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# Existence and Hyers-Ulam stability of random impulsive stochastic functional integrodifferential equations with finite delays

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**Abstract**

In this article, we study the existence and Hyers-Ulam stability of random impulsive stochastic functional integrodifferential equations with finite delays. Firstly, we prove the existence of mild solutions to the equations by using Banach fixed point theorem. In the later case we explore the Hyers Ulam stability results under the Lipschitz condition on a bounded and closed interval.

**Keywords.** Existence, Random impulsive, Hyers-Ulam stability, Integrodifferential equations.

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## 1. Introduction

Stochastic differential equations (SDEs) captures the disturbance from random factors. By the interaction of stochastic process into mathematic models yields a better understanding of the corresponding real-world system [8]. Several systems are modelled using stochastic functional differential equations with impulses. In general, impulses appears at random time points, i.e., the impulse time and the impulsive function are random variables. Sanz-Serna et al. [11] investigated the ergodicity of dissipative differential equations subject to random impulses. The random impulsive differential equations with the existence, uniqueness and stability of solutions is studied (see [3, 5, 13, 14, 19] and the reference therein). Random impulsive stochastic differential equations are widely used in the fields of medicine, biology, economics, finance and so on. For example, the classical stock price model see [15]. However, the Hyers-Ulam stability problem of SDEs have not been used in many articles. Only few works have been reported in the Hyers-Ulam stability for SDEs, refer to [2, 9, 18].

To the best of our knowledge, there has not been much of a study relating to Ulam-Hyers stability for SDEs with random impulses has not been investigated. Motivated by the above studies, in this paper, we investigate the Hyers-Ulam stability for d-dimensional random impulsive stochastic functional integrodifferential equations of the form:

$$d\left[u(t)\right] = \left[\mathtt{f}(t,u_t) + \int_0^t \mathtt{P}(t,s,u_s)ds\right]dt + \mathtt{g}(t,u_t)dw(t), t \geq t_0, t \neq \zeta_k, \tag{1.1}$$

$$u(\zeta_k) = b_k(\tau_k)u(\zeta_k^-),\ k = 1,2,..., \tag{1.2}$$

$$u_{t_0} = \zeta = \left\{\zeta(\theta) : -\tau \leq \theta \leq 0\right\}, \tag{1.3}$$

where $\tau_k$ is a random variable defined from $\Omega$ to $\mathtt{D}_k \stackrel{def}{=} (0, d_k)$ for $k = 1, 2, ...,$ where $0 < d_k < +\infty$. Suppose that $\tau_i$ and $\tau_j$ are independent of each other as $i \neq j$ for $i, j = 1, 2, ...,$ Here, $\mathtt{f} : [t_0, T] \times \mathscr{C} \to \mathtt{R}^d$, $\mathtt{g} : [t_0, T] \times \mathscr{C} \to \mathtt{R}^{d\times m}$,

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$\mathtt{P} : [t_0, T] \times [t_0, T] \times \mathscr{C} \to \mathtt{R}^d$ and $b_k : \mathtt{D}_k \to \mathtt{R}^{d\times d}$ are Borel measurable functions, and $u_t$ is $\mathtt{R}^d$-valued stochastic process such that

$$u_t \quad = \quad \{u(t+\theta) : -\tau \leq \theta \leq 0\}, \; u_t \in \mathtt{R}^d.$$

The impulsive moments $\zeta_k$ from a strictly increasing sequence, i.e. $\zeta_0 < \zeta_1 < \zeta_2 < \cdots < \zeta_k < \cdots < \lim_{k\to\infty} \zeta_k = \infty$, and $u(\zeta_k^-) = \lim_{t\to\zeta_k - 0} u(t)$. We assume that $\zeta_0 = t_0$ and $\zeta_k = \zeta_{k-1} + \tau_k$ for $k = 1, 2, ....$ Obviously, $\{\zeta_k\}$ is a process with independent increments. We suppose that $\{\mathtt{N}(t), t \geq 0\}$ is the simple counting process generated by $\{\zeta_k\}$, and $\{\mathcal{W}(t), t \geq 0\}$ is a given m-dimensional Wiener process. We denote $\mathscr{F}_t^{(1)}$ the $\sigma$-algebra generated by $\{\mathtt{N}(t), t \geq 0\}$, and denote $\mathscr{F}_t^{(2)}$ the $\sigma$-algebra generated by $\{\mathcal{W}(s), s \leq t\}$.

