula) for the Caputo-Fabrizio fractional derivative of order α is derived. The coefficient properties together with truncation error analysis of the formula are given. Section 3 deals with the stability analysis of the new formula(CF2-formula). The region of stability of the presented method is given in section 4. Two test examples are used to confirm the accuracy of the approximate solution of the new formula in section 5. Conclusions are given in the last Section.

2. Extraction of the new fractional numerical differentiation formula

In this section, we mainly explain the process of deriving the new numerical fractional differentiation formula for solving the following initial value problem where ${}_0^{CF}D_t^{\alpha}y(t)$, denotes the Caputo-Fabrizio fractional derivative.

$$ {}_0^{CF}D_t^{\alpha}y(t)\mid_{t=t_k} = f(t,y(t)), \quad 0 \le t \le T, \quad n-1 < \alpha \le n, \tag{2.1}$$

$$ y^{(k)}(t_0) = y_0^{(k)}, k = 0,1,...,[\alpha]-1. \tag{2.2}$$

Now define $t_k = k\Delta t, \quad t_{k+\frac{1}{2}} = (t_{k+1}+t_k)/2$, $k \ge 0$, wherever Δt is the time step duration. We present the following difference operators:

$$ \delta_t y_{k-\frac{1}{2}} = \frac{y(t_k)-y(t_{k-1})}{\Delta t}, \qquad \delta_t^2 y_k = \frac{1}{\Delta t}(\delta_t y_{k+\frac{1}{2}} - \delta_t y_{k-\frac{1}{2}}), \quad k \ge 1.$$

Assume $y(t) \in C^1[0,t_k], \quad (k \ge 0)$. From the definition of the Caputo-Fabrizio fractional derivative, for any α, $(0 < \alpha < 1)$, we have

$$ \begin{aligned} {}_0^{CF}D_t^{\alpha}y(t)\mid_{t=t_k} &= \frac{1}{1-\alpha}\int_0^{t_k} y'(x)\exp\left(\frac{-\alpha(t_k-x)}{1-\alpha}\right)dx \\ &= \frac{1}{1-\alpha}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j} y'(x)\exp\left(\frac{-\alpha(t_k-x)}{1-\alpha}\right)dx. \end{aligned} \tag{2.3}$$

on each interval $[t_{j-1},t_j]$ for $(1 \le j \le k)$, the piecewise Lagrange interpolation polynomial of degree one will be used to approximate of $y(t)$ as $\Pi_{1,j}y(t)$, i.e.,

$$ \Pi_{1,j}y(t) = y(t_{j-1})\frac{t_j-t}{\Delta t} + y(t_j)\frac{t-t_{j-1}}{\Delta t}. \tag{2.4}$$

It follows from the linear interpolation theory that (see theorem 2.1.4.1 in [11])

$$ y(t) - \Pi_{1,j}y(t) = \frac{y''(\zeta_j)}{2}(t-t_{j-1})(t-t_j), \quad t \in [t_{j-1},t_j], \zeta_j \in (t_{j-1},t_j), 1 \le j \le k. \tag{2.5}$$

For $j \geq 2$, we make a quadratic interpolation function $\Pi_{2,j}y(t)$ of $y(t)$ using three points $(t_{j-1}, y(t_{j-1})), (t_{j-2}, y(t_{j-2}))$ and $(t_j, y(t_j))$ and obtaining a constraint of the result onto interval $[t_{j-2}, t_j]$, we get

$$\begin{aligned}\Pi_{2,j}y(t) &= y(t_{j-2})\frac{(t-t_{j-1})(t-t_j)}{2\Delta t^2} + y(t_{j-1})\frac{(t-t_{j-2})(t_j-t)}{\Delta t^2} + y(t_j)\frac{(t-t_{j-1})(t-t_{j-2})}{2\Delta t^2}\\ &= \sum_{l=0}^{2} y(t_{j-l})\prod_{\substack{i=0\\ i\neq l}}^{2}\frac{t-t_{j-i}}{t_{j-l}-t_{j-i}} = \Pi_{1,j}y(t) + \frac{1}{2}(\delta_t^2 y_{j-1})(t-t_{j-1})(t-t_j), \quad t\in[t_{j-1}, t_j],\end{aligned}$$

where

$$\begin{aligned}(\Pi_{2,j}y(t))' &= \delta_t y_{j-\frac{1}{2}}\frac{t-t_{j-\frac{3}{2}}}{\Delta t} + \delta_t y_{j-\frac{3}{2}}\frac{t_{j-\frac{1}{2}}-t}{\Delta t}\\ &= \delta_t y_{j-\frac{1}{2}} + (\delta_t^2 y_{j-1})(t-t_{j-\frac{1}{2}}), \quad t\in[t_{j-1}, t_j],\end{aligned} \tag{2.6}$$

and

$$y(t) - \Pi_{2,j}y(t) = \frac{y'''(\eta_j)}{6}(t-t_{j-2})(t-t_{j-1})(t-t_j), \quad t\in[t_{j-1}, t_j], \tag{2.7}$$

wherever $\eta_j \in (t_{j-2}, t_j)$, $2 \leq j \leq k$. In (2.1), we use $\Pi_{1,1}y(t)$ to approximate $y(t)$ on the first interval $[t_0, t_1]$ and $\Pi_{2,j}y(t)$ to approximate $y(t)$ on the interval $[t_{j-1}, t_j]$ for $(j \geq 2)$. Now, consider

$$\int_{t_{j-1}}^{t_j} \exp\left(\frac{-\alpha(t_k-x)}{1-\alpha}\right)(x-t_{j-\frac{1}{2}})dx = \frac{(1-\alpha)\Delta t}{\alpha} b_{k-j}^{(\alpha)}, \qquad 2\leq j\leq k, \tag{2.8}$$

with

$$b_m^{(\alpha)} = \frac{1-\alpha}{\alpha\Delta t}\left(\exp\frac{-\alpha(m+1)\Delta t}{1-\alpha} - \exp\frac{-\alpha m\Delta t}{1-\alpha}\right) + \left(\frac{\exp\frac{-\alpha(m+1)\Delta t}{1-\alpha} + \exp\frac{-\alpha m\Delta t}{1-\alpha}}{2}\right), \quad m\geq 0,$$

By (2.2) and (2.4), we can take a new numerical approximation of the Caputo-Fabrizio fractional derivative of order α for function $y(t)$ in the following form:

$$\begin{aligned}{}_0^{CF}D_t^{\alpha}y(t)\mid_{t=t_k} &= \frac{1}{1-\alpha}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j} y'(x)\exp\left(\frac{-\alpha(t_k-x)}{1-\alpha}\right)dx\\ &\approx \frac{1}{1-\alpha}\left[\int_0^{t_1}(\Pi_{1,1}y(x))'\exp\left(\frac{-\alpha(t_k-x)}{1-\alpha}\right)dx\right.\\ &\left.+\sum_{j=2}^{k}\int_{t_{j-1}}^{t_j}(\Pi_{2,j}y(x))'\exp\left(\frac{-\alpha(t_k-x)}{1-\alpha}\right)dx\right]\\ &= \frac{1}{1-\alpha}\left[\int_{t_0}^{t_1}\delta_t y_{\frac{1}{2}}\exp\left(\frac{-\alpha(t_k-x)}{1-\alpha}\right)dx\right.\\ &\left.+\sum_{j=2}^{k}\int_{t_{j-1}}^{t_j}\exp\left(\frac{-\alpha(t_k-x)}{1-\alpha}\right)\left(\delta_t y_{j-\frac{1}{2}} + \delta_t^2 y_{j-1}(x-t_{j-\frac{1}{2}}\right)dx\right]\\ &= \frac{1}{1-\alpha}\left[\sum_{j=1}^{k}\delta_t y_{j-\frac{1}{2}}\int_{t_{j-1}}^{t_j}\exp\left(\frac{-\alpha(t_k-x)}{1-\alpha}\right)dx\right.\\ &\left.+\sum_{j=2}^{k}\delta_t^2 y_{j-1}\int_{t_{j-1}}^{t_j}\exp\left(\frac{-\alpha(t_k-x)}{1-\alpha}\right)(x-t_{j-\frac{1}{2}})dx\right]\\ &= {}_0^{CF1}D_t^{\alpha}y(t)\mid_{t=t_k} + \frac{\Delta t}{\alpha}\sum_{j=2}^{k} b_{k-j}^{(\alpha)}\delta_t^2 y_{j-1},\end{aligned}$$

