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Modulation instability analysis, optical solitons and other solutions to the (2+1)-dimensional hyperbolic nonlinear Schrodinger's equation

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Abstract

The current study utilizes the extended sinh-Gordon equation expansion and $(\frac{G'}{G^2})$-expansion function methods in constructing various optical soliton and other solutions to the (2+1)-dimensional hyperbolic nonlinear Schrödinger's equation which describes the elevation of water wave surface for slowly modulated wave trains in deep water in hydrodynamics. We secure different kinds of solutions like optical dark, bright, singular, combo solitons as well as hyperbolic and trigonometric functions solutions. Moreover, singular periodic wave solutions are recovered and the constraint conditions which provide the guarantee to the soliton solutions are also reported. In order to shed more light on these novel solutions, graphical features 3D, 2D and contour with some suitable choice of parameter values have been depicted. We also discuss the stability analysis of the studied nonlinear model with aid of modulation instability analysis.

Keywords. NLSE, Optical soliton, Extended sinh-Gordon equation expansion method, $(\frac{G'}{G^2})$-expansion function method, Stability analysis.
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1. Introduction

The non-linear partial differential equations (NLPDEs) have remarkable importance because of its broad range usages and applications. Non-linear phenomena have become one of the great impressive field for the researchers in this modern era of science. NLPDEs are largely used in diverse scientific fields such as biology, physics, geochemistry, ocean engineering, fluid mechanics, solid state physics, geophysics, optical fibers, plasma physics and many other fields to describe the physical mechanisms of natural phenomena and dynamical processes. NLPDEs are often used to explain the behaviour of waves in diverse fields [1–4, 17, 18, 28].

In order to understand these intricate phenomena, it is key to construct more exact solutions of NLPDEs. By using the obtained exact solutions one can understand the complex structure of physical phenomena. It is notable that many NLPDEs in diverse fields like biology, physics and chemistry consist of unknown functions and parameters and the study of exact solutions provides the guidance to the researchers to maintain and design the experiments, by producing the suitable natural environment, to obtain the these unknown function and parameters. The betterment of mathematical approaches for finding out a general and compact class of exact traveling wave solutions is one of the most basic task to observe the whole dynamical process modeling by complicated NLPDEs from the recent few years. Finding the exact solutions of NLPDEs has the importance to discuss the stability of numerical solutions and also development of a broad range of new scholar to simplify the routine calculation. Exact solutions to NLPDEs play an

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important role in nonlinear science, since they can provide much physical information and more insight of the physical aspects of the problem and thus lead to further applications. Wave phenomena in dispersion, dissipation, diffusion, reaction and convection are very much important.

Therefore, the foremost concern for the researchers is to find the exact solutions of NLPDEs. For this sake different powerful techniques such that Backlund transformation, Kudryashov method and its extended form, improved fractional sub-equation method, the unified method and its generalized scheme, the Darboux transformation method, expansion function method, the simple equation method, trial equation method, first integral method, the exponential function method, the inverse scattering method, mapping method, F-expansion method, modified extended tanh-function method, first integral method improved Bernoulli sub-ODE method, new auxiliary equation, newly extended direct algebraic method, Fan extended sub equation method and several others have been developed for finding the analytic solutions of NLPDEs by using various symbolic computation like Mathematica, Matlab and Maple [5–12, 19–22, 25, 26, 32–35, 41, 42].

Futhermore, the theory of optical solitons draw the attention of the researchers and scientific community, because it is an active area of research in the fields of telecommunication engineering and mathematical physics. Optical solitons are type of solitary waves which have the capability of propagation of waves without scattering over long distance i.e., they maintain their shape over long distance. They are significant in optical fibre communication due to this feature. The nonlinear wave phenomena can be examined in various scientific fields such as, fluid dynamics, plasma physics, solitary waves, optical fibres etc [23, 30, 36–40].

However, the goal of this article is to seek optical and other wave solutions in the forms of single (bright, dark) and combo as well as periodic solutions. These different types of solutions will be beneficial in the area where (2+1)-dimensional hyperbolic nonlinear Schr$\ddot{o}$dinger's equation is modeled for discussing the wave phenomena. We utilize the extended sinh-Gordon equation expansion method (ShGEEM)[29, 31] and $(\frac{G'}{G^2})$-expansion function[13, 14] method in this work for extracting optical solitons and other solutions to the (2+1)-dimensional hyperbolic NLSE [15]. The basic feature of proposed techniques is to observe some elementary relationships between NLPDEs and others simple NLODEs. It has been examined that with the aid of simple solutions and solvable ODEs, different kind of traveling wave solutions of some complicated NLPDEs can be easily constructed. The primary benefit of applying these techniques is that we have succeeded in a single move, to gather various types of new solutions. An important aspect of mentioned techniques is to provide us a guideline that how to organize these solutions. Such type of solutions have great significant role in the formation of deep water waves as these are created by modulation instability, they appear from nowhere. .