## 2. Preliminaries

Let $(\Omega, \mathscr{F}, \mathtt{P})$ is a probability space with filtration $\{\mathscr{F}_t\}$, $t \geq 0$ satisfying $\mathscr{F}_t = \mathscr{F}_t^{(1)} \vee \mathscr{F}_t^{(2)}$. Let $\mathcal{L}^p(\Omega, \mathtt{R}^d)$ be the collection of all strongly measurable, p-th integrable, $\mathscr{F}_t$ measurable, $\mathtt{R}^d$-valued random variables $x$ with norm $\|u\|_{\mathcal{L}^p} = (\mathbf{E}\,\|u\|^p)^{\frac{1}{p}}$, Let $\tau > 0$ and denote the Banach space of all piecewise continuous $\mathtt{R}^d$-valued stochastic process $\{\zeta(t), t \in [-\tau, 0]\}$ by $\mathscr{C}([-\tau, 0], \mathcal{L}(\Omega, \mathtt{R}^d))$ equipped with the norm

$$\|\psi\|_{\mathscr{C}} \quad = \quad \sup_{\theta\in[-\tau,0]} (\mathbf{E}\,\|\psi(\theta)\|^p)^{\frac{1}{p}}\,.$$

The initial data

$$u_{t_0} = \zeta = \{\zeta(\theta) : -\tau \leq \theta \leq 0\} \tag{2.1}$$

is an $\mathscr{F}_{t_0}$ measurable, $[-\tau, 0]$ to $\mathtt{R}^d$-valued random variable such that $\mathbf{E}\,\|\zeta\|^p < \infty$.

**Definition 2.1.** For a given $T \in (t_0, +\infty)$, a $\mathtt{R}^d$-valued stochastic process $u(t)$ on $t_0 - \tau \leq t \leq T$ is called a solution to (1.1)-(1.3) with the initial data (2.1) if for every $t_0 \leq t \leq T$, $u_{t_0} = \zeta$, $\{u_t\}_{t_0\leq t\leq T}$ is $\mathscr{F}_t$-adapted and

$$\begin{aligned} u(t) = \sum_{k=0}^{+\infty} \Bigg[ & \prod_{i=1}^{k} b_i(\tau_i)\zeta(0) + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j) \int_{\zeta_{i-1}}^{\zeta_i} \mathtt{f}(s, u_s)ds + \int_{\zeta_k}^{t} \mathtt{f}(s, u_s)ds \\ & + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j) \int_{\zeta_{i-1}}^{\zeta_i}\int_0^s \mathtt{P}(s, r, u_r)drds + \int_{\zeta_k}^{t}\int_0^s \mathtt{P}(s, r, u_r)drds \\ & + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j) \int_{\zeta_{i-1}}^{\zeta_i} \mathtt{g}(s, u_s)dw(s) + \int_{\zeta_k}^{t} \mathtt{g}(s, u_s)dw(s) \Bigg] I_{(\zeta_k, \zeta_{k-1}]}(t) \; a.s. \end{aligned} \tag{2.2}$$

where $\prod_{j=i}^{k} b_j(\tau_j) = b_k(\tau_k)b_{k-1}(\tau_k - 1)\cdots b_i(\tau_i)$, and $\mathtt{I}_{\mathtt{G}}(\cdot)$ is the index function, i.e.

$$\mathtt{I}_{\mathtt{G}}(t) = \begin{cases} 1, & if \;\, t \in \mathtt{G}, \\ 0, & if \;\, t \notin \mathtt{G}. \end{cases}$$

**Definition 2.2. Hyers-Ulam stability**: Suppose that $v(t)$ is a $\mathtt{R}^d$-valued stochastic process. If there exists a real number $M > 0$, such that for arbitrary $\epsilon \geq 0$, satisfying

$$\begin{aligned} \mathbf{E}\Bigg\| v(t) - \sum_{k=0}^{+\infty} \Bigg[ & \prod_{i=1}^{k} b_i(\tau_i)\zeta(0) + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j) \int_{\zeta_{i-1}}^{\zeta_i} \mathtt{f}(s, v_s)ds + \int_{\zeta_k}^{t} \mathtt{f}(s, v_s)ds \\ & + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j) \int_{\zeta_{i-1}}^{\zeta_i}\int_0^s \mathtt{P}(s, r, v_r)drds + \int_{\zeta_k}^{t}\int_0^s \mathtt{P}(s, r, v_r)drds \\ & + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j) \int_{\zeta_{i-1}}^{\zeta_i} \mathtt{g}(s, v_s)dw(s) + \int_{\zeta_k}^{t} \mathtt{g}(s, v_s)dw(s) \Bigg] I_{(\zeta_k, \zeta_{k-1}]}(t) \Bigg\|^p \leq \epsilon. \end{aligned}$$

For each solution $u(t)$ with the initial value $v_{t_0} = u_{t_0} = \zeta$, if there exists a solution $u(t)$ of equations (1.1)-(1.3) with

$$\mathbf{E}\,\|v(t) - u(t)\|^p \quad \leq \quad M\epsilon,\ \forall t \in (t_0 - \tau, T),$$

then equations (1.1)-(1.3) has the Hyers-Ulam stability.