wherever ${}_0^{CF1}D_t^{\alpha}y(t)$ is the classical $CF1$ operator which is derived from a piecewise linear interpolation approximation of $y(t)$ on each interval $[t_{j-1},t_j]$ defined by

$$
\begin{aligned}
{}_0^{CF1}D_t^{\alpha}y(t) &= \frac{1}{\alpha}\sum_{j=1}^{k} a_{k-j}^{(\alpha)}\delta_t y_{j-\frac{1}{2}} \\
&= \frac{1}{\alpha\Delta t}\sum_{j=1}^{k} a_{k-j}^{(\alpha)}(y_j - y_{j-1}) \\
&= \frac{1}{\alpha\Delta t}\left[a_0^{(\alpha)}y(t_k) - \sum_{j=1}^{k-1}(a_{k-j-1}^{(\alpha)} - a_{k-j}^{(\alpha)})y(t_j) - a_{k-1}^{(\alpha)}y(t_0)\right].
\end{aligned}
\tag{2.9}
$$

with

$$
a_j^{(\alpha)} = \left(\exp\frac{-\alpha j\Delta t}{1-\alpha} - \exp\frac{-\alpha(j+1)\Delta t}{1-\alpha}\right), \quad 0 \le j \le k-1.
$$

Now we define

$$
{}_0^{CF2}D_t^{\alpha}y(t)\mid_{t=t_k} := {}_0^{CF1}D_t^{\alpha}y(t)\mid_{t=t_k} + \frac{\Delta t}{\alpha}\sum_{j=2}^{k} b_{k-j}^{(\alpha)}\delta_t^2 y_{j-1}. \tag{2.10}
$$

The operator ${}_0^{CF2}D_t^{\alpha}y(t)$ is just the new numerical fractional differentiation operator for the Caputo-Fabrizio fractional derivative ${}_0^{CF}D_t^{\alpha}y(t)$ and we call it $CF2$ operator. Observing (2.9) and (2.10), we find that at $t = t_k$, the new $CF2$ operator ${}_0^{CF2}D_t^{\alpha}y(t)$ is actually a modification $CF1$ operator ${}_0^{CF1}D_t^{\alpha}y(t)\mid_{t=t_k}$ by adding a correction term $\frac{\Delta t}{\alpha}\sum_{j=2}^{k} b_{k-j}^{(\alpha)}\delta_t^2 y_{j-1}$ for $k \ge 2$. The following Lemma states the properties of coefficients $b_j^{(\alpha)}$ in the correction term.

Lemma 2.1. *For any α, $(0 < \alpha < 1)$, let*

$$
b_j^{(\alpha)} = \frac{1-\alpha}{\alpha\Delta t}\left(\exp\frac{-\alpha(j+1)\Delta t}{1-\alpha} - \exp\frac{-\alpha j\Delta t}{1-\alpha}\right) + \left(\frac{\exp\frac{-\alpha(j+1)\Delta t}{1-\alpha} + \exp\frac{-\alpha j\Delta t}{1-\alpha}}{2}\right), \quad j \ge 0.
$$

It holds

(1) $b_j^{(\alpha)} > 0$,

(2) $b_0^{(\alpha)} \ge b_1^{(\alpha)} \ge b_2^{(\alpha)} \ge \cdots$, *i.e.* $b_j^{(\alpha)}$ *is strictly monotone decreasing for* $j (j \ge 0)$.

Proof. From the definition of $b_j^{(\alpha)}$ and the error representation of the trapezoidal formula for function $y(x) = -\exp\frac{-\alpha x\Delta t}{1-\alpha}$ on the interval $[j, j+1]$, it gives

$$
\begin{aligned}
b_j^{(\alpha)} &= \int_j^{j+1}\exp\frac{-\alpha x\Delta t}{1-\alpha}dx - \frac{1}{2}[-\exp\frac{-\alpha(j+1)\Delta t}{1-\alpha} - \exp\frac{-\alpha j\Delta t}{1-\alpha}] \\
&= \frac{-1}{12}(-\exp\frac{-\alpha x\Delta t}{1-\alpha})''\mid_{x=\zeta_j} = \frac{1}{12}(\frac{\alpha\Delta t}{1-\alpha})^2\exp\frac{-\alpha\zeta_j\Delta t}{1-\alpha},
\end{aligned}
$$

where $j < \zeta_j < j+1$. It is easy to see that $b_j^{(\alpha)} > 0$ for $j \ge 0$. Besides, the second deduction of this Lemma can be seen from the monotone decreasing property of the function $\exp\frac{-\alpha x\Delta t}{1-\alpha}$ concerning x on $[0,\infty)$. $\square$

In addition, by (2.9), the new obtained fractional numerical differentiation formula (2.10), can be rewritten as

$$\begin{aligned}
{}_0^{CF2}D_t^{\alpha}y(t)\mid_{t=t_k} :&= {}_0^{CF1}D_t^{\alpha}y(t)\mid_{t=t_k} + \frac{\Delta t}{\alpha}\sum_{j=2}^{k} b_{k-j}^{(\alpha)}\delta_t^2 y_{j-1} \\
&= \frac{1}{\alpha}\sum_{j=1}^{k} a_{k-j}^{(\alpha)}\delta_t y_{j-\frac{1}{2}} + \frac{\Delta t}{\alpha}\sum_{j=2}^{k} b_{k-j}^{(\alpha)}\delta_t^2 y_{j-1} \\
&= \frac{1}{\alpha}\left[\sum_{j=1}^{k} a_{k-j}^{(\alpha)}\delta_t y_{j-\frac{1}{2}} + \sum_{j=2}^{k} b_{k-j}^{(\alpha)}\left(\delta_t y_{j-\frac{1}{2}} - \delta_t y_{j-\frac{3}{2}}\right)\right] \\
&= \frac{1}{\alpha}\left[\sum_{j=1}^{k} a_{k-j}^{(\alpha)}\delta_t y_{j-\frac{1}{2}} + \sum_{j=2}^{k} b_{k-j}^{(\alpha)}\delta_t y_{j-\frac{1}{2}} - \sum_{j=1}^{k-1} b_{k-j-1}^{(\alpha)}\delta_t y_{j-\frac{1}{2}}\right] \\
&= \frac{1}{\alpha}\sum_{j=1}^{k}(a_{k-j}^{(\alpha)} + b_{k-j}^{(\alpha)} - b_{k-j-1}^{(\alpha)})\delta_t y_{j-\frac{1}{2}} \\
&= \frac{1}{\alpha}\sum_{j=1}^{k} d_{k-j}^{(\alpha)}\delta_t y_{j-\frac{1}{2}},
\end{aligned}$$

$${}_0^{CF2}D_t^{\alpha}y(t)\mid_{t=t_k} = \frac{1}{\alpha\Delta t}\left[d_0^{(\alpha)}y(t_k) - \sum_{j=1}^{k-1}(d_{k-j-1}^{(\alpha)} - d_{k-j}^{(\alpha)})y(t_j) - d_{k-1}^{(\alpha)})y(t_0)\right], \tag{2.11}$$

wherever $d_0^{(\alpha)} = a_0^{(\alpha)} = 1$ for $k = 1$ and for $k \geq 2$,

$$d_j^{(\alpha)} = \begin{cases} a_0^{(\alpha)} + b_0^{(\alpha)}, & j = 0, \\ a_j^{(\alpha)} + b_j^{(\alpha)} - b_{j-1}^{(\alpha)}, & 1 \leq j \leq k-2, \\ a_j^{(\alpha)} - b_{j-1}^{(\alpha)}, & j = k-1. \end{cases} \tag{2.12}$$