The (2+1)-dimensional hyperbolic NLSE is given by [15]

$$i\Theta_y + \frac{1}{2}\Big(\Theta_{xx} - \Theta_{tt}\Big) + |\Theta|^2\Theta = 0, \quad i = \sqrt{-1}. \tag{1.1}$$

Eq. (1.1) expresses of the elevation of water wave surface for slowly modulated wave trains in deep water in hydrodynamics, and also governs the propagation of electromagnetic fields in self-focusing and normally dispersive planar wave guides in optics [15]. In Eq. (1.1), $\Theta(x, y, t)$ is a complex-valued function, x, y, t represent the transverse (in-plane) coordinate, the propagation coordinate and time, respectively. They are normlized with beam width x_0, the differection lenght $L_D = kx_0$, and the time $t_0 = (k''L)^{\frac{1}{2}}$, where k represents the guided-mode propagation constant while $k'' = \frac{\partial k}{\partial \omega^2} > 0$ represents the GVD coefficient. The amplitude Θ is linked to the slowly varying envelope of the electromagnetic field E by $\Theta = (\gamma L_D)^{\frac{1}{2}}E$, where $\gamma = \frac{2\pi n_2}{\lambda L_{eff}}$, where λ represents the wavelength, n_2 denotes the positive Kerr coefficient, and L_{eff} is the effective core thickness [16].

This piece of article is discussed as sequence: In section 2, application of methods to extract the soliton solutions. In section 3, MI analysis. In section 4, graphical description of solutions and finally paper comes to end with conclusion in section 5.

2. Applications

2.1. **Extended ShGEEM.** In this section, we study the application of extended ShGEEM. We first discuss the mathematical analysis of the (2+1)-dimensional hyperbolic NLSE. For solving the Eq. (1.1), we start with wave transformation as:

$$\Theta(x,y,t)=\Phi(\pi)e^{i\psi},\quad \pi=\tau(x+by-\omega t),\quad \psi=a(x+y)-\chi t+k, \tag{2.1}$$

where $\Phi(\pi)$ is an unknown function, ψ denotes the function of phase shift, a is wave number, χ is the frequency and k is the phase constant. By putting Eq. (2.1) into Eq. (1.1), we decompose Eq. (1.1) into the following real and imaginary parts:

$$(a(2+a)-\chi^2)\Phi-2\Phi^3+\tau^2(\omega^2-1)\Phi^{''}=0, \tag{2.2}$$

and

$$2\tau(a+b-\omega\chi)\Phi^{'}=0, \tag{2.3}$$

respectively. From Eq. (2.3), we get the constraint condition as:

$$\omega=\frac{a+b}{\chi}. \tag{2.4}$$

Consider the following trial solutions generated from the well-known sinh-Gordon equation by Xian-Lin and Jia-Shi [31]:

$$\Phi(w)=\sum_{j=1}^{n}\Big(b_j\sinh(w)+a_j\cosh(w)\Big)^j+a_0, \tag{2.5}$$

$$\Theta(\pi)=\sum_{j=1}^{n}\Big(\pm b_j\ i\ \operatorname{sech}(\pi)\pm a_j\tanh(\pi)\Big)^j+a_0, \tag{2.6}$$

$$\Theta(\pi)=\sum_{j=1}^{n}\Big(\pm b_j\ \operatorname{csch}(\pi)\pm a_j\coth(\pi)\Big)^j+a_0, \tag{2.7}$$

$$\Theta(\pi)=\sum_{j=1}^{n}\Big(\pm b_j\sec+a_j\tan(\pi)\Big)^j+a_0, \tag{2.8}$$

and

$$\Theta(\pi)=\sum_{j=1}^{n}\Big(\pm b_j\ \csc(\pi)-a_j\ \cot(\pi)\Big)^j+a_0, \tag{2.9}$$

where $i=\sqrt{-1}$, and $w^{'}=\sinh(w)$ or $w^{'}=\cosh(w)$. For details, see [31]. By making balance between the linear term Φ'' and the non-linear term Φ^3 to determine the value of n in Eq. (2.2), yields $n=1$. With $n=1$, Eqs. (2.5)-(2.9) change to

$$\Phi(w)=b_1\sinh(w)+a_1\cosh(w)+a_0, \tag{2.10}$$

$$\Theta(\pi)=\pm b_1\ i\operatorname{sech}(\pi)\pm a_1\tanh(\pi)+a_0, \tag{2.11}$$

$$\Theta(\pi)=\pm b_1\ \operatorname{csch}(\pi)\pm a_1\coth(\pi)+a_0, \tag{2.12}$$

$$\Theta(\pi)=\pm b_1\sec+a_1\tan(\pi)+a_0, \tag{2.13}$$

and

$$\Theta(\pi)=\pm b_1\ \csc(\pi)-a_1\ \cot(\pi)+a_0. \tag{2.14}$$

Substituting Eq. (2.10) and it is second derivative along with $w^{'} = \sinh(w)$ and/or $w^{'} = \cosh(w)$ into Eq. (2.2), yields a hyperbolic polynomial functions. On collecting the same power coefficients of the hyperbolic function and equating to zero, we have a set of algebraic expression. The obtained algebraic polynomial produce the values of the coefficients involved. Putting the values of the coefficients into Eqs. (2.11)-(2.14), provides the solutions to Eq. (1.1).
Case-1: When

$$a_0 = 0,\ a_1 = -\sqrt{\frac{a(2+a)-\chi^2}{2}},\ b_1 = 0,\ \ \tau = -\sqrt{\frac{\chi^2 - a(2+a)}{2(1-\omega^2)}}.$$

