Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 10, No. 1, 2022, pp. 61-76
DOI:10.22034/cmde.2020.41130.1784

A numerical technique for solving nonlinear fractional stochastic integro-differential equations with n-dimensional Wiener process

Elnaz Aryani[1], Afshin Babaei[1,*], and Ali Valinejad[2]

[1]Department of Applied Mathematics, University of Mazandaran, P.O. Box: 47416-95447, Babolsar, Iran.

[2]Department of Computer Sciences, University of Mazandaran, P.O. Box: 47416-95447, Babolsar, Iran.

Abstract

This paper deals with the numerical solution of nonlinear fractional stochastic integro-differential equations with the n-dimensional Wiener process. A new computational method is employed to approximate the solution of the considered problem. This technique is based on the modified hat functions, the Caputo derivative, and a suitable numerical integration rule. Error estimate of the method is investigated in detail. In the end, illustrative examples are included to demonstrate the validity and effectiveness of the presented approach.

Keywords. Stochastic process; Brownian motion; Caputo's derivative; Modified hat functions, Error estimate.

2010 Mathematics Subject Classification. 45J05, 60H20, 26A33, 65C30.

1. Introduction

Fractional integro-differential equations are nowadays extremely popular, due to wide range of their applications in real-life problems [13, 29, 30]. In natural dynamical processes, the next states of a system are usually dependent on its all past states. The fractional order operators can preserve the hereditary properties of a considered function. Thus, using these operators helps researchers to provide a more complete picture of real applications. Many theoretical and numerical methods have been presented in the literature for solving fractional order equations, like hybrid collocation method [19], least squares method [10], perturbation iteration algorithm [32], Taylor expansion approach [9], Sinc-collocation method [1] and so on.

Stochastic differential equations (SDEs) are basically differential equations with an additional stochastic term. The deterministic term, which is common to ordinary differential equations, describes behavior of the phenomenon and the stochastic term describes the 'noise', a random perturbation that influences the phenomenon. SDEs have considerable applications in basic fields of science and technology especially when we need to consider random perturbations in environmental conditions. In fact, for accurately describing different phenomena with random perturbations, for example in physics, finance, medicine, biology, and so on, researchers have applied stochastic differential equations or stochastic integro-differential equations [7, 21, 28]. With the advancement of SDEs theory, there have been many attempts to construct numerical methods for solving this range of equations. A computational scheme based on B-spline interpolation method [24], spectral method for stochastic fractional differential equations [2], Haar wavelet method for two dimensional stochastic integrals [6], Legendre wavelet collocation method [23], Galerkin method based on orthogonal polynomials [11], operational matrix of the Chebyshev wavelets [25], expansion method [12], a direct method based on stochastic operational matrix [27], a superconvergence Euler-Maruyama method [18], some implicit methods based on a backward approach along with some suitable discretization schemes [17], a split-step theta method [16] and the Euler method for Volterra integro-differential equations with fractional Brownian motions [34] are some of these approaches.

Received: 04 August 2020 ; Accepted: 21 December 2020.

$*$ Corresponding author. Email: babaei@umz.ac.ir.

In this investigation, we consider a nonlinear fractional stochastic integro-differential equation with n-dimensional Wiener process in the following form:

$$\begin{aligned}{}_0D_t^{\alpha}u(t) = f(t) + \zeta\int_0^{\mathtt{T}} H(t,s,u(s))\mathrm{d}s + \int_0^t F(t,s,u(s))\mathrm{d}s \\ + \sum_{j=1}^{n}\int_0^t G_j(t,s,u(s))\mathrm{d}B_j(s), \qquad t\in\Omega,\end{aligned}$$

with the initial condition

$$u(0) = u_0, \tag{1.1}$$

where ζ is a real constant, $\Omega := [0,\mathtt{T}]$ and the operator ${}_0D_t^{\alpha}(\cdot)$ denotes the Caputo fractional derivative defined as [30]:

$${}_0D_t^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u'(s)}{(t-s)^{\alpha}}\mathrm{d}s, \ \alpha\in(0,1). \tag{1.2}$$

$\Gamma(\cdot)$ represents the Gamma function. Let $(\Omega,\mathcal{F},\mathbb{P})$ is a probability space with a normal filtration $(\mathcal{F}_t)_{t\in\Omega}$. Moreover, $u(t)$ is an unknown process, while $f(t)\in C^3(\Omega)$ and the kernels $H(t,s,u)$, $F(t,s,u)$ and $G_j(t,s,u)$, $j=1,...,n$, are known processes, defined on the same probability space and satisfy the Lipschitz condition with respect to u. Also, $B_j(t)$, $j=1,...,n$, are the Brownian motions adapted to the filtration $(\mathcal{F}_t)_{t\in\Omega}$.

Eq. (1.1) covers a wide range of fractional stochastic integro-differential equations. Many dynamical systems in real-world applications are modeled by integro-differential equations. These problems are often dependent on one or several independent noise sources. Thus, using stochastic integro-differential equations to describe the behaviors of these systems have found considerable attention in recent decades. For example, in physics, the fractional Fokker-Planck equation is a special case of Eq. (1.1) [5, 33]. The exponential population growth models in biology are explained by the It-Volterra integral equations with multidimensional Wiener process [20]. Also, finding the wealth process of the consumption-investment problem in financial mathematics can be described by a nonlinear multidimensional stochastic integral equation [3, 31]. To our knowledge, it seems that the more general Eq. (1.1) is little considered in the literature. Hence, our purpose in the present work is to propose an effective numerical approach for the solution of this type of fractional order stochastic equations.

The rest of this paper is organized as follows. In section 2, some basic definitions of fractional calculus, the definition of modified hat functions (MHFs), their properties, operational matrix of integration based on the modified hat functions and Legendre-Gauss integration method are reviewed. The numerical technique is described in section 3. Error estimate of the proposed method is investigated in section 4. Some numerical experiments are presented in section 5 to illustrate the accuracy of our method. Finally, the conclusion of this work is included in section 6.

2. Preliminary concepts

In this section, some useful concepts and tools have been introduced that will be used during this paper.