Lemma 2.2. *For any α and $d_j^{(\alpha)}(0 \leq j \leq k-1, k \geq 3)$ defined in (2.12), it holds*

$(1) d_0^{(\alpha)} > |d_1^{(\alpha)}|, j \geq 0;$
$(2) d_j^{(\alpha)} > 0, j \neq 1;$
$(3) d_2^{(\alpha)} \geq_3^{(\alpha)} \geq ... \geq d_{k-1}^{(\alpha)};$
$(4) d_0^{(\alpha)} > d_2^{(\alpha)},$
$(5) \sum_{j=0}^{k-1} d_j^{(\alpha)} = -\exp\frac{-\alpha k\Delta t}{1-\alpha} + \frac{1}{2}\exp\frac{-\alpha(k-1)\Delta t}{1-\alpha} + \frac{1}{2}\exp\frac{-\alpha(k-2)\Delta t}{1-\alpha} + 1.$

Proof. For any α and $k \geq 3$, in view of the definition (2.12) of $d_j^{(\alpha)}$, we have

$$\begin{aligned}
d_0^{(\alpha)} &= a_0^{(\alpha)} + b_0^{(\alpha)} = (\frac{1-\alpha}{\alpha\Delta t} - \frac{1}{2})(\exp\frac{-\alpha\Delta t}{1-\alpha} - 1) + 1, \\
d_1^{(\alpha)} &= a_1^{(\alpha)} + b_1^{(\alpha)} - b_0^{(\alpha)} = (\frac{1-\alpha}{\alpha\Delta t} - \frac{1}{2})(\exp\frac{-\alpha\Delta t}{1-\alpha} - 1)^2.
\end{aligned}$$

Let $f(x) = (\frac{1-x}{x\Delta t} - \frac{1}{2})(\exp\frac{-x\Delta t}{1-x} - 1)^2$, then

$$f'(x) = \frac{-1}{x^2\Delta t}(\exp\frac{-x\Delta t}{1-x} - 1)^2 - \frac{2\Delta t}{(1-x)^2}\exp\frac{-x\Delta t}{1-x}(\exp\frac{-x\Delta t}{1-x} - 1)(\frac{1-x}{x\Delta t} - \frac{1}{2}) < 0, \quad x \in (0,1).$$

Therefore $f(x)$ is monotone decreasing on $(0,1)$ and then $-0.5 = d_1^{(1)} < d_1^{(\alpha)} < d_1^{(0)} = 0$, thus, $d_0^{(\alpha)} > |d_1^{(\alpha)}|$. i.e. (1) holds. Furtheremore for $2 \le j \le k-2$

$$\begin{aligned}
d_j^{(\alpha)} &= a_j^{(\alpha)} + b_j^{(\alpha)} - b_{j-1}^{(\alpha)} \\
&= \exp\frac{-\alpha j\Delta t}{1-\alpha} - \exp\frac{-\alpha(j+1)\Delta t}{1-\alpha} + \frac{1-\alpha}{\alpha\Delta t}[\exp\frac{-\alpha(j+1)\Delta t}{1-\alpha} - \exp\frac{-\alpha j\Delta t}{1-\alpha}] \\
&+ \frac{1}{2}[\exp\frac{-\alpha j\Delta t}{1-\alpha} + \exp\frac{-\alpha(j+1)\Delta t}{1-\alpha}] - \frac{1-\alpha}{\alpha\Delta t}[\exp\frac{-\alpha j\Delta t}{1-\alpha} - \exp\frac{-\alpha(j-1)\Delta t}{1-\alpha}] \\
&- \frac{1}{2}[\exp\frac{-\alpha(j-1)\Delta t}{1-\alpha} + \exp\frac{-\alpha j\Delta t}{1-\alpha}] \\
&= \{-\exp\frac{-\alpha(j+1)\Delta t}{1-\alpha} + \frac{1-\alpha}{\alpha\Delta t}[\exp\frac{-\alpha(j+1)\Delta t}{1-\alpha} - \exp\frac{-\alpha j\Delta t}{1-\alpha}] \\
&+ \frac{1}{2}[\exp\frac{-\alpha(j+1)\Delta t}{1-\alpha} + \exp\frac{-\alpha j\Delta t}{1-\alpha}]\} - \left\{-\exp\frac{-\alpha j\Delta t}{1-\alpha} + \frac{1-\alpha}{\alpha\Delta t}[\exp\frac{-\alpha j\Delta t}{1-\alpha} - \exp\frac{-\alpha(j-1)\Delta t}{1-\alpha}]\right. \\
&\left.+ \frac{1}{2}[\exp\frac{-\alpha j\Delta t}{1-\alpha} + \exp\frac{-\alpha(j-1)\Delta t}{1-\alpha}]\right\} \\
&= \left\{\frac{1-\alpha}{\alpha\Delta t}[\exp\frac{-\alpha(j+1)\Delta t}{1-\alpha} - \exp\frac{-\alpha j\Delta t}{1-\alpha}] - \frac{1}{2}[\exp\frac{-\alpha(j+1)\Delta t}{1-\alpha} - \exp\frac{-\alpha j\Delta t}{1-\alpha}]\right\} \\
&- \left\{\frac{1-\alpha}{\alpha\Delta t}[\exp\frac{-\alpha j\Delta t}{1-\alpha} - \exp\frac{-\alpha(j-1)\Delta t}{1-\alpha}] - \frac{1}{2}[\exp\frac{-\alpha j\Delta t}{1-\alpha} - \exp\frac{-\alpha(j-1)\Delta t}{1-\alpha}]\right\} \\
&:= I_j - I_{j-1},
\end{aligned}$$

with

$$\begin{aligned}
I_j &= \frac{1-\alpha}{\alpha\Delta t}[\exp\frac{-\alpha(j+1)\Delta t}{1-\alpha} - \exp\frac{-\alpha j\Delta t}{1-\alpha}] - \frac{1}{2}[\exp\frac{-\alpha(j+1)\Delta t}{1-\alpha} - \exp\frac{-\alpha j\Delta t}{1-\alpha}] \\
&= \left[\frac{1-\alpha}{\alpha\Delta t}\exp\frac{-\alpha(j+1)\Delta t}{1-\alpha} - \frac{1}{2}\exp\frac{-\alpha(j+1)\Delta t}{1-\alpha}\right] - \left[\frac{1-\alpha}{\alpha\Delta t}\exp\frac{-\alpha j\Delta t}{1-\alpha} - \frac{1}{2}\exp\frac{-\alpha j\Delta t}{1-\alpha}\right], \quad j \ge 1.
\end{aligned}$$

Let $g(x) = (\frac{1-\alpha}{\alpha\Delta t} - \frac{1}{2})\exp\frac{-\alpha x\Delta t}{1-\alpha}, \quad x \ge 1$, then $I_j = h(j+1) - h(j)$. For $x \ge 1$, it follows

$$\begin{aligned}
g'(x) &= -1 + \frac{\alpha\Delta t}{2(1-\alpha)}\exp\frac{-\alpha x\Delta t}{1-\alpha} < 0, \\
g''(x) &= (\frac{1-\alpha}{\alpha\Delta t} - \frac{1}{2})(\frac{\alpha\Delta t}{1-\alpha}^2\exp\frac{-\alpha x\Delta t}{1-\alpha} > 0, \\
g'''(x) &= (\frac{1}{2} - \frac{1-\alpha}{\alpha\Delta t})(\frac{\alpha\Delta t}{1-\alpha}^3\exp\frac{-\alpha x\Delta t}{1-\alpha} < 0.
\end{aligned}$$

Consequently,

$$\begin{aligned}
d_j^{(\alpha)} &= I_j - I_{j-1} = g(j+1) - 2g(j) + g(j-1) \\
&= g''(\xi_j) > 0, \quad j-1 < \xi_j < j+1, \\
d_j^{(\alpha)} - d_{j+1}^{(\alpha)} &= -g(j+2) + 3g(j+1) - 3g(j) + g(j-1) \\
&= -g'''(\theta_j) > 0, \quad j-1 < \xi_j < j+2.
\end{aligned}$$

So, it leads to $d_2^{(\alpha)} \ge d_3^{(\alpha)} \ge ... \ge d_{k-2}^{(\alpha)}$ and $d_{k-2}^{(\alpha)} > a_{k-1}^{(\alpha)} + b_{k-1}^{(\alpha)} - b_{k-2}^{(\alpha)} > a_{k-1}^{(\alpha)} - b_{k-2}^{(\alpha)} = d_{k-1}^{(\alpha)}$, where is used, i.e. (2) and (3) of this theorem are valid. Noticing that
$d_2^{(\alpha)} = I_2 - I_1 = g(3) - 2g(2) + g(1) = g''(\xi_2) < g''(1) < d_0^{(\alpha)}$, (4) is also apparent. The validity of (5) can be directly derived from the definition (2.12) of $d_j^{(\alpha)}$. The proof is completed. □

Now, truncation errors of the new CF2 formula (2.11) are illustrated in the following theorem.