**Lemma 2.3.** *[3] Let $\phi, \varphi \in \mathscr{C}([a,b], \mathtt{R}^d)$ be two functions. We suppose that $\phi(t)$ is nondecreasing. If $u(t) \in \mathscr{C}([a,b], \mathtt{R}^d)$ is a solution of the following inequality*

$$u(t) \leq \phi(t) + \int_a^t \varphi(s)u(s)ds,\ t \in [a,b],$$

*then $u(t) \leq \phi(t)\exp(\int_a^t \varphi(s)ds)$.*

**Lemma 2.4.** *[8] For any $p \geq 1$ and for any predictable process $u \in \mathcal{L}^p_{d\times m}[0,T]$, the inequality holds,*

$$\sup_{s\in[o,t]} \mathbf{E}\,\|u(t)dw(t)\|^p \quad \leq \quad (p/2(p-1))^{p/2}(\int_0^t (\mathbf{E}\,\|u(s)\|^p)^{2/p}ds)^{p/2},\ t \in [0,T].$$

## 3. Main Results

Here, we will consider the existence results for system (1.1)-(1.3). Now we introduce the following hypotheses used in our discussion:

(H1) The function $\mathtt{f} : [t_0, T] \times \mathscr{C} \to \mathtt{R}^d$ satisfies the following conditions: there exist positive constant $L_1, L_2 > 0$ such that, for all $t \in [t_0, T]$ and $\psi_1, \psi_2 \in \mathscr{C}$

$$\begin{aligned}\mathbf{E}\,\|\mathtt{f}(t,\psi_1) - \mathtt{f}(t,\psi_2)\|^p &\leq L_1\,\|\psi_1 - \psi_2\|^p_{\mathscr{C}}\,,\\ \mathbf{E}\,\|\mathtt{f}(t,\psi)\|^p &\leq L_2\,(1 + \|\psi\|^p_{\mathscr{C}})\,.\end{aligned}$$

(H2) For the continuous function $\mathtt{g} \in \mathcal{L}^p([t_0,T] \times \mathscr{C} : \mathtt{R}^{d\times m})$, there exists a constant $L_3, L_4 > 0$ such that, for all $t \in [t_0, T]$ and $\psi_1, \psi_2 \in \mathscr{C}$

$$\begin{aligned}\mathbf{E}\,\|\mathtt{g}(t,\psi_1) - \mathtt{g}(t,\psi_2)\|^p &\leq L_3\,\|\psi_1 - \psi_2\|^p_{\mathscr{C}}\,,\\ \mathbf{E}\,\|\mathtt{g}(t,\psi)\|^p &\leq L_4\,(1 + \|\psi\|^p_{\mathscr{C}})\,.\end{aligned}$$

(H3) The function $\mathtt{P} : [t_0, T] \times [t_0, T] \times \mathscr{C} \to \mathtt{R}^d$, there exists a constant $L_5, L_6 > 0$ such that, for all $t \in [t_0, T]$ and $\psi_1, \psi_2 \in \mathscr{C}$

$$\begin{aligned}\int_0^t \mathbf{E}\,\|\mathtt{P}(t,s,\psi_1) - \mathtt{P}(t,s,\psi_2)\|^p &\leq L_5\,\|\psi_1 - \psi_2\|^p_{\mathscr{C}}\,,\\ \int_0^t \mathbf{E}\,\|\mathtt{P}(t,s,\psi)\|^p &\leq L_6\,(1 + \|\psi\|^p_{\mathscr{C}})\,.\end{aligned}$$

(H4) The condition $\mathbf{E}\left\{\max_{i,k}\left\{\prod_{j=i}^k \|b_j(\tau_j)\|^p\right\}\right\} < \infty$. That is, there is a constant $\mathtt{B} > 0$ such that

$$\mathbf{E}\left(\max_{i,k}\left\{\prod_{j=i}^k \|b_j(\tau_j)\|\right\}\right)^p \quad \leq \quad \mathtt{B}.$$

**Theorem 3.1.** *Suppose that the assumptions (H1)-(H4) are satisfied. Then the system (1.1)-(1.3) has a unique solution in $\mathcal{B}$.*

*Proof.* Let $\mathcal{B}$ be the space $\mathcal{B} = \mathscr{C}([t_0 - \tau, T], \mathcal{L}^p(\Omega, \mathtt{R}^d))$ endowed using the norm

$$\|u\|^p_{\mathcal{B}} \quad = \quad \sup_{t\in[t_0,T]} \|u_t\|^p_{\mathscr{C}}\,,$$

where $\|u_t\|_{\mathscr{C}} = \sup_{-\tau\le s\le t}\mathbf{E}\,\|u(s)\|^p$. Denote $\mathtt{B}_r = \{u\in\mathcal{B};\|u\|^p_{\mathcal{B}}\le r\}$, which is the closed ball with center $u$ and radius $r>0$. For any initial value $(t_0,u_0)t_0\ge 0$ and $u_0\in\mathtt{B}_r$.
we define the operator $\Theta:\mathcal{B}\to\mathcal{B}$ by