The dark optical soliton solution

$$\Theta_{1.1}(x,y,t) = -\sqrt{\frac{a(2+a)-\chi^2}{2}} \tanh\left(\tau(x+by-\omega t)\right) e^{i(a(x+y)-\chi t+k)}. \tag{2.15}$$

The singular soliton solution

$$\Theta_{1.2}(x,y,t) = -\sqrt{\frac{a(2+a)-\chi^2}{2}} \coth\left(\tau(x+by-\omega t)\right) e^{i(a(x+y)-\chi t+k)}, \tag{2.16}$$

where $a(2+a) - \chi^2 > 0$ and $(1-\omega^2)(\chi^2 - a(2+a)) > 0$ for valid solutions.
Case-2: When

$$a_0 = 0,\ a_1 = 0,\ b_1 = -\sqrt{\chi^2 - a(2+a)},\ \ \tau = -\sqrt{\frac{\chi^2 - a(2+a)}{\omega^2 - 1}},$$

the bright optical soliton solution is

$$\Theta_{2.1}(x,y,t) = -\sqrt{a(2+a)-\chi^2}\, \operatorname{sech}\left(\tau(x+by-\omega t)\right) e^{i(a(x+y)-\chi t+k)}, \tag{2.17}$$

and the singular soliton solution is

$$\Theta_{2.2}(x,y,t) = -\sqrt{\chi^2 - a(2+a)}\, \operatorname{csch}\left(\tau(x+by-\omega t)\right) e^{i(a(x+y)-\chi t+k)}, \tag{2.18}$$

where $(\omega^2 - 1)(\chi^2 - a(2+a)) > 0$ for valid solutions.
Case-3: When

$$a_0 = 0,\ a_1 = -\sqrt{\frac{a(2+a)-\chi^2}{2}},\ b_1 = -\sqrt{\frac{a(2+a)-\chi^2}{2}},\ \ \tau = \sqrt{\frac{2a(2+a)-2\chi^2}{\omega^2-1}},$$

the mixed dark-bright optical soliton solution is

$$\Theta_{3.1}(x,y,t) = -\sqrt{\frac{a(2+a)-\chi^2}{2}} \times \left(i\, \operatorname{sech}\left(\tau(x+by-\omega t)\right) + \tanh\left(\tau(x+by-\omega t)\right)\right) e^{i(a(x+y)-\chi t+k)}, \tag{2.19}$$

and the mixed singular soliton solution is

$$\Theta_{3.2}(x,y,t) = -\sqrt{\frac{a(2+a)-\chi^2}{2}} \times \left(i\, \operatorname{csch}\left(\tau(x+by-\omega t)\right) + \coth\left(\tau(x+by-\omega t)\right)\right) e^{i(a(x+y)-\chi t+k)}, \tag{2.20}$$

where $a(2+a) - \chi^2 > 0$ and $(\omega^2 - 1)(a(2+a) - \chi^2) > 0$ for valid solutions.
Case-4: when

$$a_0 = 0,\ a_1 = -\sqrt{\frac{\chi^2 - a(2+a)}{2}},\ b_1 = \sqrt{\frac{\chi^2 - a(2+a)}{2}},\ \ \tau = \sqrt{\frac{2\chi^2 - 2a(2+a)}{\omega^2-1}},$$

the singular periodic wave solutions are

$$\Theta_{4.1}(x,y,t)=\sqrt{\frac{\chi^2-a(2+a)}{2}}\times \left(\sec\left(\tau(x+by-\omega t)\right)-\tan\left(\tau(x+by-\omega t)\right)\right)e^{i(a(x+y)-\chi t+k)}, \tag{2.21}$$

and

$$\Theta_{4.2}(x,y,t)=\sqrt{\frac{\chi^2-a(2+a)}{2}}\times \left(\cot\left(\tau(x+by-\omega t)\right)+\csc\left(\tau(x+by-\omega t)\right)\right)e^{i(a(x+y)-\chi t+k)}, \tag{2.22}$$

where $(\omega^2-1)(\chi^2-a(2+a))>0$ for valid solutions.