Theorem 2.3.

$$|\hat{R}(y(t_1))| \leq \frac{1}{2\alpha} max_{t_0\leq t\leq t_1}|y''(t)|\Delta t, \tag{2.13}$$

and

$$|\hat{R}(y(t_k))| \leq \frac{\alpha}{(1-\alpha)^2}\Big\{\frac{1}{2} max_{t_0\leq t\leq t_1}|y''(t)|\Delta t^3 + \frac{1}{3} max_{t_0\leq t\leq t_k}|y''(t)|(k-2)\Delta t^4 + \frac{1}{6} max_{t_0\leq t\leq t_k}|y''(t)|(1-2(\frac{1-\alpha}{\alpha}))\Delta t^3\Big\}, k\geq 2. \tag{2.14}$$

Proof. We have

$$\begin{aligned}
\hat{R}(y(t_1)) &= \frac{1}{1-\alpha}\int_{t_0}^{t_1}[y(x)-\Pi_{1,1}y(x)]'\exp\frac{-\alpha(t_1-x)}{1-\alpha}dx \\
&= \frac{1}{1-\alpha}\int_{t_0}^{t_1}\exp\frac{-\alpha(t_1-x)}{1-\alpha}d[y(x)-\Pi_{1,1}y(x)]dx \\
&= \frac{1}{1-\alpha}\Big\{[y(x)-\Pi_{1,1}y(x)]\exp\frac{-\alpha(t_1-x)}{1-\alpha}|_{t_0}^{t_1} - \int_{t_0}^{t_1}\frac{\alpha}{1-\alpha}\exp\frac{-\alpha(t_1-x)}{1-\alpha}[y(x)-\Pi_{1,1}y(x)]dx\Big\} \\
&= \frac{1}{1-\alpha}\Big\{\frac{y''(\zeta_1)}{2}(x-t_0)(x-t_1)\exp\frac{-\alpha(t_1-x)}{1-\alpha}|_{t_0}^{t_1} - \int_{t_0}^{t_1}\frac{y''(\xi_1)}{2}\frac{\alpha}{1-\alpha}(x-t_0)(x-t_1)\exp\frac{-\alpha(t_1-x)}{1-\alpha}dx\Big\} \\
&= -\frac{1}{1-\alpha}\frac{y''(\eta_1)}{2}\int_{t_0}^{t_1}(x-t_0)(x-t_1)\frac{\alpha}{1-\alpha}\exp\frac{-\alpha(t_1-x)}{1-\alpha}dx \\
&= -\frac{1}{1-\alpha}\frac{y''(\eta_1)}{2}\Big[(x-t_0)(x-t_1)\exp\frac{-\alpha(t_1-x)}{1-\alpha} - (2x-(t_0+t_1))\frac{1-\alpha}{\alpha}\exp\frac{-\alpha(t_1-x)}{1-\alpha} \\
&\quad + 2(\frac{1-\alpha}{\alpha})^2\exp\frac{-\alpha(t_1-x)}{1-\alpha}\Big]_{x=t_0}^{x=t_1} \\
&= -\frac{1}{1-\alpha}\frac{y''(\eta_1)}{2}\Big[-\frac{(1-\alpha)}{\alpha}\Delta t(1+\exp\frac{-\alpha\Delta t}{1-\alpha}) + 2(\frac{1-\alpha}{\alpha})^2(1-\exp\frac{-\alpha\Delta t}{1-\alpha})\Big] \\
&= -\frac{1}{1-\alpha}\frac{y''(\eta_1)}{2}(\frac{1-\alpha}{\alpha}\Delta t + 3\Delta t^2 + \ldots) \\
&= -\frac{1}{2\alpha}y''(\eta_1)\Delta t,
\end{aligned}$$

where $\eta_1 \in (t_0, t_1)$. Hence, (2.13) holds. For $k \geq 2$, from (2.3) and (2.8), we get

$$\begin{aligned}
\hat{R}(y(t_k)) &= \frac{1}{1-\alpha}\Big[\int_{t_0}^{t_1}[y(x)-\Pi_{1,1}y(x)]'\exp\frac{-\alpha(t_k-x)}{1-\alpha}dx + \sum_{j=2}^{k}\int_{t_{j-1}}^{t_j}[y(x)-\Pi_{2,j}y(x)]'\exp\frac{-\alpha(t_k-x)}{1-\alpha}dx\Big] \\
&= \frac{1}{1-\alpha}\Big\{[y(x)-\Pi_{1,1}y(x)]\exp\frac{-\alpha(t_k-x)}{1-\alpha}|_{x=t_0}^{t_1} - \int_{t_0}^{t_1}\frac{\alpha}{1-\alpha}\exp\frac{-\alpha(t_k-x)}{1-\alpha}[y(x)-\Pi_{1,1}y(x)]dx \\
&\quad + \sum_{j=2}^{k}\Big[[y(x)-\Pi_{2,j}y(x)]\exp\frac{-\alpha(t_k-x)}{1-\alpha}|_{x=t_{j-1}}^{t_j} - \int_{t_{j-1}}^{t_j}\frac{\alpha}{1-\alpha}\exp\frac{-\alpha(t_k-x)}{1-\alpha}[y(x)-\Pi_{2,j}y(x)]dx\Big]\Big\} \\
&= \frac{-\alpha}{(1-\alpha)^2}\Big\{\int_{t_0}^{t_1}\exp\frac{-\alpha(t_k-x)}{1-\alpha}[y(x)-\Pi_{1,1}y(x)]dx + \sum_{j=2}^{k}\int_{t_{j-1}}^{t_j}\exp\frac{-\alpha(t_k-x)}{1-\alpha}[y(x)-\Pi_{2,j}y(x)]dx\Big\},
\end{aligned} \tag{2.15}$$

where (2.3), (2.5) and

$$[y(x)-\Pi_{1,1}y(x)]\exp\frac{-\alpha(t_k-x)}{1-\alpha}\mid_{x=t_0}^{t_1}=\frac{y''(\zeta_1)}{2}(x-t_0)(x-t_1)\exp\frac{-\alpha(t_k-x)}{1-\alpha}\mid_{x=t_0}^{t_1}=0,$$
$$[y(x)-\Pi_{2,k}y(x)]\exp\frac{-\alpha(t_k-x)}{1-\alpha}\mid_{x=t_{k-1}}^{t_k}=\frac{y'''(\eta_k)}{6}(x-t_{k-2})(x-t_{k-1})(x-t_k)\mid_{x=t_{k-1}}^{t_k}=0,$$