$$
\begin{aligned}
(\Theta u)(t) = \sum_{k=0}^{+\infty}\Bigg[&\prod_{i=1}^{k} b_i(\tau_i)\zeta(0) + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\zeta_{i-1}}^{\zeta_i}\mathtt{f}(s,u_s)ds + \int_{\zeta_k}^{t}\mathtt{f}(s,u_s)ds \\
&+ \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\zeta_{i-1}}^{\zeta_i}\int_0^s \mathtt{P}(s,r,u_r)drds + \int_{\zeta_k}^{t}\int_0^s \mathtt{P}(s,r,u_r)drds \\
&+ \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\zeta_{i-1}}^{\zeta_i}\mathtt{g}(s,u_s)dw(s) + \int_{\zeta_k}^{t}\mathtt{g}(s,u_s)dw(s)\Bigg] I_{(\zeta_k,\zeta_{k-1}]}(t).
\end{aligned}
$$

The continuity of $\Theta$ can be proved easily. Now, we claim that $\Theta$ maps $\mathcal{B}$ into itself.

$$
\begin{aligned}
\mathbf{E}\,\|(\Theta u)(t)\|^p \le{}& 4^{p-1}\max_k \mathbf{E}\left\{\prod_{i=1}^{k}\|b_i(\tau_i)\|^p\right\}\|u_0\|^p + 4^{p-1}\mathbf{E}\big[\max_{i,k}\{1,\prod_{j=i}^{k}\|b_j(\tau_j)\|\,\}\big]^p\left[\int_{t_0}^{t}\|\mathtt{f}(s,u_s)\|\,dsI_{(\zeta_k,\zeta_{k+1}]}(t)\right]^p \\
&+ 4^{p-1}\mathbf{E}\big[\max_{i,k}\{1,\prod_{j=i}^{k}\|b_j(\tau_j)\|\,\}\big]^p\left[\int_{t_0}^{t}\int_0^s\|\mathtt{P}(s,r,u_r)\|\,drdsI_{(\zeta_k,\zeta_{k+1}]}(t)\right]^p \\
&+ 4^{p-1}\mathbf{E}\big[\max_{i,k}\{1,\prod_{j=i}^{k}\|b_j(\tau_j)\|\,\}\big]^p\left[\int_{t_0}^{t}\|\mathtt{g}(s,u_s)\|\,dw(s)I_{(\zeta_k,\zeta_{k+1}]}(t)\right]^p \\
\le{}& 4^{p-1}\mathtt{B}\mathbf{E}\,\|x_0\|^p + 4^{p-1}\max(1,\mathtt{B})(t-t_0)^{p-1}L_2\int_{t_0}^{t}\mathbf{E}(1+\|u_s\|^p_{\mathscr{C}})ds \\
&+ 4^{p-1}\max(1,\mathtt{B})(t-t_0)^{p-1}L_6\int_{t_0}^{t}\mathbf{E}(1+\|u_s\|^p_{\mathscr{C}})ds + 4^{p-1}\max(1,\mathtt{B})(t-t_0)^{\frac{p}{2}-1}c_pL_4\int_{t_0}^{t}\mathbf{E}(1+\|u_s\|^p_{\mathscr{C}})ds.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\sup_{s\in[t-\tau,t]}\mathbf{E}\,\|(\Theta u)(t)\|^p \le{}& 4^{p-1}\mathtt{B}\mathbf{E}\,\|u_0\|^p + \big[4^{p-1}\max(1,\mathtt{B})(t-t_0)^{p-1}L_2 \\
&+ 4^{p-1}\max(1,\mathtt{B})(t-t_0)^{\frac{p}{2}-1}c_pL_4 + 4^{p-1}\max(1,\mathtt{B})(t-t_0)^{p-1}L_6\big](t-t_0)\sup_{s\in[t-\tau,t]}\mathbf{E}(1+\|u_s\|^p_{\mathscr{C}}).
\end{aligned}
$$

Therefore $\Theta$ maps $\mathcal{B}$ into itself.
Now, we have to prove that $\Theta$ is a contraction mapping