2.2. $\frac{G'}{G^2}$**-expansion function method.** We study the application of $\frac{G'}{G^2}$-expansion function method in this section as follow: Balancing the highest power of nonlinear term and the highest derivative in Eq. (2.2) yields, $n=1$. So the solution of Eq. (2.2) can be expressed as:

$$\Phi(\pi)=\alpha_0+\alpha_1\left(\frac{G'}{G^2}\right)+\beta_1\left(\frac{G'}{G^2}\right)^{-1}, \tag{2.23}$$

where α_0,α_1 and β_1 are considered as constants and to be determined later. By following the steps of the proposed method and with the aid of Mathematica, we get the following set of solutions:

Set-1

$$\alpha_0=0,\ \alpha_1=-\frac{\sqrt{\varphi\left(\chi^2-a(a+2)\right)}}{\sqrt{2\zeta}},\ \beta_1=0,\ \tau=-\frac{\sqrt{\chi^2-a(a+2)}}{\sqrt{2\zeta\varphi\left(\omega^2-1\right)}}.$$

Set-2

$$\alpha_0=0,\ \alpha_1=0,\ \beta_1=-\frac{\sqrt{\zeta\left(\chi^2-a(a+2)\right)}}{\sqrt{2\varphi}},\ \tau=\frac{\sqrt{\chi^2-a(a+2)}}{\sqrt{2\zeta\varphi\left(\omega^2-1\right)}}.$$

Set-3

$$\alpha_0=0,\ \alpha_1=-\frac{\sqrt{\varphi\left(\chi^2-a(a+2)\right)}}{2\sqrt{2\zeta}},\ \beta_1=\frac{\sqrt{\zeta}\sqrt{\varphi\left(\chi^2-a(a+2)\right)}}{2\sqrt{2\varphi}},\ \tau=-\frac{\sqrt{\chi^2-a(a+2)}}{2\sqrt{2\zeta\varphi\left(\omega^2-1\right)}}.$$

Set-4

$$\alpha_0=0,\ \alpha_1=\frac{\sqrt{\varphi\left(\chi^2-a(a+2)\right)}}{2\sqrt{2\zeta}},\ \beta_1=-\frac{\sqrt{\zeta}\sqrt{\varphi\left(\chi^2-a(a+2)\right)}}{2\sqrt{2\varphi}},\ \tau=\frac{\sqrt{\chi^2-a(a+2)}}{2\sqrt{2\zeta\varphi\left(\omega^2-1\right)}}.$$

For Set-1, we have the following solutions as:

__**Case-1**__: When $\zeta\varphi>0$, the trigonometric solution can be shown as:

$$\Theta_{1,1}(x,y,t)=-\frac{\sqrt{\varphi\left(\chi^2-a(a+2)\right)}}{\sqrt{2\zeta}}\times\left(\sqrt{\frac{\zeta}{\varphi}}\left(\frac{E\cos(\sqrt{\zeta\varphi}\ \pi)+F\sin(\sqrt{\zeta\varphi}\ \pi)}{F\cos(\sqrt{\zeta\varphi}\pi)-E\sin(\sqrt{\zeta\varphi}\ \pi)}\right)\right)\times e^{i(a(x+y)-\chi t+k)}. \tag{2.24}$$

__**Case-2**__: When $\zeta\varphi<0$, the hyperbolic solution can be expressed:

$$\Theta_{1,2}(x,y,t)=\frac{\sqrt{\varphi\left(\chi^2-a(a+2)\right)}}{\sqrt{2\zeta}}\times\left(\frac{\sqrt{|\zeta\varphi|}}{\varphi}\left(\frac{E\sinh(2\sqrt{|\zeta\varphi|}\ \pi)+E\cosh(2\sqrt{|\zeta\varphi|}\ \pi)+F}{E\sinh(2\sqrt{|\zeta\varphi|}\ \pi)+E\cosh(2\sqrt{|\zeta\varphi|}\ \pi)-F}\right)\right)\times e^{i(a(x+y)-\chi t+k)}. \tag{2.25}$$

For soliton solution, take $E=F$, we get singular soliton solution as:

$$\Theta_{1,2}(x,y,t)=\sqrt{\frac{\chi^2-a(a+2)}{2}}\coth\left(\sqrt{|\zeta\varphi|}\ \pi\right)\times e^{i(a(x+y)-\chi t+k)}. \tag{2.26}$$

For Set-2, we have the following solutions as:
Case-1: When $\zeta\varphi > 0$, the trigonometric solution can be written as:

$$\Theta_{2,1}(x,y,t) = -\frac{\sqrt{\zeta\,(\chi^2 - a(a+2))}}{\sqrt{2\varphi}} \times \left(\sqrt{\frac{\zeta}{\varphi}}\left(\frac{E\cos(\sqrt{\zeta\varphi}\,\pi) + F\sin(\sqrt{\zeta\varphi}\,\pi)}{F\cos(\sqrt{\zeta\varphi}\,\pi) - E\sin(\sqrt{\zeta\varphi}\,\pi)}\right)\right)^{-1} \times e^{i(a(x+y)-\chi t+k)}. \tag{2.27}$$