are used.By (2.5), it follows

$$\begin{aligned}
|\int_{t_0}^{t_1}[y(x)-\Pi_{1,1}y(x)]\exp\frac{-\alpha(t_k-x)}{1-\alpha}dx| &= |\int_{t_0}^{t_1}\frac{y''(\zeta_1)}{2}(x-t_0)(x-t_1)\exp\frac{-\alpha(t_k-x)}{1-\alpha}dx| \\
&= |\frac{y''(\eta_1)}{2}\int_{t_0}^{t_1}(x-t_0)(x-t_1)\exp\frac{-\alpha(t_k-x)}{1-\alpha}dx| \\
&\le \frac{1}{2}|y''(\eta_1)|\Delta t^2\int_{t_0}^{t_1}\exp\frac{-\alpha(t_k-x)}{1-\alpha}dx \\
&\le \frac{1}{2}|y''(\eta_1)|(\frac{(1-\alpha}{\alpha})\Delta t^2\exp\frac{-\alpha(t_k-x)}{1-\alpha}|_{x=t_0}^{t_1} \\
&\le (\frac{(1-\alpha}{2\alpha})\Delta t^2|y''(\eta_1)|\left\{\exp\frac{-\alpha(t_k-t_1)}{1-\alpha}-\exp\frac{-\alpha(t_k-t_0)}{1-\alpha}\right\} \\
&\le (\frac{1-\alpha}{2\alpha})\Delta t^2|y''(\eta_1)|\left\{\exp\frac{-\alpha(k-1)\Delta t}{1-\alpha}-\exp\frac{-\alpha k\Delta t}{1-\alpha}\right\} \\
&\le \frac{1}{2}|y''(\eta_1)|\Delta t^3, \qquad (2.16)
\end{aligned}$$

where $\eta_1\in(t_0,t_1)$ and

$$\begin{aligned}
|\sum_{j=2}^{k-1}\int_{t_{j-1}}^{t_j}[y(x)-\Pi_{2,j}y(x)]\exp\frac{-\alpha(t_k-x)}{1-\alpha}dx| &= |\sum_{j=2}^{k-1}\int_{t_{j-1}}^{t_j}\frac{y'''(\eta_j)}{6}(x-t_{j-2})(x-t_{j-1})(x-t_j)\exp\frac{-\alpha(t_k-x)}{1-\alpha}dx| \\
&= \frac{1}{6}|\sum_{j=2}^{k-1}y'''(\vartheta_j)\int_{t_{j-1}}^{t_j}(x-t_{j-2})(x-t_{j-1})(x-t_j)\exp\frac{-\alpha(t_k-x)}{1-\alpha}dx| \\
&= \frac{1}{6}|y'''(\vartheta)|\sum_{j=2}^{k-1}\int_{t_{j-1}}^{t_j}(x-t_{j-2})(x-t_{j-1})(x-t_j)\exp\frac{-\alpha(t_k-x)}{1-\alpha}dx \\
&\le \frac{1}{3}|y'''(\vartheta)|\Delta t^3\sum_{j=2}^{k-1}\int_{t_{j-1}}^{t_j}\exp\frac{-\alpha(t_k-x)}{1-\alpha}dx \\
&\le \frac{1}{3}|y'''(\vartheta)|\Delta t^3\int_{t_1}^{t_{k-1}}\exp\frac{-\alpha(t_k-x)}{1-\alpha}dx \\
&\le \frac{1-\alpha}{3\alpha}|y'''(\vartheta)|\Delta t^3(\exp\frac{-\alpha\Delta t}{1-\alpha}-\exp\frac{-\alpha(k-1)\Delta t}{1-\alpha}) \\
&\le \frac{1}{3}|y'''(\vartheta)|(k-2)\Delta t^4, \qquad (2.17)
\end{aligned}$$

where $\vartheta_j \in (t_{j-2}, t_j)$, $2 \leq j \leq k-1$, $\vartheta \in (t_0, t_{k-1})$. In addition,

$$\begin{aligned}
\int_{t_{k-1}}^{t_k} [y(x) - \Pi_{2,k}y(x)] \exp\frac{-\alpha(t_k - x)}{1-\alpha} dx &= \int_{t_{k-1}}^{t_k} \frac{y'''(\eta_k)}{6}(x - t_{k-2})(x - t_{k-1})(x - t_k) \exp\frac{-\alpha(t_k - x)}{1-\alpha} dx \\
&= \frac{1}{6} y'''(\vartheta_k) \int_{t_{k-1}}^{t_k} (x - t_{k-2})(x - t_{k-1})(x - t_k) \exp\frac{-\alpha(t_k - x)}{1-\alpha} dx \\
&\leq \frac{1}{6} y'''(\vartheta_k) \Delta t \int_{t_{k-1}}^{t_k} (x - t_{k-2})(x - t_k) \exp\frac{-\alpha(t_k - x)}{1-\alpha} dx \\
&\leq \frac{1}{6} y'''(\vartheta_k) \Delta t \Big\{ \Big(\frac{1-\alpha}{\alpha}(x - t_{k-2})(x - t_k) \\
&- (2x - (t_{k-2} + t_k))(\frac{1-\alpha}{\alpha})^2 + 2(\frac{1-\alpha}{\alpha})^3 \Big) \exp\frac{-\alpha(t_k - x)}{1-\alpha} |_{x=t_{k-1}}^{t_k} \Big\} \\
&= \frac{1}{6} y'''(\vartheta_k)(1 - 2(\frac{1-\alpha}{\alpha}))\Delta t^3, \qquad (2.18)
\end{aligned}$$

$\vartheta_k \in (t_{k-2}, t_k)$. The substitution of (2.16), (2.17) and (2.18) into (2.15) will lead to (2.14). The proof ends. □

3. Stability analysis of CF_2 new formula

Stability analysis of the new formula considering the fact that, a numerical method is stable if small change in the initial conditions results only small changes in the computed solution [2], we deal with the stability analysis of the new scheme. Assume that y_k and $\hat{y}_k$ $\quad (k = 1, 2, \cdots, N)$

are two solutions of the formula (2.11) with different initial values $y_0^{(i)}$ and $\hat{y_0}^{(i)} (i = 0, 1, \cdots, [\alpha] - 1)$ respectively. Then the presented method is stable if there exists a positive constant $C_{\alpha,T}$ independent of h and k, such that [7]

$$|y_k - \hat{y}_k| \leq C_{\alpha,T} \sum_{i=0}^{[n]-1} |y_0^{(i)} - \hat{y_0}^{(i)}|, \quad k = 0, \cdots, N.$$

Theorem 3.1. *Let y_{k+1} and $\tilde{y_{k+1}}$ are numerical solutions for (2.1), which the initial conditions are given by $y_0^{(i)}$ and $\tilde{y_0}^{(i)}$ respectively. Then*

$$|y_{k+1} - \tilde{y_{k+1}}| \leq C_{\alpha,T} ||y_0 - \tilde{y_0}||_\infty, \qquad (3.1)$$

i.e. the new CF2 (2.11) is numerically stable.

To prove this theorem, we need the following Lemma.

Lemma 3.2. *Set $d_j^{(\alpha)}$ be introduced in (2.11), then there is a constant C such that*

$$\sum_{j=0}^{k+1} |d_j^{(\alpha)}| \leq CT. \qquad (3.2)$$

Proof. We first derive the estimate for $|d_0|$

$$\begin{aligned}
|d_0^{(\alpha)}| &= |\int_{t_0}^{t_1} \exp\frac{-\alpha(t_{k+1} - \tau)}{1-\alpha} (\frac{\tau - t_1}{t_0 - t_1})' d\tau + \int_{t_1}^{t_2} \exp\frac{-\alpha(t_{k+1} - \tau)}{1-\alpha} (\frac{\tau - t_1}{t_0 - t_1} \frac{\tau - t_2}{t_0 - t_2})'| \\
&= |\int_{t_0}^{t_1} \exp\frac{-\alpha(t_{k+1} - \tau)}{1-\alpha} \frac{1}{t_0 - t_1} d\tau + \int_{t_1}^{t_2} \exp\frac{-\alpha(t_{k+1} - \tau)}{1-\alpha} \frac{(\tau - t_2) + (\tau - t_1)}{(t_0 - t_1)(t_0 - t_2)} d\tau|.
\end{aligned}$$