2.1. Fractional calculus.

Definition 2.1. [30] The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, is defined as

$${}_0I_t^{\alpha}g(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds, \ t>0. \tag{2.1}$$

Lemma 2.2. *[30] Assume that $r\in\mathbb{N}$, $r-1<\alpha\leq r$ and $g\in C^{r-1}[0,b]$ where b is a positive real constant. Then*

$$\begin{aligned}&{}_0I_t^{\alpha}({}_0I_t^{\beta}g(t)) = {}_0I_t^{\beta}({}_0I_t^{\alpha}g(t)) = {}_0I_t^{\alpha+\beta}g(t),\\ &{}_0I_t^{\alpha}({}_0D_t^{\alpha}g(t)) = g(t) - \sum_{k=0}^{r-1} g^{(k)}(t)\Big|_{t=0}\frac{t^k}{\Gamma(k+1)},\\ &{}_0D_t^{\alpha}({}_0D_t^{r}g(t)) = {}_0D_t^{r}({}_0D_t^{\alpha}g(t)) = {}_0D_t^{r+\alpha}g(t).\end{aligned} \tag{2.2}$$

2.2. **Properties of modified hat functions.** Let the interval Ω is divided into N subintervals with equidistant size $h = \frac{\mathtt{T}}{N}$. Hat functions are defined as [22]

$$\psi_0(t) = \begin{cases} 1 - \frac{t}{h}, & t \in [0, h], \\ \\ 0, & otherwise, \end{cases}$$

$$\psi_i(t) = \begin{cases} \frac{t}{h} - (i-1), & t \in [(i-1)h, ih], \\ \\ (i+1) - \frac{t}{h}, & t \in [ih, (i+1)h], \qquad i = 1, 2, \ldots, N-1, \\ \\ 0, & otherwise, \end{cases}$$

$$\psi_N(t) = \begin{cases} \frac{t - \mathtt{T}}{h} + 1, & t \in [\mathtt{T} - h, \mathtt{T}], \\ \\ 0, & otherwise. \end{cases}$$

Definition 2.3. When $h = \frac{\mathtt{T}}{N}$ and $N \geq 2$ is an even integer, MHFs are defined on Ω as follows [26]

$$\theta_0(t) = \begin{cases} \frac{1}{2h^2}(t-h)(t-2h), & t \in [0, 2h], \\ \\ 0, & otherwise, \end{cases}$$

when i be odd and $i = 1, 3, ..., N-1$,

$$\theta_i(t) = \begin{cases} \frac{-1}{h^2}(t - (i-1)h)(t - (i+1)h), & t \in [(i-1)h, (i+1)h], \\ \\ 0, & otherwise, \end{cases}$$

when i be even and $i = 2, 4, ..., N-2$,

$$\theta_i(t) = \begin{cases} \frac{1}{2h^2}(t - (i-1)h)(t - (i-2)h), & t \in [(i-2)h, ih], \\ \\ \frac{1}{2h^2}(t - (i+1)h)(t - (i+2)h), & t \in [ih, (i+2)h], \\ \\ 0, & otherwise, \end{cases}$$

and

$$\theta_N(t) = \begin{cases} \frac{1}{2h^2}(t - (\mathtt{T} - h))(t - (\mathtt{T} - 2h)), & t \in [\mathtt{T} - 2h, \mathtt{T}], \\ \\ 0, & otherwise. \end{cases}$$

According to the definition of MHFs, we have

$$\theta_i(jh) = \begin{cases} 1, & i = j, \\ \\ 0, & i \neq j, \end{cases} \tag{2.3}$$

$$\theta_i(t)\theta_j(t) = \begin{cases} 0, & i \text{ is even and } |i-j| \geq 3, \\ \\ 0, & i \text{ is odd and } |i-j| \geq 2. \end{cases} \tag{2.4}$$

and

$$\sum_{i=0}^{N} \theta_i(t) = 1. \tag{2.5}$$

An arbitrary function $g(t) \in L^2(\Omega)$ can be expanded in terms of MHFs as

$$g(t) \simeq g_N(t) = \sum_{i=0}^{N} c_i\theta_i(t) = \mathbf{C}^T\Theta(t) = \Theta^T(t)\mathbf{C}, \tag{2.6}$$

where $\Theta(t)$ is defined as follows:

$$\Theta(t) = [\theta_0(t), \ldots, \theta_i(t), \ldots, \theta_N(t)]^T, \tag{2.7}$$

and

$$\mathbf{C} = [c_0, \ldots, c_i, \ldots, c_N]^T,$$

in which $c_i = g(ih),\ i = 0, 1, \ldots, N$.

Now, we review the operational matrix based on MHFs that will be used in our proposed method.

Theorem 2.4. *Let $\Theta(t)$ is the MHFs vector given by Eq. (2.7). Then*

$$ {}_0I_t^{\alpha}\Theta(t) \simeq L^{\alpha}\Theta(t), \tag{2.8}$$

L^{α} is the $(N+1)\times(N+1)$ operational matrix of fractional integration of order α from $\Theta(t)$ which is defined as

$$L^{\alpha} = \begin{pmatrix} 0 & \phi_1 & \phi_2 & \phi_3 & \phi_4 & \ldots & \phi_{N-1} & \phi_N \\ 0 & \vartheta_0 & \vartheta_1 & \vartheta_2 & \vartheta_3 & \ldots & \vartheta_{N-2} & \vartheta_{N-1} \\ 0 & \eta_{-1} & \eta_0 & \eta_1 & \eta_2 & \ldots & \eta_{N-3} & \eta_{N-2} \\ 0 & 0 & 0 & \vartheta_0 & \vartheta_1 & \ldots & \vartheta_{N-4} & \vartheta_{N-3} \\ 0 & 0 & 0 & \eta_{-1} & \eta_0 & \ldots & \eta_{N-5} & \eta_{N-4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & \vartheta_0 & \vartheta_1 \\ 0 & 0 & 0 & 0 & 0 & \ldots & \eta_{-1} & \eta_0 \end{pmatrix},$$

in which

$$\phi_1 = \frac{h^{\alpha}\alpha(3+2\alpha)}{2\Gamma(\alpha+3)}, \qquad \vartheta_0 = \frac{2h^{\alpha}(1+\alpha)}{\Gamma(\alpha+3)},$$

$$\eta_{-1} = -\frac{h^{\alpha}\alpha}{2\Gamma(\alpha+3)}, \qquad \eta_0 = \frac{h^{\alpha}2^{\alpha+1}(2-\alpha)}{2\Gamma(\alpha+3)},$$

$$\eta_1 = \frac{h^{\alpha}}{2\Gamma(\alpha+3)}\Big(3^{\alpha+1}(4-\alpha) - 6(2+\alpha)\Big),$$

$$\phi_k = \frac{h^{\alpha}}{2\Gamma(\alpha+3)}\Big(k^{\alpha+1}(2k-6-3\alpha) + 2j^{\alpha}(1+\alpha)(2+\alpha) \\ -(k-2)^{(\alpha+1)}(2k-2+\alpha)\Big), \qquad k = 2, 3, \ldots, N,$$

$$\vartheta_k = \frac{2h^\alpha}{\Gamma(\alpha+3)}\Big((k-1)^{\alpha+1}(k+1+\alpha) \\ -(k+1)^{\alpha+1}(k-1-\alpha)\Big), \qquad k=1,2,\dots,N-1,$$

and

$$\eta_k = \frac{h^\alpha}{2\Gamma(\alpha+3)}\Big((k+2)^{\alpha+1}(2k+2-\alpha) - 6k^{\alpha+1}(2+\alpha) \\ -(k-2)^{\alpha+1}(2k-2+\alpha)\Big), \qquad k=2,3,\dots,N-2.$$