Case-2: When $\zeta\varphi < 0$, the hyperbolic solution can be written as:

$$\Theta_{2,2}(x,y,t) = \frac{\sqrt{\zeta\,(\chi^2 - a(a+2))}}{\sqrt{2\varphi}} \times \left(\frac{\sqrt{|\zeta\varphi|}}{\varphi}\left(\frac{E\sinh(2\sqrt{|\zeta\varphi|}\,\pi + E\cosh(2\sqrt{|\zeta\varphi|}\,\pi) + F}{E\sinh(2\sqrt{|\zeta\varphi|}\,\pi) + E\cosh(2\sqrt{|\zeta\varphi|}\,\pi) - F}\right)\right)^{-1} \times e^{i(a(x+y)-\chi t+k)}. \tag{2.28}$$

For soliton solution, take $E = F$, we get dark soliton solution as:

$$\Theta_{2,2}(x,y,t) = \sqrt{\frac{\chi^2 - a(a+2)}{2}}\tanh(\sqrt{|\zeta\varphi|}\,\pi) \times e^{i(a(x+y)-\chi t+k)}. \tag{2.29}$$

For Set-3, we have the following solutions as:
Case-1: When $\zeta\varphi > 0$, the trigonometric solution can be shown as:

$$\begin{aligned}\Theta_{3,1}(x,y,t) = &-\frac{\sqrt{\varphi\,(\chi^2 - a(a+2))}}{2\sqrt{2\zeta}} \times \left(\sqrt{\frac{\zeta}{\varphi}}\left(\frac{E\cos(\sqrt{\zeta\varphi}\,\pi) + F\sin(\sqrt{\zeta\varphi}\,\pi)}{F\cos(\sqrt{\zeta\varphi}\,\pi) - E\sin(\sqrt{\zeta\varphi}\,\pi)}\right)\right)\\ &+\frac{\sqrt{\zeta}\sqrt{\varphi\,(\chi^2 - a(a+2))}}{2\sqrt{2}\varphi} \times \left(\sqrt{\frac{\zeta}{\varphi}}\left(\frac{E\cos(\sqrt{\zeta\varphi}\,\pi) + F\sin(\sqrt{\zeta\varphi}\,\pi)}{F\cos(\sqrt{\zeta\varphi}\,\pi) - E\sin(\sqrt{\zeta\varphi}\,\pi)}\right)\right)^{-1}\\ &\times e^{i(a(x+y)-\chi t+k)}.\end{aligned} \tag{2.30}$$

By choosing, $E = F$, we have periodic travelling wave solution as:

$$\Theta_{3,1}(x,y,t) = -\sqrt{\frac{\chi^2 - a(a+2)}{2}}\tan(2\sqrt{\zeta\varphi}\,\pi) \times e^{i(a(x+y)-\chi t+k)}. \tag{2.31}$$

Case-2:
When $\zeta\varphi < 0$, the hyperbolic solution can be expressed:

$$\begin{aligned}\Theta_{3,2}(x,y,t) = &\left(\frac{\sqrt{\varphi\,(\chi^2 - a(a+2))}}{2\sqrt{2\zeta}} \times \left(\frac{\sqrt{|\zeta\varphi|}}{\varphi}\left(\frac{E\sinh(2\sqrt{|\zeta\varphi|}\,\pi) + E\cosh(2\sqrt{|\zeta\varphi|}\,\pi) + F}{E\sinh(2\sqrt{|\zeta\varphi|}\,\pi) + E\cosh(2\sqrt{|\zeta\varphi|}\,\pi) - F}\right)\right)\right.\\ &-\frac{\sqrt{\zeta}\sqrt{\varphi\,(\chi^2 - a(a+2))}}{2\sqrt{2}\varphi} \times \left(\frac{\sqrt{|\zeta\varphi|}}{\varphi}\left(\frac{E\sinh(2\sqrt{|\zeta\varphi|}\,\pi) + E\cosh(2\sqrt{|\zeta\varphi|}\,\pi) + F}{E\sinh(2\sqrt{|\zeta\varphi|}\,\pi) + E\cosh(2\sqrt{|\zeta\varphi|}\,\pi) - F}\right)\right)^{-1}\\ &\times e^{i(a(x+y)-\chi t+k)}.\end{aligned} \tag{2.32}$$

For soliton solution, take $E = F$, we get combo dark-singular soliton solution as:

$$\Theta_{3,2}(x,y,t) = \frac{\sqrt{(\chi^2 - a(a+2))}}{2\sqrt{2}}\left(\coth(\sqrt{|\zeta\varphi|}\,\pi) - \tanh(\sqrt{|\zeta\varphi|}\,\pi)\right) \times e^{i(a(x+y)-\chi t+k)}. \tag{2.33}$$