$$
\begin{aligned}
\mathbf{E}\,\|(\Theta u)(t)-(\Theta v)(t)\|^p \le{}& 3^{p-1}\mathbf{E}\big[\max_{i,k}\{1,\prod_{j=i}^{k}\|b_j(\tau_j)\|\,\}\big]^p\left[\int_{t_0}^{t}\mathbf{E}\,\|\mathtt{f}(s,u_s)-\mathtt{f}(s,v_s)\|\,ds\right]^p \\
&+ 3^{p-1}\mathbf{E}\big[\max_{i,k}\{1,\prod_{j=i}^{k}\|b_j(\tau_j)\|\,\}\big]^p\left[\int_{t_0}^{t}\int_0^s\mathbf{E}\,\|\mathtt{P}(s,r,u_r)-\mathtt{P}(s,r,v_r)\|\,drds\right]^p \\
&+ 3^{p-1}\mathbf{E}\big[\max_{i,k}\{1,\prod_{j=i}^{k}\|b_j(\tau_j)\|\,\}\big]^p\left[\int_{t_0}^{t}\mathbf{E}\,\|\mathtt{g}(s,u_s)-\mathtt{g}(s,v_s)\|\,dw(s)\right]^p \\
\le{}& 3^{p-1}\max\{1,\mathtt{B}\}(t-t_0)^pL_1\,\|u_s-v_s\|^p_{\mathscr{C}} + 3^{p-1}\max\{1,\mathtt{B}\}(t-t_0)^pL_5\,\|u_s-v_s\|^p_{\mathscr{C}} \\
&+ 3^{p-1}\max\{1,\mathtt{B}\}c_p(t-t_0)^{\frac{p}{2}}L_3\,\|u_s-v_s\|^p_{\mathscr{C}} \\
\le{}& 3^{p-1}\max_{1,\mathtt{B}}\big[L_1(t-t_0)^p + L_5(t-t_0)^p + c_pL_3(t-t_0)^{\frac{p}{2}}\big]\sup_{\theta\in[-\tau,0]}\mathbf{E}\,\|u(t+\theta)-v(t+\theta)\|^p \\
\le{}& 3^{p-1}\max_{1,\mathtt{B}}\big[L_1(t-t_0)^p + L_5(t-t_0)^p + c_pL_3(t-t_0)^{\frac{p}{2}}\big]\sup_{s\in[t-\tau],t}\mathbf{E}\,\|u(s)-v(s)\|^p.
\end{aligned}
$$

Taking the supremum over t, we obtain

$$\|(\Theta x)(t) - (\Theta y)(t)\|_{\mathcal{B}}^p \quad \leq \quad Q(T)\mathbf{E}\,\|u-v\|_{\mathcal{B}}^p\,,$$

where $Q(T) = 3^{p-1}\max_{1,\mathtt{B}}\left(L_1(t-t_0)^p + L_5(t-t_0)^p + c_pL_3(t-t_0)^{p/2}\right)$. By selecting a suitable $0 < T_1 < T$ sufficiently small enough such that $Q(T) < 1$, thus $\Theta$ is a contraction mapping. $\Theta u = u$ is a unique solution of equations (1.1)-(1.3) by Banach fixed point theorem. □

## 4. Stability

Here, we will study the stability of the system (1.1)-(1.3) by the continuous dependence of solutions with initial condition.

**Definition 4.1.** A mild solution $u(t)$ of the system (1.1) and (1.2) with initial value $\zeta$ satisfies (2.2) is said to be stable in the mean square if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\mathbf{E}\,\|u(t)-v(t)\|^p \quad \leq \quad \epsilon \text{ whenever } \mathbf{E}\,\|u-v\|^p < \delta, \text{ for all } t \in [t_0, T].$$

where $v(t)$ is another mild solution of the system (1.1) and(1.2) with initial value $\hat{\zeta}$ defined in (3).

**Theorem 4.2.** *The hypotheses of Theorem 3.1 are satisfied when the solution of the system(1.1)-(1.3) is stable in the mean square provided $u(t)$ and $v(t)$ being the mild solutions of the system (1.1)-(1.3) whose initial values being $\zeta_1$ and $\zeta_2$.*

*Proof.* Let $u$ and $v$ be the two solutions of the system(1.1)-(1.3) with initial values $\zeta_1$ and $\zeta_2$ respectively, then

$$\begin{aligned}
\mathbf{E}\,\|u(t)-v(t)\|^p \leq 4^{p-1}\max_k\left\{\prod_{i=1}^k \|b_i(\tau_i)\|^p\right\}\mathbf{E}\,\|\zeta_1-\zeta_2\|^p \\
&+ 4^{p-1}\mathbf{E}\big[\max_{i,k}\{1,\prod_{j=i}^k\|b_j(\tau_j)\|\,\}\big]^p\left[\int_{t_0}^t \mathbf{E}\,\|\mathtt{f}(s,u_s)-\mathtt{f}(s,v_s)\|\,ds\right]^p \\
&+ 4^{p-1}\mathbf{E}\big[\max_{i,k}\{1,\prod_{j=i}^k\|b_j(\tau_j)\|\,\}\big]^p\left[\int_{t_0}^t\int_0^s \mathbf{E}\,\|\mathtt{P}(s,r,u_r)-\mathtt{P}(s,r,v_r)\|\,drds\right]^p \\
&+ 4^{p-1}\mathbf{E}\big[\max_{i,k}\{1,\prod_{j=i}^k\|b_j(\tau_j)\|\,\}\big]^p\left[\int_{t_0}^t \mathbf{E}\,\|\mathtt{g}(s,u_s)-\mathtt{g}(s,v_s)\|\,dw(s)\right]^p \\
&\leq 4^{p-1}\mathtt{B}\mathbf{E}\,\|\zeta_1-\zeta_2\|^p + \big[4^{p-1}\max(1,\mathtt{B})(t-t_0)^{p-1}L_1 + 4^{p-1}\max(1,\mathtt{B})(t-t_0)^{p-1}L_5 \\
&+ 4^{p-1}\max(1,\mathtt{B})c_p(t-t_0)^{\frac{p}{2}-1}L_3\big]\int_{t_0}^t \sup_{s\in[t-\tau,t]}\mathbf{E}\,\|u(s)-v(s)\|^p\,.
\end{aligned}$$