Proof. See [26]. □

2.3. **Legendre-Gauss integration rule.** The Legendre-Gauss rule is one of the mostly used methods for computing numerical integration. Suppose $L_{p+1}(t)$ is Legendre polynomial of order $p+1$ on $[-1,1]$. For any function $g(t) \in C^{2p}[a,b]$, the Legendre-Gauss quadrature formula is as:

$$\int_a^b g(t)dt = \frac{b-a}{2}\sum_{\tau=0}^{p}\omega_\tau g(\frac{b-a}{2}\sigma_\tau + \frac{b+a}{2}) + \mathrm{E}^p_{\mathtt{Gauss}}(g), \tag{2.9}$$

in which distinct nodes $\{\sigma_\tau\}_{\tau=0}^p$ are the zeros of $L_{p+1}(t)$ and $\{\omega_\tau\}_{\tau=0}^p$ are the corresponding weights [15]

$$\omega_\tau = \frac{2}{(1-\sigma_\tau^2)[L'_{p+1}(\sigma_\tau)]^2}, \qquad \tau=0,1,\dots,p, \tag{2.10}$$

and

$$\mathrm{E}^p_{\mathtt{Gauss}}(g) = \frac{(b-a)^{2p+1}(p!)^4}{(2p!)^3(2p+1)}g^{(2p)}(\varsigma), \tag{2.11}$$

for some $\varsigma \in (a,b)$.

2.4. **It$\hat{o}$ approximation.** If a stochastic process $\{g(t)\}_{t\in\Omega}$ is measurable on the filtration $\{\mathcal{F}_t\}$ for any $t\in\Omega$, then the It$\hat{o}$ integral of this process is defined by [4]:

$$\int_a^b g(t)\mathrm{d}B(t) = \lim_{p\longrightarrow\infty}\sum_{i=0}^{p-1} g(t_i)(B(t_{i+1})-B(t_i)), \tag{2.12}$$

where $t_i = a + \frac{b-a}{p}i,\ i=0,1,\dots,p$, and $B(t)$ is Wiener process on the filtered probability space $(\Omega,\mathcal{F},\mathbb{P})$. The limit of relation (2.12) is defined on $L^2(\Omega,\mathbb{P})$ space and the approximation of this relation is calculated at the left end point of interval $[t_i,t_{i+1}]$. In the other words,

$$\mathbb{E}\left[\left|\int_a^b g(t)\mathrm{d}B(t) - \sum_{i=0}^{p-1} g(t_i)\left(B(t_{i+1})-B(t_i)\right)\right|^2\right] \longrightarrow 0,$$

as $p\longrightarrow\infty$.

3. Description of the proposed method

In this section, to present a numerical method for the problem (1.1)-(1.1). We consider an approximation of the fractional derivative of the unknown function as follow:

$${}_0D_t^\alpha u(t) = \mathbf{C}^T\Theta(t), \tag{3.1}$$

where $\mathbf{C}$ is the coefficients vector with the unknown elements $c_i,\ i=1,2\dots,N$,

$$\mathbf{C} = [c_0,c_1,\dots,c_N]^T, \tag{3.2}$$

and $\Theta(t)$ is correspond to the MHFs basis functions. By the relation (2.8) and the initial condition (1.1), we can write

$$u(t) = \mathbf{C}^T L^\alpha \Theta(t) + u_0. \tag{3.3}$$

Also, let $f(t) = \mathbf{f}^T \Theta(t)$, now substituting (3.1) and (3.3) in Eq. (1.1), results

$$\begin{aligned} \mathbf{C}^T \Theta(t) = \mathbf{f}^T \Theta(t) + \zeta \int_0^{\mathtt{T}} H(t, s, \mathbf{C}^T L^\alpha \Theta(s) + u_0) \mathrm{d}s \\ + \int_0^t F(t, s, \mathbf{C}^T L^\alpha \Theta(s) + u_0) \mathrm{d}s \\ + \sum_{j=1}^{n} \int_0^t G_j(t, s, \mathbf{C}^T L^\alpha \Theta(s) + u_0) \mathrm{d}B_j(s), \end{aligned} \tag{3.4}$$

Then, by setting $t = ih,\ i = 0, 1, \ldots, N$, (3.4) change to

$$\begin{aligned} \mathbf{C}^T \Theta(ih) = \mathbf{f}^T \Theta(ih) + \zeta \underbrace{\int_0^{\mathtt{T}} H(ih, s, \mathbf{C}^T L^\alpha \Theta(s) + u_0) \mathrm{d}s}_{I_0} \\ + \underbrace{\int_0^{ih} F(ih, s, \mathbf{C}^T L^\alpha \Theta(s) + u_0) \mathrm{d}s}_{I_1} \\ + \sum_{j=1}^{n} \underbrace{\int_0^{ih} G_j(ih, s, \mathbf{C}^T L^\alpha \Theta(s) + u_0) \mathrm{d}B_j(s)}_{I_2}. \end{aligned} \tag{3.5}$$

Due to (2.9), I_0, and I_1 can be approximated as

$$\begin{aligned} I_0 &= \int_0^{\mathtt{T}} H(ih, s, \mathbf{C}^T L^\alpha \Theta(s) + u_0) \mathrm{d}s \\ &\simeq \frac{\mathtt{T}}{2} \sum_{\tau=0}^{p} \omega_\tau H(ih, \kappa_\tau, \mathbf{C}^T L^\alpha \Theta(\kappa_\tau) + u_0), \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} I_1 &= \int_0^{ih} F(ih, s, \mathbf{C}^T L^\alpha \Theta(s) + u_0) \mathrm{d}s \\ &\simeq \frac{ih}{2} \sum_{\tau=0}^{p} \omega_\tau F(ih, \sigma_{\tau,i}, \mathbf{C}^T L^\alpha \Theta(\sigma_{\tau,i}) + u_0), \end{aligned} \tag{3.7}$$

where

$$\sigma_{\tau,i} = \frac{ih}{2}\sigma_\tau + \frac{ih}{2}, \qquad \kappa_\tau = \frac{\mathtt{T}}{2}\sigma_\tau + \frac{\mathtt{T}}{2}, \qquad \omega_\tau = \frac{2}{(1-\sigma_\tau^2)[L'_{p+1}(\sigma_\tau)]^2}. \tag{3.8}$$

Also, due to (2.12)

$$\begin{aligned} I_2 &= \int_0^{ih} G_j(ih, s, \mathbf{C}^T L^\alpha \Theta(s) + u_0) \mathrm{d}B_j(s) \\ &\simeq \sum_{k=0}^{M} G_j(ih, s_{i,k}, \mathbf{C}^T L^\alpha \Theta(s_{i,k}) + u_0)(B_j(s_{i,k+1}) - B_j(s_{i,k})), \end{aligned} \tag{3.9}$$

where

$$s_{i,k} = \frac{ih}{M}k, \ \ k = 0, \dots, M. \tag{3.10}$$

So, the second part of (3.5) can be written as

$$\begin{aligned} &\sum_{j=1}^{n} \int_0^{ih} G_j(ih, s, \mathbf{C}^T L^\alpha \Theta(s) + u_0) \mathrm{d}B_j(s) = \\ &\sum_{j=1}^{n} \Big(\sum_{k=0}^{M} G_j(ih, s_{i,k}, \mathbf{C}^T L^\alpha \Theta(s_{i,k}) + u_0) \left(B_j(s_{i,k+1}) - B_j(s_{i,k})\right) \Big). \end{aligned} \tag{3.11}$$