For Set-4, we have the following solutions as:
Case-1: When $\zeta\varphi > 0$, the trigonometric solution can be described as:

$$\begin{aligned}\Theta_{4,1}(x,y,t) = &\frac{\sqrt{\varphi\,(\chi^2 - a(a+2))}}{2\sqrt{2\zeta}} \times \left(\sqrt{\frac{\zeta}{\varphi}}\left(\frac{E\cos(\sqrt{\zeta\varphi}\,\pi) + F\sin(\sqrt{\zeta\varphi}\,\pi)}{F\cos(\sqrt{\zeta\varphi}\,\pi) - E\sin(\sqrt{\zeta\varphi}\,\pi)}\right)\right)\\ &-\frac{\sqrt{\zeta}\sqrt{\varphi\,(\chi^2 - a(a+2))}}{2\sqrt{2}\varphi} \times \left(\sqrt{\frac{\zeta}{\varphi}}\left(\frac{E\cos(\sqrt{\zeta\varphi}\,\pi) + F\sin(\sqrt{\zeta\varphi}\,\pi)}{F\cos(\sqrt{\zeta\varphi}\,\pi) - E\sin(\sqrt{\zeta\varphi}\,\pi)}\right)\right)^{-1} \times e^{i(a(x+y)-\chi t+k)}.\end{aligned} \tag{2.34}$$

By choosing, $E = F$, we have periodic traveling wave solution as:

$$\Theta_{4,1}(x,y,t) = \sqrt{\frac{\chi^2 - a(a+2)}{2}}\tan(2\sqrt{\zeta\varphi}\,\pi) \times e^{i(a(x+y)-\chi t+k)}. \tag{2.35}$$

Case-2: When $\zeta\varphi < 0$, the hyperbolic solution can be expressed:

$$\Theta_{4,2}(x,y,t) = \Bigg(\frac{\sqrt{\varphi\,(\chi^2 - a(a+2))}}{2\sqrt{2\zeta}} \times \Bigg(\frac{\sqrt{|\zeta\varphi|}}{\varphi}\Bigg(\frac{E\sinh(2\sqrt{|\zeta\varphi|}\ \pi) + E\cosh(2\sqrt{|\zeta\varphi|}\ \pi) + F}{E\sinh(2\sqrt{|\zeta\varphi|}\ \pi) + E\cosh(2\sqrt{|\zeta\varphi|}\ \pi) - F}\Bigg) - \frac{\sqrt{\zeta}\sqrt{\varphi\,(\chi^2 - a(a+2))}}{2\sqrt{2\varphi}} \times \Bigg(\frac{\sqrt{|\zeta\varphi|}}{\varphi}\Bigg(\frac{E\sinh(2\sqrt{|\zeta\varphi|}\ \pi) + E\cosh(2\sqrt{|\zeta\varphi|}\ \pi) + F}{E\sinh(2\sqrt{|\zeta\varphi|}\ \pi) + E\cosh(2\sqrt{|\zeta\varphi|}\ \pi) - F}\Bigg)\Bigg)^{-1}\Bigg) \times e^{i(a(x+y)-\chi t+k)}. \tag{2.36}$$

For soliton solution, take $E = F$, we get combo dark-singular soliton solution as:

$$\Theta_{4,2}(x,y,t) = \frac{\sqrt{(\chi^2 - a(a+2))}}{2\sqrt{2}}\Big(\tanh(\sqrt{|\zeta\varphi|}\ \pi) - \coth(\sqrt{|\zeta\varphi|}\ \pi)\Big) \times e^{i(a(x+y)-\chi t+k)}, \tag{2.37}$$

where $\pi = \tau(x + by - \omega t)$.

3. Modulation Instability Analysis

In this section, we discuss modulation instability (MI) of the (2+1)-dimensional hyperbolic NLSE by utilizing the standard linear stability analysis [24, 27]. Suppose the following form of steady-state solutions

$$\Theta(x,y,t) = \Big(\sqrt{q_0} + U(x,y,t)\Big)e^{iq_0 x}, \tag{3.1}$$

where q_0 is the normalized optical power. Putting Eq. (3.1) into Eq. (1.1) and linearizing, gives

$$iU_y + \frac{1}{2}\Big(U_{xx} - U_{tt}\Big) + i\ q_0 U_x + (q_0 + q_0^2)(U + U^*) = 0, \tag{3.2}$$

where $*$ indicates the conjugate of the unknown complex function $U(x,y,t)$. Suppose the solution Eq. (3.2) can be expressed in the following form

$$U(x,y,t) = \delta_1 e^{i(\kappa_1 x + \kappa_2 y - \phi t)} + \delta_2 e^{-i(\kappa_1 x + \kappa_2 y - \phi t)}, \tag{3.3}$$

where ϕ, κ_1 and κ_2 represent the frequency of perturbation, normalized wave number and nonzero constant, respectively. Substituting Eq. (3.3) into Eq. (3.2), splitting the coefficients of $e^{i(\kappa_1 x + \kappa_2 y - \phi t)}$ and $e^{-i(\kappa_1 x + \kappa_2 y - \phi t)}$ yields, the dispersion relation:

$$\frac{\kappa_1^4}{4} - \kappa_2^2 - \frac{\kappa_1^2\phi^2}{2} + \frac{\phi^4}{4} - \kappa_1^2 q_0 - 2\kappa_1\kappa_2 q_0 + \phi^2 q_0 - 2\kappa_1^2 q_0^2 - \phi^2 q_0^2 = 0. \tag{3.4}$$