Using Grownwall's inequality, we have

$$\begin{aligned}
\sup_{s\in[t-\tau,t]}\mathbf{E}\,\|u(t)-v(t)\|^p &\leq 4^{p-1}\mathtt{B}\mathbf{E}\,\|\zeta_1-\zeta_2\|^p\exp\big[4^{p-1}\max(1,\mathtt{B})(t-t_0)^{p-1}L_1 \\
&+ 4^{p-1}\max(1,\mathtt{B})(t-t_0)^{p-1}L_5 + 4^{p-1}\max(1,\mathtt{B})c_p(t-t_0)^{\frac{p}{2}-1}L_3\big] \\
&\leq \Gamma\mathbf{E}\,\|\zeta_1-\zeta_2\|^p\,,
\end{aligned}$$

where

$$\Gamma = 4^{p-1}\mathtt{B}\exp\big[4^{p-1}\max(1,\mathtt{B})(t-t_0)^{p-1}L_1 + 4^{p-1}\max(1,\mathtt{B})(t-t_0)^{p-1}L_5 + 4^{p-1}\max(1,\mathtt{B})c_p(t-t_0)^{\frac{p}{2}-1}L_3\big]$$

Given $\epsilon > 0$, determine $\delta = \frac{\epsilon}{\Gamma}$ such that $\mathbf{E}\,\|\zeta_1-\zeta_2\|^p < \delta$. Then

$$\sup_{s\in[t-\tau,t]}\mathbf{E}\,\|u(t)-v(t)\|^p \quad \leq \quad \epsilon.$$

Which complete the proof. □

## 5. Hyers-Ulam stability

Here, we will prove the Hyers-Ulam stability of Equation (1.1)-(1.3) with the assumptions (H1)-(H4).

**Theorem 5.1.** *If the hypotheses of Theorem 3.1 are satisfied, then Equation (1.1)-(1.3) has the Ulam-Hyers stability.*

*Proof.* We know that, $x(t)$ is the solution of equations (1.1)-(1.3).

$$
\begin{aligned}
u(t) = \sum_{k=0}^{+\infty}\Bigg[&\prod_{i=1}^{k} b_i(\tau_i)\zeta(0) + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\zeta_{i-1}}^{\zeta_i} \mathtt{f}(s,u_s)ds + \int_{\zeta_k}^{t} \mathtt{f}(s,u_s) \\
&+ \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\zeta_{i-1}}^{\zeta_i}\int_0^s \mathtt{P}(s,r,u_r)drds + \int_{\zeta_k}^{t}\int_0^s \mathtt{P}(s,r,u_r)drds \\
&+ \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\zeta_{i-1}}^{\zeta_i} \mathtt{g}(s,u_s)dw(s) + \int_{\zeta_k}^{t} \mathtt{g}(s,u_s)dw(s)\Bigg] I_{(\zeta_k,\zeta_{k-1}]}(t).
\end{aligned}
$$

It follows from the condition that

$$
\begin{aligned}
\mathbf{E}\Bigg\| v(t) - \sum_{k=0}^{+\infty}\Bigg[&\prod_{k=0}^{k} b_i(\tau_i)\zeta(0) + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\zeta_{i-1}}^{\zeta_i} \mathtt{f}(s,v_s)ds + \int_{\zeta_k}^{t} \mathtt{f}(s,v_s)ds \\
&+ \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\zeta_{i-1}}^{\zeta_i}\int_0^s \mathtt{P}(s,r,v_r)dw(s) + \int_{\zeta_k}^{t}\int_0^s \mathtt{P}(s,r,v_r)drds \\
&+ \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\zeta_{i-1}}^{\zeta_i} \mathtt{g}(s,v_s)dw(s) + \int_{\zeta_k}^{t} \mathtt{g}(s,v_s)dw(s)\Bigg\|^p \le \epsilon.
\end{aligned}
$$