Now, by using Eqs. (3.7) and (3.11), Eq. (3.4) truns into

$$\begin{aligned} &\mathbf{C}^T \Theta(ih) = \mathbf{f}^T \Theta(ih) + \frac{\mathrm{T}}{2} \sum_{\tau=0}^{p} \omega_\tau H(ih, \kappa_\tau, \mathbf{C}^T L^\alpha \Theta(\kappa_\tau) + u_0) \\ &+ \frac{ih}{2} \sum_{\tau=0}^{p} \omega_\tau F(ih, \sigma_{\tau,i}, \mathbf{C}^T L^\alpha \Theta(\sigma_{\tau,i}) + u_0) \\ &+ \sum_{j=1}^{n} \Big(\sum_{k=0}^{M} G_j(ih, s_{i,k}, \mathbf{C}^T L^\alpha \Theta(s_{i,k}) + u_0) \left(B_j(s_{i,k+1}) - B_j(s_{i,k})\right) \Big). \end{aligned} \tag{3.12}$$

Solving this nonlinear system, leads to an approximate solution for (1.1)-(1.1).

4. Error estimate

In this section, error estimate of the proposed method have been discussed. Here we consider the norm

$$\|g\| = \mathbb{E}[\sup_{t \in \Omega} |g(t)|], \tag{4.1}$$

where $\mathbb{E}[.]$ is the mathematical expectation.

Theorem 4.1. *Suppose $g(t) \in C^3(\Omega)$ and $g_N(t)$ is the MHFs expansion of $g(t)$ that defined in (2.6). Then we have*

$$\sup_{t \in \Omega} |g(t) - g_N(t)| \leq ch^3, \tag{4.2}$$

in which c is a constant value.

Proof. See [22].

Theorem 4.2. *Suppose $u(t)$ be the exact solution and $u_N(t)$ is the numerical solution of (1.1)-(1.1). Also, let $H(t, s, u(s))$, $F(t, s, u(s))$ and $G_j(t, s, u(s))$ are sufficiently continuously differentiable on Ω and satisfy the following*

Lipschitz conditions as

$$\begin{aligned}
\Big\|H(t,s,u_1)-H(t,s,u_2)\Big\| &\le L_H\Big\|u_1-u_2\Big\|, && (4.3)\\
\Big\|F(t,s,u_1)-F(t,s,u_2)\Big\| &\le L_F\Big\|u_1-u_2\Big\|, && (4.4)\\
\Big\|G_j(t,s,u_1)-G_j(t,s,u_2)\Big\| &\le L_{G_j}\Big\|u_1-u_2\Big\|, \qquad j=1,...,n, && (4.5)
\end{aligned}$$

where L_H, L_F and L_{G_j} are some positive constants. Also, assume that $\hat{\mu}\hat{\eta}<1$, where $\hat{\mu}:=\frac{\mathtt{T}^{\alpha}}{\Gamma(\alpha+1)}$, $\hat{\eta}:=\sigma_1(L_H+L_F)+\delta\bar{L}$ and $\bar{L}=\max\{L_{G_j},\ j=1,\dots,n\}$. Then

$$\|u-u_N\|\le\frac{\hat{\mu}}{1-\hat{\mu}\hat{\eta}}\Big(ch^3+\mathrm{E}^p_{Gauss}\Big),$$

where $\mathrm{E}^p_{Gauss}=max\{\mathrm{E}^p_{Gauss}(H),\mathrm{E}^p_{Gauss}(F)\}$.

Proof. First, by employing Riemann-Liouville fractional integrating, Eq. (1.1) changes into

$$\begin{aligned}
u(t) &= u_0+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\mathrm{d}s+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left(\int_0^{\mathtt{T}}H(s,r,u(r))\mathrm{d}r\right)\mathrm{d}s\\
&+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left(\int_0^{s}F(s,r,u(r))\mathrm{d}r\right)\mathrm{d}s\\
&+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left(\sum_{j=1}^n\int_0^{s}G_j(s,r,u(r))\mathrm{d}B_j(r)\right)\mathrm{d}s. && (4.6)
\end{aligned}$$

Also, $u_N(t)$ satisfies the equation

$$\begin{aligned}
u_N(t) &= u_0+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f_N(s)\mathrm{d}s+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left(\int_0^{\mathtt{T}}H(s,r,u_N(r))\mathrm{d}r\right)\mathrm{d}s\\
&+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left(\int_0^{s}F(s,r,u_N(r))\mathrm{d}r\right)\mathrm{d}s\\
&+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left(\sum_{j=1}^n\int_0^{s}G_j(s,r,u_N(r))\mathrm{d}B_j(r)\right)\mathrm{d}s, && (4.7)
\end{aligned}$$

where $f_N(t)$ is the MHFs expansion of $f(t)$ in the form (2.6). Let

$$\begin{aligned}
\mathcal{J}_H(t) &= \int_0^{\mathtt{T}}\Big(H(t,s,u(s))-H(t,s,u_N(s))\Big)\mathrm{d}s, && (4.8)\\
\mathcal{J}_F(t) &= \int_0^{t}\Big(F(t,s,u(s))-F(t,s,u_N(s))\Big)\mathrm{d}s, && (4.9)\\
\mathcal{J}_G(t) &= \sum_{j=1}^n\int_0^{t}\Big(G_j(t,s,u(s))-G_j(t,s,u_N(s))\Big)\mathrm{d}B_j(s). && (4.10)
\end{aligned}$$

Now, we define

$$\begin{aligned}
\mathbf{e}_N(t) &= u(t)-u_N(t), && (4.11)\\
\mathbf{e}_f(t) &= \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left(f(s)-f_N(s)\right)\mathrm{d}s, && (4.12)\\
\mathbf{e}_H(t) &= \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\mathcal{J}_H(s)\mathrm{d}s, && (4.13)\\
\mathbf{e}_F(t) &= \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\mathcal{J}_F(s)\mathrm{d}s, && (4.14)
\end{aligned}$$

and

$$\mathbf{e}_G(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mathcal{J}_G(s)\mathrm{d}s. \tag{4.15}$$

By these definitions, subtracting (4.7) from (4.6) yields

$$\mathbf{e}_N(t) = \mathbf{e}_f(t) + \mathbf{e}_H(t) + \mathbf{e}_F(t) + \mathbf{e}_G(t), \tag{4.16}$$

From Theorem 4.1, we get

$$\|\mathbf{e}_f\| \leq \frac{\sup_{t\in\Omega} |f(t) - f_N(t)|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\mathrm{d}s \leq \frac{\mathtt{T}^\alpha}{\Gamma(\alpha+1)} ch^3. \tag{4.17}$$