Describing the dispersion relation (3.4) for ϕ, we get

$$\phi = \sqrt{\kappa_1^2 - 2q_0 - 2q_0^2 - 2\sqrt{\kappa_2^2 + 2\kappa_1\kappa_2 q_0 + q_0^2 + \kappa_1^2 q_0^2 + 2q_0^3 + q_0^4}}. \tag{3.5}$$

In a situation whereby $(\phi)^2 \geq 0$, the wave number ϕ is real for all κ_1, κ_2 and the steady state is stable against small perturbations. While, in contradiction to the above statement, the steady-state solution turns to be unstable in condition whereby $\kappa_1^2 - 2q_0 - 2q_0^2 < 2\sqrt{\kappa_2^2 + 2\kappa_1\kappa_2 q_0 + q_0^2 + \kappa_1^2 q_0^2 + 2q_0^3 + q_0^4}$, the wave number ϕ becomes imaginary, and the perturbation grows exponentially. The growth rate of MI gain spectrum $G(\kappa)$ can be written as:

$$G(\kappa) = 2Im(\phi) = 2Im\Bigg(\sqrt{\kappa_1^2 - 2q_0 - 2q_0^2 - 2\sqrt{\kappa_2^2 + 2\kappa_1\kappa_2 q_0 + q_0^2 + \kappa_1^2 q_0^2 + 2q_0^3 + q_0^4}}\Bigg). \tag{3.6}$$

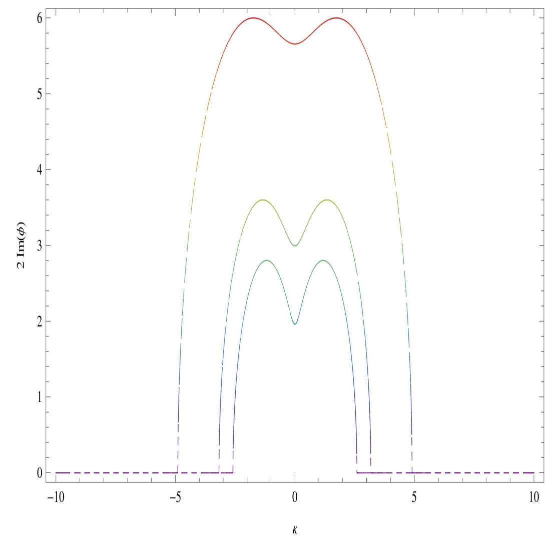
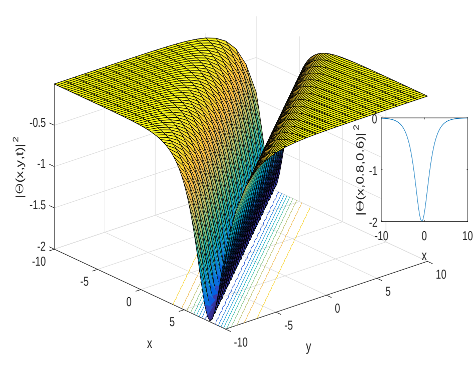


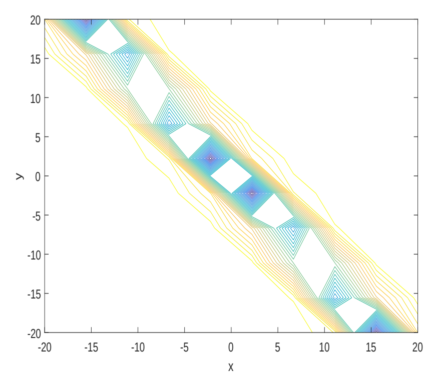
FIGURE 1. Gain spectrum of modulation instability for three distinct values of the parameters.

4. Graphical representation of solutions

The graphical description of derived solitons and other solutions have been expressed in the mentioned figures by allotting the different values of the parameters. The graphs show that these solutions have different physical meanings. For example, hyperbolic functions such as, the hyperbolic tangent appears in the calculation and rapidity of special relativity while, the hyperbolic cotangent arises in the Langevin function for magnetic polarization. It is worthy to note that the bright soliton describes the solitary waves whose peak intensity is larger than the background while dark soliton describes the solitary waves with lower intensity than the background, and the singular soliton solutions is a solitary wave with discontinuous derivatives; examples of such solitary waves include compactions, which have finite (compact) support, and peakons, whose peaks have a discontinuous first derivative.