When $t \in [t_0-\tau, t_0]$, we have, $\mathbf{E}\left\|v(t)-u(t)\right\|^p = 0$. And when $t \in [t_0, T]$, we have

$$
\begin{aligned}
\mathbf{E}\left\|v(t)-u(t)\right\|^p \le 2^{p-1}\mathbf{E}\Bigg\| v(t) - \sum_{k=0}^{+\infty}\Bigg[&\prod_{k=0}^{k} b_i(\tau_i)\zeta(0) + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\zeta_{i-1}}^{\zeta_i} \mathtt{f}(s,v_s)ds + \int_{\zeta_k}^{t} \mathtt{f}(s,v_s)ds \\
&+ \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\zeta_{i-1}}^{\zeta_i}\int_0^s \mathtt{P}(s,r,v_r)drds + \int_{\zeta_k}^{t}\int_0^s \mathtt{P}(s,r,v_r)drds \\
&+ \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\zeta_{i-1}}^{\zeta_i} \mathtt{g}(s,v_s)dw(s) + \int_{\zeta_k}^{t} \mathtt{g}(s,v_s)dw(s)\Bigg] I_{(\zeta_k,\zeta_{k-1}]}(t)\Bigg\|^p \\
&+ 2^{p-1}\mathbf{E}\Bigg\|\sum_{k=0}^{+\infty}\Bigg[\sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\zeta_{i-1}}^{\zeta_i}[\mathtt{f}(s,u_s)-\mathtt{f}(s,v_s)]ds + \int_{\zeta_k}^{t}[\mathtt{f}(s,u_s)-\mathtt{f}(s,v_s)]ds \\
&+ \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\zeta_{i-1}}^{\zeta_i}\int_0^s[\mathtt{P}(s,r,u_r)-\mathtt{P}(s,r,v_r)]drds + \int_{\zeta_k}^{t}\int_0^s[\mathtt{P}(s,r,u_r)-\mathtt{P}(s,r,v_r)]drds \\
&+ \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\zeta_{i-1}}^{\zeta_i}[\mathtt{g}(s,u_s)-\mathtt{g}(s,v_s)]dw(s) + \int_{\zeta_k}^{t}[\mathtt{g}(s,u_s)-\mathtt{g}(s,v_s)]dw(s)\Bigg] I_{(\zeta_k,\zeta_{k-1}]}(t)\Bigg\|^p \\
&\le 2^{p-1}\epsilon + 2^{p-1}\mathfrak{G},
\end{aligned}
$$

where

$$\begin{aligned}
\mathfrak{G} &= \mathbf{E}\Bigg\| \sum_{k=0}^{+\infty} \Bigg[ \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\zeta_{i-1}}^{\zeta_i} [\mathtt{f}(s,u_s) - \mathtt{f}(s,v_s)]ds + \int_{\zeta_k}^{t} [\mathtt{f}(s,u_s) - \mathtt{f}(s,v_s)]ds \\
&+ \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\zeta_{i-1}}^{\zeta_i} \int_0^s [\mathtt{P}(s,r,u_r) - \mathtt{P}(s,r,v_r)]drds + \int_{\zeta_k}^{t} \int_0^s [\mathtt{P}(s,r,u_r) - \mathtt{P}(s,r,v_r)]drds \\
&+ \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\zeta_{i-1}}^{\zeta_i} [\mathtt{g}(s,u_s) - \mathtt{g}(s,v_s)]dw(s) + \int_{\zeta_k}^{t} [\mathtt{g}(s,u_s) - \mathtt{g}(s,v_s)]dw(s) \Bigg] I_{(\zeta_k,\zeta_{k-1}]}(t) \Bigg\|^p \\
&\leq 3^{p-1} \left( \mathfrak{A} + \mathfrak{B} + \mathfrak{C} \right).
\end{aligned}$$

First,

$$\begin{aligned}
\mathfrak{A} &= \mathbf{E}\Bigg\| \sum_{k=0}^{+\infty} \Bigg[ \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\zeta_{i-1}}^{\zeta_i} [\mathtt{f}(s,u_s) - \mathtt{f}(s,v_s)]ds + \int_{\zeta_k}^{t} [\mathtt{f}(s,u_s) - \mathtt{f}(s,v_s)]ds \Bigg] I_{(\zeta_k,\zeta_{k-1}]}(t) \Bigg\|^p \\
&\leq (B^p+1)(T-t_0)^{p-1} \int_{t_0}^{t} \mathbf{E} \left\| \mathtt{f}(s,u_s) - \mathtt{f}(s,v_s) \right\|^p ds \\
&\leq (B^p+1)L_1(T-t_0)^{p-1} \int_{t_0}^{t} \left\| u_s - v_s \right\|^p ds.
\end{aligned}$$