From (4.13)

$$\|\mathbf{e}_H\| \leq \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\alpha-1} \|\mathcal{J}_H\| \mathrm{d}s \right) \leq \frac{\mathtt{T}^\alpha}{\Gamma(\alpha+1)} \|\mathcal{J}_H\|. \tag{4.18}$$

Now, by using Legendre-Gauss quadrature formula (2.9)

$$\mathcal{J}_H(t) = \frac{\mathtt{T}}{2} \sum_{\tau=0}^{p} \omega_\tau (H(t, \kappa_\tau, u(\kappa_\tau)) - H(t, \kappa_\tau, u_N(\kappa_\tau))) + \mathrm{E}^p_{\mathtt{Gauss}}(H),$$

Moreover, the function H satisfies the Lipschitz condition (4.3), hence

$$\begin{aligned} \|\mathcal{J}_H\| \quad &\leq \quad \frac{\mathtt{T}}{2} \sum_{\tau=0}^{p} \omega_\tau \|H(t, \kappa_\tau, u(\kappa_\tau)) - H(t, \kappa_\tau, u_N(\kappa_\tau))\| + \mathrm{E}^p_{\mathtt{Gauss}}(H) \\ &\leq \quad \sigma_1 L_H \|u - u_N\| + \mathrm{E}^p_{\mathtt{Gauss}}(H), \end{aligned} \tag{4.19}$$

where $\sigma_1 = \frac{\mathtt{T}}{2} \sum_{\tau=0}^{p} \omega_\tau$. So, we obtain

$$\|\mathbf{e}_H\| \leq \frac{\mathtt{T}^\alpha}{\Gamma(\alpha+1)} \Big(\sigma_1 L_H \|u - u_N\| + \mathrm{E}^p_{\mathtt{Gauss}}(H) \Big). \tag{4.20}$$

In a similar manner, we have

$$\|\mathbf{e}_F\| \leq \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\alpha-1} \|\mathcal{J}_F\| \mathrm{d}s \right) \leq \frac{\mathtt{T}^\alpha}{\Gamma(\alpha+1)} \|\mathcal{J}_F\|. \tag{4.21}$$

Since $t \leq \mathtt{T}$ and the function F satisfies the Lipschitz condition (4.4), using Legendre-Gauss quadrature formula (2.9) yields

$$\|\mathcal{J}_F\| \leq \sigma_1 L_F \|u - u_N\| + \mathrm{E}^p_{\mathtt{Gauss}}(F). \tag{4.22}$$

Thus

$$\|\mathbf{e}_F\| \leq \frac{\mathtt{T}^\alpha}{\Gamma(\alpha+1)} \Big(\sigma_1 L_F \|u - u_N\| + \mathrm{E}^p_{\mathtt{Gauss}}(F) \Big). \tag{4.23}$$

Also, we have

$$\|\mathbf{e}_G\| \leq \frac{\|\mathcal{J}_G\|}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\alpha-1} \mathrm{d}s \right) \leq \frac{\mathtt{T}^\alpha}{\Gamma(\alpha+1)} \|\mathcal{J}_G\|. \tag{4.24}$$

Since the functions G_j, $j = 1, ..., n$, satisfy the Lipschitz conditions (4.5), for all $0 \leq s \leq t \leq \mathtt{T}$, we have

$$\begin{aligned} \|\mathcal{J}_G\| &\leq \Big(\sum_{j=1}^{n} \sum_{k=0}^{M} \|G_j(t, s_{t,k}, u(s_{t,k})) - G_j(t, s_{t,k}, u_N(s_{i,k}))\| \times \|B_j(s_{t,k+1}) - B_j(s_{t,k})\| \Big) \\ &\leq \sum_{j=1}^{n} \sum_{k=0}^{M} L_{G_j} \|u - u_N\| \times \|B_j(s_{t,k+1}) - B_j(s_{t,k})\|, \end{aligned}$$

where $s_{t,k} = \frac{t}{M}k$. Moreover, $B(t)$, $0 \le t \le \mathtt{T}$, is a continuous bounded function, thus we can let

$$\delta = \max_{j=1,\dots,n} \left\{ \sup_{0\le t\le \mathtt{T}} \left\{ \sum_{k=0}^{M} \|B_j(s_{t,k+1}) - B_j(s_{t,k})\| \right\} \right\},$$

then, we have

$$\|\mathcal{J}_G\| \le \delta \bar{L} \|u - u_N\|, \tag{4.25}$$

where $\bar{L} = \max\{L_{\mathrm{G}_j},\ j = 1, \dots, n\}$. Thus, we obtain

$$\|\mathbf{e}_G\| \le \frac{\mathtt{T}^{\alpha}}{\Gamma(\alpha+1)} \delta \bar{L} \|u - u_N\|. \tag{4.26}$$

Thus, according to Eq. (4.16) and the relations (4.17), (4.20), (4.23) and (4.26), it can be concluded that

$$\|\mathbf{e}_N\| \le \frac{\hat{\mu}}{1 - \hat{\mu}\hat{\eta}} \Big(ch^3 + \mathrm{E}^{p}_{\mathtt{Gauss}} \Big), \tag{4.27}$$

□

So, from the above theorem, it is clear that $\mathbb{E}\|u - u_N\|$ tends to zero, when $h \to 0$ (or $N \to \infty$) and $p \to \infty$.

Algorithm

Input: $\mathtt{T} \in \mathbb{R}^+$, $\zeta \in \mathbb{R}$, $n, N, p \in \mathbb{Z}^+$, $\alpha \in (0,1)$, functions f, H, F, G_j, $j = 1, ..., n$ u_0 and Brownian motion processes $B_j(t)$, $j = 1, ..., n$. Let $h = \frac{\mathtt{T}}{N}$.
Step 1: Compute the MHFs $\theta_i(t)$, $i = 0, ..., N$, from Definition 2.3.
Step 2: Compute the vector of MHFs $\Theta(t)$ from Eq. (2.7) and let the coefficients vector $\mathbf{C}$ from Eq. (3.2).
Step 3: Compute the vector $\mathbf{f} = [f_0, ..., f_N]^T$ where $f_i = f(ih)$, $i = 0, ..., N$.
Step 4: Compute the operational matrix of fractional order L^{α}, from Theorem 2.4.
Step 5: Compute the Legendre polynomial L_{p+1} on the interval $[-1, 1]$.
Step 6: Let $\{\sigma_\tau\}_{\tau=0}^{p}$ the zeros of $L_{p+1}(t)$ and Compute the weights $\{\omega_\tau\}_{\tau=0}^{p}$ from Eq. (2.10).
Step 7: Compute $\sigma_{\tau,i}$ and κ_τ from Eq. (3.8) and $s_{i,k}$ from Eq. (3.10).
Step 8: Assign I_0, I_1 and I_2 at collocation points $t = ih$, by using Eqs. (3.6), (3.7) and (3.9), respectively.
Step 9: Solve the nonlinear system (3.12) by applying **Step 3** and **Step 8**. Then, obtain the unknown vector $\mathbf{C}$.
Output: The approximate solution: $u(t) \simeq \mathbf{C}^T L^{\alpha} \theta(t) + u_0$ from (3.3).