By utilizing the integral mean value theorem, for $\tilde{\tau}_j \in [t_{j-1}, t_j]$, $j = 1, 2$, the above equation can be rewritten as follows:

$$\begin{aligned}
|d_0^{(\alpha)}| &\le |\frac{1}{-h}| \int_{t_0}^{t_1} \exp\frac{-\alpha(t_{k+1}-\tau)}{1-\alpha} d\tau + |\frac{(\tilde{\tau}_2 - t_2) + (\tilde{\tau}_2 - t_1)}{(t_0 - t_1)(t_0 - t_2)}| \int_{t_1}^{t_2} \exp\frac{-\alpha(t_{k+1}-\tau)}{1-\alpha} d\tau \\
&\le \frac{1}{h}\frac{1-\alpha}{\alpha}[(\exp\frac{-\alpha(t_{k+1}-t_1)}{1-\alpha} - \exp\frac{-\alpha(t_{k+1}-t_0)}{1-\alpha}) + (\exp\frac{-\alpha(t_{k+1}-t_2)}{1-\alpha} - \exp\frac{-\alpha(t_{k+1}-t_1)}{1-\alpha})] \\
&\le \frac{1}{h}\frac{1-\alpha}{\alpha}[\exp\frac{-\alpha t_{k-1}}{1-\alpha} - \exp\frac{-\alpha t_{k+1}}{1-\alpha}] \\
&\le \frac{1}{h} t_2.
\end{aligned}$$

In a similar way, we can derive $|d_1^{(\alpha)}| \le CT, |d_k^{(\alpha)}| \le CT$ and $|d_{k+1}^{(\alpha)}| \le CT$, where C have dissimilar values at different formulae. For $j = 2, \cdots, k-1$, we have

$$\begin{aligned}
\sum_{j=2}^{k-1} |d_j^{(\alpha)}| \le \sum_{j=2}^{k-1} [&|\frac{\tilde{\tau}_1 - t_{j-2}}{2h}\frac{\tilde{\tau}_1 - t_{j-1}}{h}| \int_{t_{j-1}}^{t_j} \exp\frac{-\alpha(t_{k+1}-\tau)}{1-\alpha} d\tau \\
&+ |\frac{\tilde{\tau}_2 - t_{j-1}}{h}\frac{\tilde{\tau}_2 - t_{j+1}}{-h}| \int_{t_j}^{t_{j+1}} \exp\frac{-\alpha(t_{k+1}-\tau)}{1-\alpha} d\tau \\
&+ |\frac{\tilde{\tau}_3 - t_{j-+1}}{-h}\frac{\tilde{\tau}_3 - t_{j+2}}{-2h}| \int_{t_{j+1}}^{t_{j+2}} \exp\frac{-\alpha(t_{k+1}-\tau)}{1-\alpha} d\tau],
\end{aligned}$$

where $\tilde{\tau}_1 \in [t_{j-1}, t_j]$, $\tilde{\tau}_2 \in [t_j, t_{j+1}]$ and $\tilde{\tau}_3 \in [t_{j+1}, t_{j+2}]$. Hence, above equation has the simple form

$$\begin{aligned}
\sum_{j=2}^{k-1} |d_j^{(\alpha)}| \le& |\frac{2h^2}{2h^2}| \sum_{j=2}^{k-1} [\frac{1-\alpha}{\alpha} \exp\frac{-\alpha(t_{k+1}-t_j)}{1-\alpha} - \exp\frac{-\alpha(t_{k+1}-t_{j-1})}{1-\alpha}]] \\
&+ |\frac{2h^2}{-h^2}| \sum_{j=2}^{k-1} [\frac{1-\alpha}{\alpha} \exp\frac{-\alpha(t_{k+1}-t_{j+1})}{1-\alpha} - \exp\frac{-\alpha(t_{k+1}-t_j)}{1-\alpha}] \\
&+ |\frac{h^2}{2h^2}| \sum_{j=2}^{k-1} [\frac{1-\alpha}{\alpha} \exp\frac{-\alpha(t_{k+1}-t_{j+2})}{1-\alpha} - \exp\frac{-\alpha(t_{k+1}-t_{j+1})}{1-\alpha}] \\
=& \frac{1-\alpha}{\alpha}[(\exp\frac{-\alpha(t_{k+1}-t_{k-1})}{1-\alpha} - \exp\frac{-\alpha(t_{k+1}-t_1)}{1-\alpha}) \\
&+ 2(\exp\frac{-\alpha(t_{k+1}-t_k)}{1-\alpha} - \exp\frac{-\alpha(t_{k+1}-t_2)}{1-\alpha}) + \frac{1}{2}(1 - \exp\frac{-\alpha(t_{k+1}-t_3)}{1-\alpha})] \\
\le& t_{k-2} + 2t_{k-2} + \frac{1}{2}t_{k-2} = \frac{7}{2}t_{k-2} = \frac{7}{2}T.
\end{aligned}$$

Due to the values of d_k and d_{k+1} in (2.12), it is obvious that

$$d_k \le CT, \quad d_{k+1} \le CT.$$

In summary, combining all the above results, by choosing sufficiently large C one can reach the estimate of (3.2). $\square$

4. Stability region

Consider the following test problem to investigate stability region of the proposed numerical method:

$${}_0^{CF}D_t^{\alpha} y(t) \mid_{t=t_k} = \lambda y(t), \quad y(t_0) = y_0, \quad 0 < \alpha < 1. \tag{4.1}$$

The new method gives the following new numerical fractional differentiation the formulae for solving the test problem:

$${}_0^{CF_2}D_t^{\alpha} y(t_k) = \lambda y(t_k),$$

$$\frac{1}{\alpha h}\left[d_0^{(\alpha)}y(t_k)-\sum_{j=1}^{k-1}(d_{k-j-1}^{(\alpha)}-d_{k-j}^{(\alpha)})y(t_j)-d_{k-1}^{(\alpha)})y(t_0)\right]=\lambda y(t_k),$$

$$d_0^{(\alpha)}y_k-\sum_{j=1}^{k-1}(d_{k-j-1}^{(\alpha)}-d_{k-j}^{(\alpha)})y_j-d_{k-1}^{(\alpha)}y_0=\alpha\lambda h y_k, \tag{4.2}$$

Assume that $z=\lambda h$. Then we have

$$z=\frac{\sum_{j=0}^{k}\gamma_j^{(\alpha)}y_j}{\alpha y_k}, \tag{4.3}$$

where

$$\gamma_j^{(\alpha)}=\begin{cases}-d_{k-1}^{(\alpha)}, & j=0,\\ (d_{k-j-1}^{(\alpha)}-d_{k-j}^{(\alpha)}), & 1\le j\le k-1,\\ d_0^{(\alpha)}, & j=k.\end{cases}$$

Let $y_k=\xi^k$, then by assuming $\xi=e^{i\theta}$ with $0\le\theta\le 2\pi$ we get the stability region

$$S=\{z:z=\frac{\sum_{j=0}^{k}\gamma_j^{(\alpha)}\xi^j}{\alpha\xi^k}\} \tag{4.4}$$

In Fig. 4, we present stability region (the gray areas) of the new method (2.11) at $\alpha=0.1,0.3,0.5,0.7$ for $N=160$. In this figures, we can observe that, as increases α, the stability region of the new method (2.11) increases.

5. Numerical results and discussion

Now, with some examples, let's examine the accuracy of the obtained formulas. Take a positive integer N, let $T_0=1$, $\Delta t=T_0/N=1/N$ and denote

$$E^N(CF_1)=|{}_0^{CF}D_t^{\alpha}y(t)\mid_{t=t_k}-D_{1t}^{\alpha}y(t)\mid_{t=t_k}|, 0\le k\le N,$$
$$E^N(CF_2)=|{}_0^{CF}D_t^{\alpha}y(t)\mid_{t=t_k}-D_{2t}^{\alpha}y(t)\mid_{t=t_k}|, 0\le k\le N.$$

Example 5.1. Suppose $0<\alpha<1$. Consider the following fractional differential equation of order α

$${}_0^{CF}D_t^{\alpha}y(t)=\frac{2\left(\exp\left(\frac{\alpha t}{\alpha-1}\right)-\exp(2t)\right)}{\alpha-2}$$

The exact solution is given by $y(t)=\exp(2t)$. Taking different temporal stepsizes $\Delta t=1/10,1/20,1/40,1/80,1/160,1/320,1/640,1/1280$, we compute the example by the formulae (2.9) and (2.11) respectively. Table (1) lists the computational results at $t_N=T_0=1$ with different parameters $\alpha=0.9,0.5,0.1$. From the results presented in Table (1), we find that the computational errors by the formula (2.11) are much smaller than that by the formula (2.9).