By (H3), we have

$$\begin{aligned}
\mathfrak{B} &= \mathbf{E}\Bigg\| \sum_{k=0}^{+\infty} \Bigg[ \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\zeta_{i-1}}^{\zeta_i} \int_0^s [\mathtt{P}(s,r,u_r) - \mathtt{P}(s,r,v_r)]ds + \int_{\zeta_k}^{t} \int_0^s [\mathtt{P}(s,r,u_r) - \mathtt{P}(s,r,v_r)]ds \Bigg] I_{(\zeta_k,\zeta_{k-1}]}(t) \Bigg\|^p \\
&\leq (B^p+1)(T-t_0)^{p-1} \int_{t_0}^{t} \mathbf{E} \left\| \mathtt{P}(s,r,u_r) - \mathtt{P}(s,r,v_r) \right\|^p ds \\
&\leq (B^p+1)L_5(T-t_0)^{p-1} \int_{t_0}^{t} \left\| u_s - v_s \right\|^p ds.
\end{aligned}$$

From Lemma 2.4 we have

$$\begin{aligned}
\mathfrak{C} &= \mathbf{E}\Bigg\| \sum_{k=0}^{+\infty} \Bigg[ \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\zeta_{i-1}}^{\zeta_i} [\mathtt{g}(s,u_s) - \mathtt{g}(s,v_s)]ds + \int_{\zeta_k}^{t} [\mathtt{g}(s,u_s) - \mathtt{g}(s,v_s)]ds \Bigg] I_{(\zeta_k,\zeta_{k-1}]}(t) \Bigg\|^p \\
&\leq (B^p+1)(p(p-1))^{p/2}(T-t_0)^{(p-2)/2} \int_{t_0}^{t} \mathbf{E} \left\| \mathtt{g}(s,u_s) - \mathtt{g}(s,v_s) \right\|^p ds \\
&\leq (B^p+1)L_3(p(p-1)/2)^{p/2}(T-t_0)^{(p-2)/2} \int_{t_0}^{t} \left\| u_s - v_s \right\|^p ds.
\end{aligned}$$

Therefore,

$$H \quad \leq \quad K \int_{t_0}^{t} \| u(s) - v(s) \|_{\mathscr{C}}^p \, ds,$$

where

$$K = 3^{p-1}(B^p+1)(T-t_0)^{p/2-1}\left[ L_1(T-t_0)^{p/2} + L_5(T-t_0)^{p/2} + (p(p-1)/2)^{p/2} L_3 \right].$$

Then, we obtain that

$$\mathbf{E} \| u(s) - v(s) \|^p \quad \leq \quad 2^{p-1}\epsilon + 2^{p-1} K \int_{t_0}^{t} \| v(s) - u(s) \|_{\mathscr{C}}^p \, ds.$$

Considering,

$$\begin{aligned}\int_{t_0}^{t} \|v(s)-u(s)\|_{\mathscr{C}}^{p}\, ds &= \int_{t_0}^{t} \sup_{\theta\in[-\tau,0]} \mathbf{E}\,\|v(s+\theta)-u(s+\theta)\|^{p}\, ds \\ &= \sup_{\theta\in[-\tau,0]} \int_{t_0}^{t} \mathbf{E}\,\|v(s+\theta)-u(s+\theta)\|^{p}\, ds \\ &= \sup_{\theta\in[-\tau,0]} \int_{t_0+\theta}^{t+\theta} \mathbf{E}\,\|v(l)-u(l)\|^{p}\, dl.\end{aligned}$$

Notice that, when $t \in [t_0-\tau, t_0]$,

$$\mathbf{E}\,\|v(l)-u(l)\|^{p} = 0.$$

Therefore,

$$\int_{t_0}^{t} \|v_s-u_s\|_{\mathscr{C}}^{p}\, ds = \sup_{\theta\in[-\tau,0]} \int_{t_0}^{t+\theta} \mathbf{E}\,\|v(l)-u(l)\|^{p}\, dl = \int_{t_0}^{t} \mathbf{E}\,\|v(l)-u(l)\|^{p}\, dl.$$

So, we get

$$\mathbf{E}\,\|v(t)-u(t)\|^{p} \leq 2^{p-1}\epsilon + 2^{p-1}K\int_{t_0}^{t} \mathbf{E}\,\|v(l)-u(l)\|^{p}\, dl.$$

From Lemma 2.3 we have

$$\mathbf{E}\,\|v(t)-u(t)\|^{p} \leq 2^{p-1}\epsilon.\exp(2^{p-1}K).$$

Therefore, there exists $M = 2^{p-1}.\exp(2^{p-1}K)$ such that

$$\mathbf{E}\,\|v(t)-u(t)\|^{p} \leq M\epsilon.$$

Thus the proof gets completed. □

## 6. Conclusion

This manuscript addresses the existence and Hyers-Ulam stability of random impulsive stochastic functional integrodifferential equations with finite delays. Firstly, we prove the existence of mild solutions to the equations by using Banach fixed point theorem. In the later case we explore the Hyers Ulam stability results under the Lipschitz condition on a bounded and closed interval. As further direction, researchers are invited to investigate the controllability of random impulsive stochastic functional integrodifferential equations with finite delays.

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