---

5. Numerical implementation

In this section, we assess the applicability of our proposed approach to solve nonlinear fractional stochastic integro-differential equations with the n-dimensional Wiener process.

To simulate the Brownian motion $B(t)$, we employ the approach described in [8]. To this aim, we consider a discretization of $B(t)$. We set $t_0 = 0$ and $t_j = jh, j = 1, ..., N$, where $t_i < t_j$ for $i < j$. Also, let $B_j = B(t_j)$ and

$$\Delta_j = t_j - t_{j-1}, \qquad j = 1, ..., N. \tag{5.1}$$

From the definition of Brownian motion $B(t)$ on $(\Omega, \mathcal{F}, \mathbb{P})$, we know that $B(0) = 0$ with the probability 1 and $B(\tau) - B(r) \sim \sqrt{\tau - r}\mathcal{N}(0,1)$, for $0 \le r < \tau \le T$, where $\mathcal{N}(0,1)$ is a normally distributed random variable with zero mean and unit variance. Also, $B(\tau_2) - B(\tau_1)$ and $B(\nu_2) - B(\nu_1)$ are independent for $0 \le \tau_1 < \tau_2 < \nu_1 < \nu_2 \le T$. Thus, we let $B_0 = t_0$ with the probability 1, and

$$B_j = B_{j-1} + \mathrm{d}B_j, \qquad j = 1, ..., M, \tag{5.2}$$

where each $\mathrm{d}B_j$ is an independent random variable of the form $\sqrt{\Delta_j}\mathcal{N}(0,1)$. For testing the presented approach, we run our algorithm for $\hat{p}$ different iterations. Then, the arithmetic mean of these obtained approximate solutions will be considered as the numerical solution of the problem. The computations have been executed on a personal computer using a 2.20 GHz processor and the codes are written in Matlab software 2017.

Example 5.1. Consider the fractional stochastic integro-differential equation

$$\begin{aligned} {}_0D_t^{\alpha}u(t) &= f(t) + \int_0^1 tu(s)\mathrm{d}s + \int_0^t t\cos(\pi s)u(s)\mathrm{d}s \\ &+ \int_0^t tu(s)\mathrm{d}B_1(s) + \int_0^t e^{t-u(s)}\mathrm{d}B_2(s) + \int_0^t \sin(t)su^2(s)\mathrm{d}B_3(s), \end{aligned}$$

where

$$\begin{aligned} f(t) = &\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{t}{3} - \frac{t}{\pi^3}\Big(2\pi t\cos(\pi t) + (\pi^2 t^2 - 2)\sin(\pi t)\Big) \\ &-t^3B_1(t) - e^{t-t^2}B_2(t) - t^5\sin(t)B_3(t) + 2t\int_0^t sB_1(s)\mathrm{d}s \\ &-2\int_0^t se^{s-s^2}B_2(s)\mathrm{d}s + 5\sin(t)\int_0^t s^4B_3(s)\mathrm{d}s, \end{aligned}$$

with the initial condition $u(0) = 0$. The exact solution of this problem is $u(t) = t^2$.

Figure 1 shows the approximate solutions over $\hat{p} = 50$ different discretized Brownian paths (blue) with $\alpha = 0.5$, $N = 32$ and $p = 8$ and their arithmetic mean (red). Table 1 displays the absolute errors of the numerical solution for $u(t)$, when $\alpha = 0.6$, $p = 8$ and $\hat{p} = 80$. Figure 2 indicates the behaviour of the obtained approximate solutions with different values of $\hat{p}$, when $\alpha = 0.5$, $N = 32$ and $p = 10$. Also, Figure 3 displays the exact and numerical solutions and the absolute errors of $u(t)$, when $\alpha = 0.45$, $N = 48$, $p = 10$ and $\hat{p} = 200$.

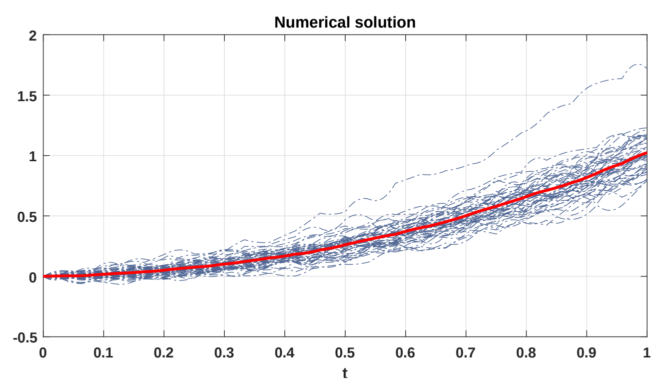

FIGURE 1. The obtained numerical solutions over $\hat{p} = 50$ different discretized Brownian paths (blue) and their mean (red) in Example 5.1 with $\alpha = 0.5$.

Example 5.2. Consider the fractional stochastic integro-differential equation

$$ {}_0D_t^{\alpha}u(t) = f(t) + \int_0^t su^2(s)\mathrm{d}s + \int_0^t e^{t-u(s)}\mathrm{d}B_1(s) + \int_0^t \sin(u(s))\mathrm{d}B_2(s)$$

TABLE 1. Absolute errors of the numerical solution of Example 5.1 for several values of N.

t	$N=24$	$N=36$	$N=48$
0.1	6.2471×10^{-2}	3.2113×10^{-2}	7.2741×10^{-3}
0.2	5.1044×10^{-2}	2.0016×10^{-2}	2.1541×10^{-3}
0.3	2.5102×10^{-2}	8.4452×10^{-3}	4.2636×10^{-3}
0.4	4.2677×10^{-2}	2.1378×10^{-2}	2.2741×10^{-3}
0.5	2.5103×10^{-2}	9.9194×10^{-3}	4.0113×10^{-3}
0.6	4.2812×10^{-2}	1.2631×10^{-2}	6.1760×10^{-3}
0.7	5.2331×10^{-2}	2.4134×10^{-2}	2.4579×10^{-3}
0.8	4.1359×10^{-2}	2.8891×10^{-2}	3.3215×10^{-3}
0.9	3.2261×10^{-2}	7.3101×10^{-3}	3.1103×10^{-3}

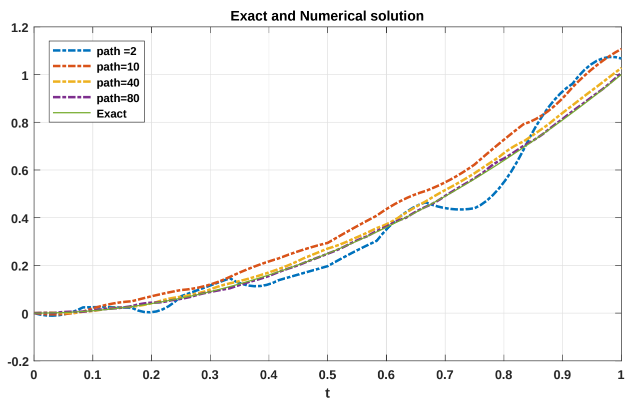

FIGURE 2. The exact and numerical solution in Example 5.1 for different values of $\hat{p}$.