Example 5.2. Consider the following initial value problem for $0<\alpha<1$:

$$\begin{cases}{}_0^{CF}D_t^{\alpha}y(t)-y(t)=-\exp\frac{\alpha t}{\alpha-1}, & t\in[0,1]\\ y(0)=1.\end{cases}$$

The exact solution to this initial value problem is $y(t)=\exp(t)$. The interval $[0,1]$ is divided into $N+1$ equi-spaced nodes t_k, given by $t_k=k\Delta t$, $\quad k=0,1,2,\dots,N$, in which $\Delta t=1/N$ denotes the time step size, where y_k is the numerical approximation to $y(t_k)$. For these points we have the above initial value problem:

$$\begin{cases}{}_0^{CF}D_t^{\alpha}y_k-y_k=-\exp\frac{\alpha t_k}{\alpha-1}, & t_k\in[0,1],\\ y_0=1.\end{cases}$$

By setting the approximation obtained by formula (2.11) for the Caputo-Fabrizio, a system of linear equations with unknown values y_k achieved. By solving this linear equation, the approximate values for the problem at node points t_k was achieved. In Table 1, we list the absolute error of the proposed method (2.11) at some node points t_k for various values of α. From the results presented in Table (2), the accuracy of the approximate solution increases by increasing the number of nodes points t_k. We compared the results of the presented formula (CF2) with the errors of the numerical approximation reported in [9] in the examples 3-5 listed below with their numerical simulations recorded in Tables 3-5. (The absolute errors of the CF2 formula and the scheme of [13] in the examples 3-5 are shown in Tables 3-5 and they are compared for different values of h and α for $t = 1$)

Consider the following fractional order differential equations for $0 < \alpha < 1$:

Example 5.3.

$$\begin{cases} {}_0^{CF}D_t^{\alpha} y(t) = \frac{-\alpha \exp\left(\frac{\alpha}{\alpha-1}t\right) + \alpha cos(t) + (1-\alpha) sin(t)}{1-2\alpha+2\alpha^2}, & t \in [0,1] \\ y(0) = 0, \end{cases}$$

The exact solution to this initial value problem is $y(t) = sin(t)$.

Example 5.4.

$$\begin{cases} {}_0^{CF}D_t^{\alpha} y(t) = \exp\left(\frac{\alpha}{\alpha-1}t\right)[-\alpha + (\alpha+t)\exp\left(\frac{1}{1-\alpha}t\right)], & t \in [0,1] \\ y(0) = 0, \end{cases}$$

The exact solution to this initial value problem is $y(t) = texp(t)$.

Example 5.5.

$$\begin{cases} {}_0^{CF}D_t^{\alpha} y(t) = \frac{-\alpha \exp\left(\frac{\alpha}{\alpha-1}t\right) + \alpha cos(2t) - 2(\alpha-1) sin(2t)}{4-8\alpha+5\alpha^2}, & t \in [0,1] \\ y(0) = 0, \end{cases}$$

The exact solution to this initial value problem is $y(t) = sin(t)cos(t)$.

It is note worthy that the error of the presented scheme is always smaller than the error of literature in all given cases. So the new formula is more accurate.

6. Conclusion

A new fractional numerical differentiation formula (called the CF2 formula) to approximate the Caputo-Fabrizio fractional derivative of order α is established. The new formula is obtained by a piecewise quadratic interpolation approximation for the integrand $y(t)$, in detail, by the constraint on to the small interval $[t_{j-1}, t_j]$ of the quadratic interpolation approximation using the three points $(t_{j-2}, y(t_{j-2}))$, $(t_{j-1}, y(t_{j-1}))$ and $(t_j, y(t_j))$, $j \geq 2$. As a result, the new CF2 formula can be viewed formally as the modification of the CF1 formula by adding some correction terms when $k \geq 2$. We then make some analysis for the coefficient features and truncation errors of the resulting CF2 formula. Two test examples are carried out to effectively confirm the computational validity and numerical accuracy of the CF2 formula. Stability analysis of the presented new fractional numerical differentiation formula (called the CF2 formula) to approximate the Caputo-Fabrizio fractional derivative is investigated. The stability region of the new formula for different values of N is achieved. Numerical simulations are carried out for a hypothetical set of parameter values to validate the new numerical scheme and substantiate our analytical findings.

TABLE 1. The absolute errors whit different temporal stepsizes for example (5.1).

α	Δt	$E^N(CF_1)$	$E^N(CF_2)$
0.9	1/10	0.196395	0.0188106
	1/20	0.0500519	0.00244953
	1/40	0.0125742	0.000311033
	1/80	0.00314741	$3.91377e-005$
	1/160	0.00087093	$4.90696e-006$
	1/320	0.000196788	$6.14246e-007$
	1/640	$4.9198e-005$	$7.68341e-008$
	1/1280	$1.22996e-005$	$9.60755e-009$
0.5	1/10	0.0155689	0.00171214
	1/20	0.00389854	0.000219741
	1/40	0.000975031	$2.78359e-005$
	1/80	0.000243783	$3.50283e-006$
	1/160	$6.09472e-005$	$4.39323e-007$
	1/320	$1.52369e-005$	$5.50076e-008$
	1/640	$3.80923e-006$	$6.88171e-009$
	1/1280	$9.52308e-007$	$8.60569e-010$
0.1	1/10	0.0012649	0.000154894
	1/20	0.000316418	$1.98788e-005$
	1/40	$7.91165e-005$	$2.51862e-006$
	1/80	$1.97799e-005$	$3.16986e-007$
	1/160	$4.94502e-006$	$3.97595e-008$
	1/320	$1.23626e-006$	$4.9785e-009$
	1/640	$3.09065e-007$	$6.22866e-010$
	1/1280	$7.72662e-008$	$7.79181e-011$

TABLE 2. The absolute errors of example (5.2) by the proposed method (2.11) at some node points t_k for $\alpha = 0.1$

$\alpha = 0.1$	$N = 10$	$N = 20$	$N = 40$
t_k	$\lvert y(t_k) - y_k \rvert$	$\lvert y(t_k) - y_k \rvert$	$\lvert y(t_k) - y_k \rvert$
0	0.00×10^0	0.00×10^0	0.00×10^0
0.1	1.02×10^{-4}	2.56×10^{-5}	6.40×10^{-6}
0.2	2.27×10^{-4}	5.66×10^{-5}	1.41×10^{-5}
0.3	3.76×10^{-4}	9.38×10^{-5}	2.34×10^{-5}
0.4	5.53×10^{-4}	1.38×10^{-4}	3.45×10^{-5}
0.5	7.65×10^{-4}	1.91×10^{-4}	4.77×10^{-5}
0.6	1.01×10^{-3}	2.53×10^{-4}	6.33×10^{-5}
0.7	1.31×10^{-3}	3.26×10^{-4}	8.16×10^{-5}
0.8	1.65×10^{-3}	4.12×10^{-4}	1.03×10^{-4}
0.9	2.05×10^{-3}	5.13×10^{-4}	1.28×10^{-4}
1	2.52×10^{-3}	6.30×10^{-4}	1.57×10^{-4}

TABLE 3. Absolute errors of the present scheme ($CF2$) and the numerical method of [9] for Example 5.3.