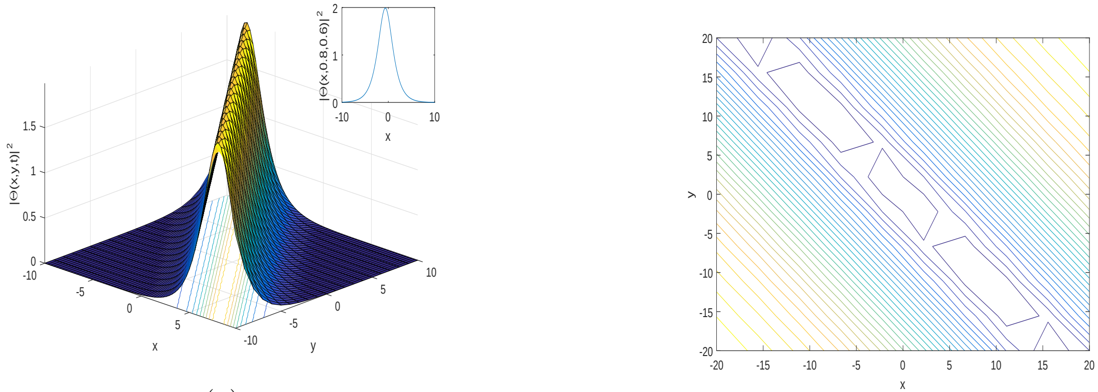


(A)



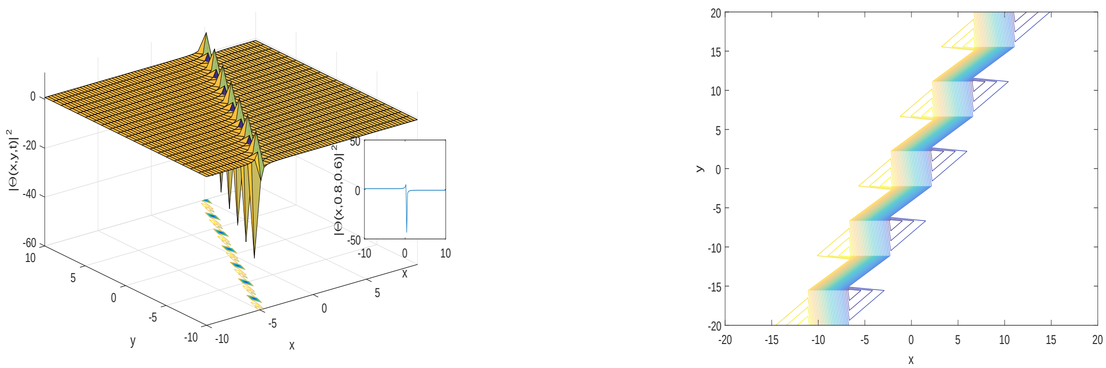
(B)

FIGURE 2. The (A) 2D, 3D surfaces and (B) contour profile of Eq. (2.15).



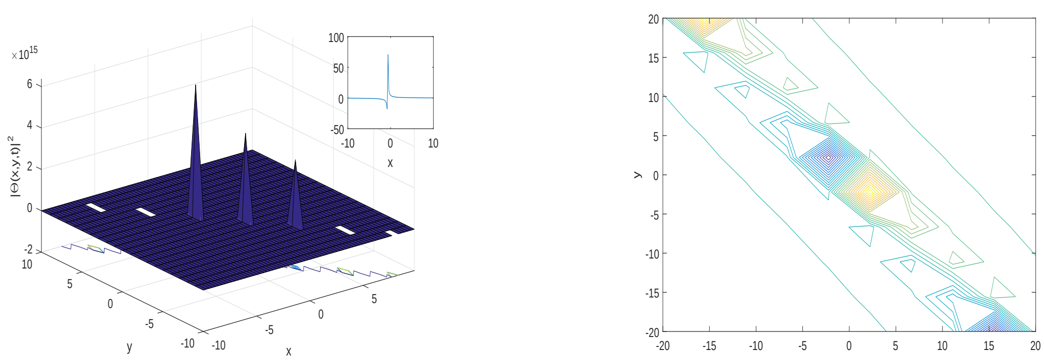
(A)

FIGURE 3. The (A) 2D, 3D surfaces and (B) contour profile of Eq. (2.17).



(A) (B)

FIGURE 4. The (A) 2D, 3D surfaces and (B) contour profile of Eq. (2.26).



(A) (B)

FIGURE 5. The (A) 2D, 3D surfaces and (B) contour profile of Eq. (2.33).

5. Conclusions

In this piece of research, we investigated the (2+1)-dimensional hyperbolic NLSE with the help of extended sinh-Gordon equation expansion and $(\frac{G'}{G^2})$-expansion function approaches. Various kinds of solutions like bright, dark, singular, combo dark-singular, mixed dark-bright, mixed singular solitons as well as periodic wave solutions in singular form are obtained. Moreover, the modulation instability analysis to the proposed model is also observed. The graphical view of some solutions have been depicted by 3D,2D and contour profile, respectively. The results are new, interesting and have a great impact in the field of nonlinear sciences where the (2+1)-dimensional hyperbolic NLSE will be used for the dynamics of optical solitons and other solutions.

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