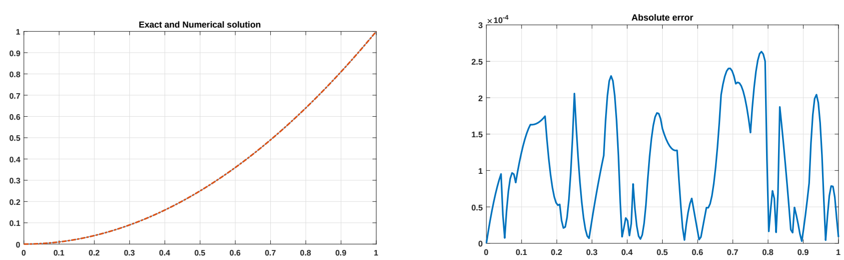

FIGURE 3. The exact and numerical solutions of Example 5.1 (left) and the absolute errors (right) when $\hat{p}=200$.

with the initial condition $u(0)=0$, and

$$\begin{aligned} f(t) &= \frac{\Gamma(4)t^{3-\alpha}}{\Gamma(4-\alpha)} - \frac{\Gamma(5)t^{4-\alpha}}{\Gamma(5-\alpha)} - \frac{t^8}{360}\Big(36t^2-80t+45\Big) \\ &+ \int_0^t \Big(s^3-3s^2(1-s)\Big)\Big[e^{t-s^3(1-s)}B_1(s) - \cos(s^3(1-s))B_2(s)\Big]\mathrm{d}s \\ &-e^{t-t^3(1-t)}B_1(t) - \sin(t^3(1-t))B_2(t) \end{aligned}$$

The exact solution of this problem is $u(t)=t^3(1-t)$.

TABLE 2. Absolute errors of the numerical solution of Example 5.2 for several values of N.

t	$N=12$	$N=24$	$N=36$	$N=48$
0.1	3.2113×10^{-2}	7.2231×10^{-3}	2.4101×10^{-3}	7.2544×10^{-4}
0.3	5.4023×10^{-2}	1.2434×10^{-2}	5.3324×10^{-3}	2.2001×10^{-3}
0.5	6.2774×10^{-2}	3.5302×10^{-2}	1.4452×10^{-3}	5.6117×10^{-4}
0.7	3.6314×10^{-2}	4.2417×10^{-3}	4.5278×10^{-3}	1.2976×10^{-3}
0.9	5.1224×10^{-2}	2.8132×10^{-2}	2.0114×10^{-3}	6.3566×10^{-4}

Figure 4 shows the approximate solution over $\hat{p}=100$ discretized Brownian paths (blue) and their arithmetic mean (red) when $\alpha=0.3$, $N=32$ and $p=10$. Also, Figure 5 displays the behaviour of the approximate solutions for different values of $\hat{p}$, when $\alpha=0.5$, $N=32$ and $p=7$. Table 2 indicates the absolute errors of the obtained numerical solution when $\alpha=0.6$, $p=10$ and $\hat{p}=150$. Figure 6 displays the absolute errors when $\alpha=0.5$, $N=40$ $p=10$ and $\hat{p}=200$. Finally, Figure 7 shows the logarithm of absolute error when $N=20,30,40$, $\alpha=0.6$, $p=8$ and $\hat{p}=100$.

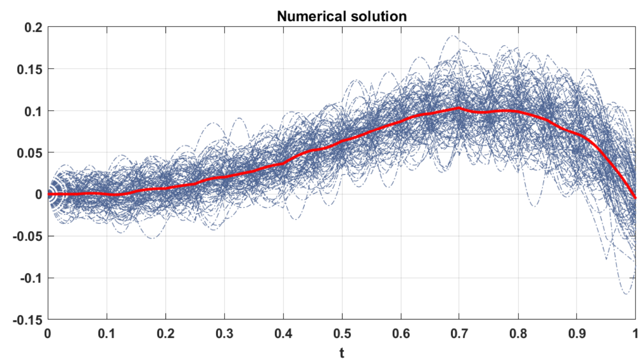

FIGURE 4. The numerical solutions of Example 5.2 over $\hat{p}=100$ different discretized Brownian paths (blue) and the obtained mean solution (red) when $\alpha=0.3$.

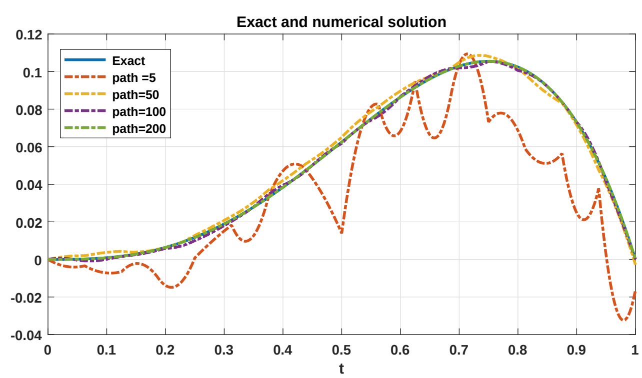

FIGURE 5. The exact and numerical solutions in Example 5.2 for different values of $\hat{p}$.

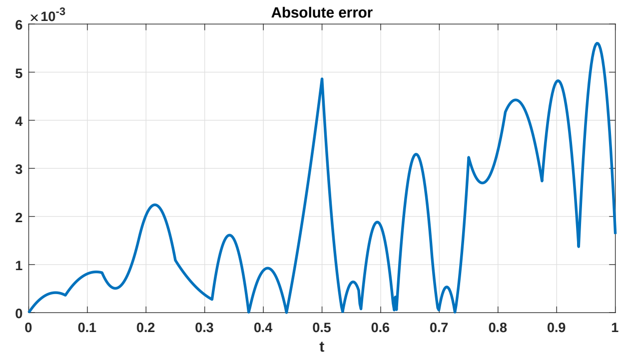

FIGURE 6. The absolute error of $u_N(t)$ in Example 5.2 with $\hat{p} = 200$.

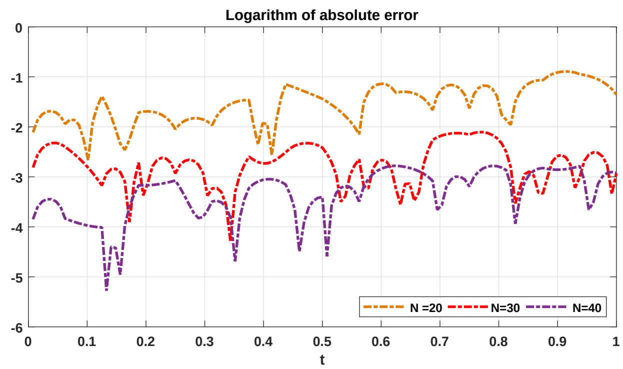

FIGURE 7. The logarithm of absolute errors for several values of N in Example 5.2.