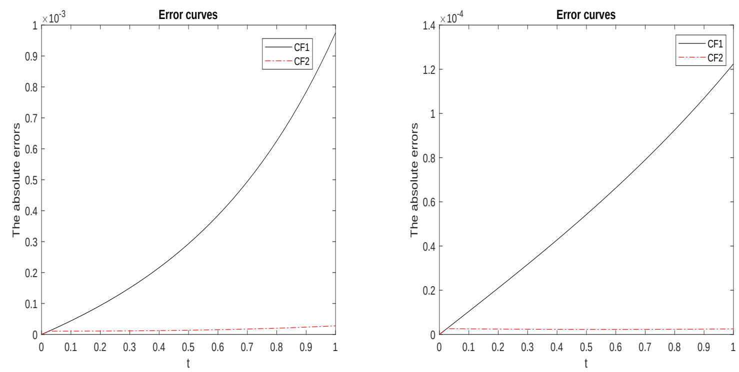
	$\alpha = 0.25$		$\alpha = 0.5$		$\alpha = 0.75$	
h	$E(CF2)$	$E[9]$	$E(CF2)$	$E[9]$	$E(CF2)$	$E[9]$
10^{-2}	$1.30747e-08$	$4.1345e-03$	$4.23617e-08$	$7.4921e-04$	$1.16139e-07$	$6.0877e-03$
10^{-3}	$1.30472e-11$	$4.0793e-04$	$4.2256e-11$	$6.7530e-05$	$1.157e-10$	$6.1385e-04$

TABLE 4. Absolute errors of the present scheme $(CF2)$ and the numerical method of [9] for Example 5.4.

	$\alpha = 0.25$		$\alpha = 0.5$		$\alpha = 0.75$	
h	$E(CF2)$	$E[9]$	$E(CF2)$	$E[9]$	$E(CF2)$	$E[9]$
10^{-2}	$1.53017e-07$	$6.1334e-02$	$4.79786e-07$	$6.9383e-02$	$1.35355e-06$	$7.5564e-02$
10^{-3}	$1.53342e-10$	$6.1716e-03$	$4.80629e-10$	$6.9755e-03$	$1.35635e-09$	$7.5842e-03$

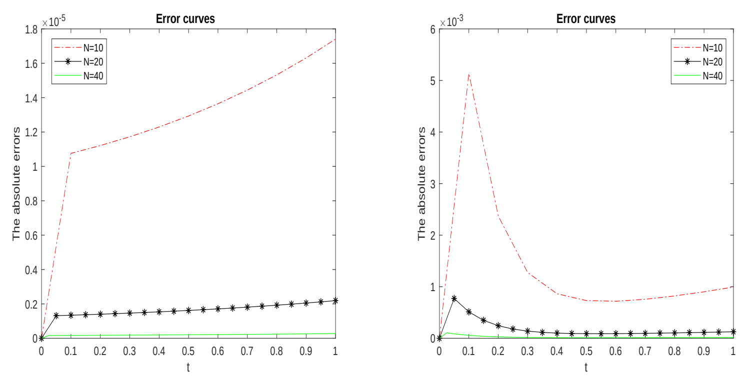
TABLE 5. Absolute errors of the present scheme $(CF2)$ and the numerical method of [12] for Example 5.5.

	$\alpha = 0.25$		$\alpha = 0.5$		$\alpha = 0.75$	
h	$E(CF2)$	$E[9]$	$E(CF2)$	$E[9]$	$E(CF2)$	$E[9]$
10^{-2}	$2.63062e-08$	$7.0012e-03$	$7.02057e-08$	$1.2357e-02$	$6.8956e-08$	$2.0529e-02$
10^{-3}	$2.59975e-11$	$7.0960e-04$	$6.90947e-11$	$1.2451e-03$	$6.51574e-11$	$2.0530e-03$



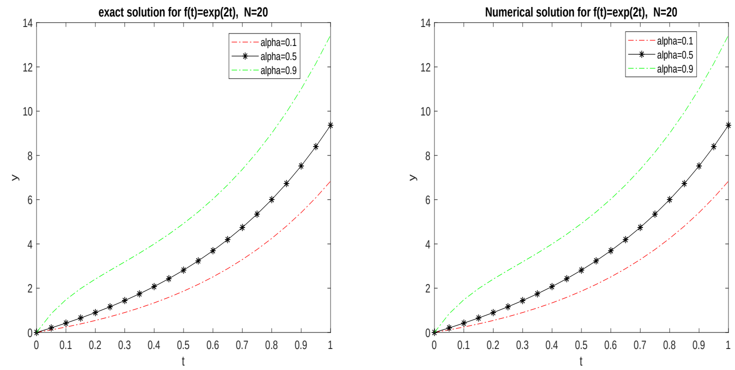
(a) (b)

FIGURE 1. The absolute errors with $\alpha = 0.5$, $N = 40$ ((a): example5.1; (b): example5.2).



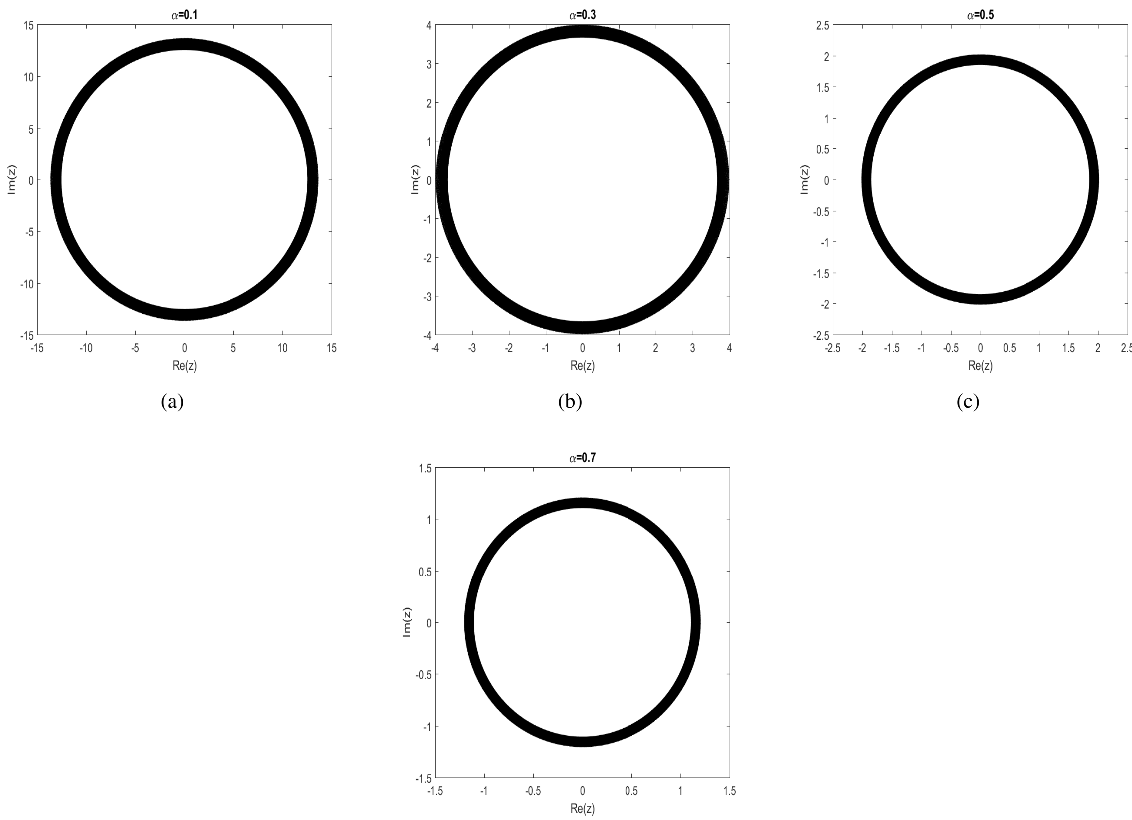
(a) (b)

FIGURE 2. The absolute errors with ($\alpha = 0.1$ by the scheme (a) and $\alpha = 0.9$ by the scheme (b))for different N for example (5.2).



(a) (b)

FIGURE 3. The solution curves by the formula (2.11) for different α for example (5.1).



(a) (b) (c)

(d)

FIGURE 4. Stability regions of the new method CF2 to approximate the Caputo-Fabrizio derivative with $\alpha = 0.1, 0.3, 0.5, 0.7$.

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