6. Conclusion

In this work, by using the properties of modified hat functions and some suitable numerical integration rules, a numerical scheme is introduced for solving a class of nonlinear fractional stochastic integro-differential equations with n-dimensional Wiener process. Error estimate of the method was discussed. Furthermore, two numerical examples have been prepared to illustrate the effectiveness and ability of this algorithm. The obtained results confirm the good accuracy and reliability of the proposed method.

References

[1] A. Babaei, H. Jafari, and S. Banihashemi, *Numerical solution of variable order fractional nonlinear quadratic integro-differential equations based on the sixth-kind Chebyshev collocation method*, Journal of Computational and Applied Mathematics (2020), 112908.

[2] A. Cardone, R. D'Ambrosio, and B. Patersone, *A spectral method for stochastic fractional differential equations*, Applied Numerical Mathematics, *139*(1) (2019), 115-119.

[3] F. R. Chang, *Stochastic Optimization in Continuous Time*, Cambridge University Press, Cambridge, 2004.

[4] G. H. Choe, *Stochastic analysis for finance with simulations*, Universitext, Springer, 2016.

[5] S. I. Denisov, P. Hanggi, and H. Kantz, *Parameters of the fractional Fokker-Planck equation*, Europhys. Lett, *85*(4) (2009), 40007.

[6] M. Fallahpour, K. Maleknejad, and M. Khodabin, *Approximation solution of two-dimensional linear stochastic fredholm integral equation by applying the haar wavelet*, International Journal of Mathematical Modelling & Computations, *5*(4) (2015), 361-372.

[7] M. Girgoriu, *Stochastic calculus: Applications in Science and Engineering*, Springer, LLC, 2000.

[8] D. J. Higham, *An algorithmic introduction to numerical simulation of stochastic differential equations*, Society for Industrial and Applied Mathematics, *43*(3) (2001), 525-546.

[9] L. Huang, X. F. Li, Y. Zhao, and X. Y. Duan, *Approximate solution of fractional integro-differential equations by Taylor expansion method*, Computers and Mathematics with Applications, *62*(3) (2011), 1127-1134.

[10] D. Jabari Sabegh, R. Ezzati, and K. Maleknejad, *Approximate solution of fractional integro-differentail equations by least squares method*, International Journal of Analysis and Applications, *17*(2) (2019), 303-310.

[11] M. Kamrani, *Numerical solution of stochastic fractional differential equations*, Numerical Algorithms, *68*(1) (2014), 81-93.

[12] M. Khodabin, K. Maleknejad, and T. Damercheli, *Approximate solution of stochastic Volterra integral equations via expansion method*, Int. J. Industeria Mathematics, *6*(1) (2014), 41-48.

[13] A. A Kilbas, H. M. Sirvastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, 204 of North-Holland Mathematics Studies. Elesiver, Amesterdam, 2006.

[14] P.E. Kloden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, Springer-Velarge Berlin Heidelbreg, New York, 1995.

[15] P. K. Kythe and M. R. Schaferkotter, *Handbook of computational methods for integration*, Chapman and Hall/CRC Press, New York, (2005).

[16] M. Li, C. Huang, P. Hu, and J. Wen, *Mean-square stability and convergence of a split-step theta method for stochastic Volterra integral equations*, Journal of Computational and Applied Mathematics, *382*(1) (2021), 113077.

[17] L. Li and J. G. Liu, *A discretization of Caputo derivatives with application to time fractional SDEs and gradient flows*, SIAM Journal on Numerical Analysis, *57*(5) (2019), 2095-2120.

[18] H. Liang, Z. Yang, and J. Gao, *Strong superconvergence of the EulerMaruyama method for linear stochastic Volterra integral equations*, Journal of Computational and Applied Mathematics, *317*(1) (2017), 447-457.

[19] X. Ma and Ch. Huang, *Numerical solution of fractional integro differential equations by hybrid collocation method*, Applied Mathematics and Computation, *219*(12) (2013), 6750-6760.

[20] K. Maleknejad, M. Khodabin, and M. Rostami, *Numerical solution of stochastic Volterra integral equations by a stochastic operational matrix based on block pulse functions*, Math. Comput. Model, *55*(3-4) (2015), 791-800.

[21] X. Mao, *Stochastic Differential Equations and Applications*, Second edition, Elsevier, New York, 2007.

[22] F. Mirzaee and E. Hadadiyan, *Numerical solution of Volterra-Fredholm integral equations via modification of hat functions*, Applied Mathematics and Computations, *280*(1) (2016), 110-123.

[23] F. Mirzaee and N. Samadyar, *Convergence of Legendre wavelet collocation method for solving nonlinear Stratonovich Volterra integral equations*, Computational Methods for Differential Equations, *6*(1) (2018), 80-97.

[24] B. P. Moghadam, A. Mendes Lopes, J. A. Tenreiro Machado, and Z. S. Mostaghim, *Computational scheme for solving nonlinear fractional stochastic differential equations with delay*, Stochastic Analysis and Applications, *37*(6) (2019), 893-908.

[25] F. Mohammadi, *A Chebyshev wavelet operational method for solving stochastic Volterra-Fredholm integral equations*, International Journal of Applied Mathematics Research, *4*(2) (2015), 217-227.

[26] S. Nemati and P. M. Lima, *Numerical solution of nonlinear fractional integro-differential equations with weakly singular kernels via a modification of hat functions*, Applied Mathematics and Computation, *327*(1) (2018), 79-92 .

[27] M. Nouri, *Solving Ito integral equations with time delay via basis functions*, Computational Methods for Differential Equations, *8*(2) (2020), 268-281.

[28] B. Oksendal, *Stochastic Differential Equations: An Introduction with Applications*, 5th edition, Springer-Velarge, Berlin, 1998.

[29] R. Panda and M. Dash, *Fractional generalized splines and signal processing*, Signal Process, *86*(9) (2006), 2340-2350.

[30] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.

[31] M. Saffarzadeh, G. B. Loghmani, and M. Heydari, *An iterative technique for the numerical solution of non-linear Ito-stochastic integral equations*, Journal of Computational and Applied Mathematics, *333*(1) (2018), 74-86.

[32] M. Senol and H. D. Kasmaei, *On the numerical solution of nonlinear fractional integro-differential equations*, New Trends in Mathematical Sciences, *5*(3) (2017), 118-127.

[33] Z. Taheri, S. Javadi, and E. Babolian, *Numerical solution of stochastic fractional integro-differential equation by the spectral collocation method*, Journal of Computational and Applied Mathematics, *321*(1) (2017), 336-347.

[34] Z. Yang, H. Yang, and Z. Yao, *Strong convergence analysis for Volterra integro-differential equations with fractional Brownian motions*, Journal of Computational and Applied Mathematics, *383*(1) (2021), 113156.