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New midpoint type inequalities for generalized fractional integral

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Abstract

In this paper, we first establish two new identities for differentiable function involving generalized fractional integrals. Then, by utilizing these equalities, we obtain some midpoint type inequalities involving generalized fractional integrals for mappings whose derivatives in absolute values are convex. We also give several results as special cases of our main results.

Keywords. Hermite-Hadamard inequality, Generalized fractional integral, Convex function.
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1. Introduction

In recent years, the Hermite-Hadamard inequality, which is the first fundamental result for convex mappings with a natural geometrical interpretation and many applications, has drawn attention much interest in elementary mathematics.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significance in the literature (see, e.g., [13, 29]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\kappa_1, \kappa_2 \in I$ with $\kappa_1 < \kappa_2$, then

$$f\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{1}{\kappa_2-\kappa_1}\int\limits_{\kappa_1}^{\kappa_2} f(\kappa)dx \leq \frac{f(\kappa_1)+f(\kappa_2)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave.

Over the last twenty years, the numerous studies have focused on to obtain new bound for left hand side and right and side of the inequality (1.1). For some examples, please refer to ([6, 8, 11, 13, 14, 24, 31–34, 40]).

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

Definition 1.1. [23] Let $f \in L_1[\kappa_1, \kappa_2]$. The Riemann-Liouville integrals $J^{\alpha}_{\kappa_1+}f$ and $J^{\alpha}_{\kappa_2-}f$ of order $\alpha > 0$ with $\kappa_1 \geq 0$ are defined by

$$J^{\alpha}_{\kappa_1^+}f(\kappa) = \frac{1}{\Gamma(\alpha)}\int_{\kappa_1}^{\kappa} (\kappa-\xi)^{\alpha-1} f(\xi)d\xi, \quad \kappa > \kappa_1$$

and

$$J^{\alpha}_{\kappa_2-}f(\kappa) = \frac{1}{\Gamma(\alpha)}\int_{\kappa}^{\kappa_2} (\xi-\kappa)^{\alpha-1} f(\xi)d\xi, \quad \kappa < \kappa_2$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J^{0}_{\kappa_1+}f(\kappa) = J^{0}_{\kappa_2-}f(\kappa) = f(\kappa)$.

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It is remarkable that Sarikaya et al.[37] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1.2. *Let $f:[\kappa_1,\kappa_2]\to\mathbb{R}$ be a positive function with $0\le\kappa_1<\kappa_2$ and $f\in L_1[\kappa_1,\kappa_2]$. If f is a convex function on $[\kappa_1,\kappa_2]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{\kappa_1+\kappa_2}{2}\right) \le \frac{\Gamma(\alpha+1)}{2\left(\kappa_2-\kappa_1\right)^{\alpha}}\left[J^{\alpha}_{\kappa_1^+}f(\kappa_2)+J^{\alpha}_{\kappa_2^-}f(\kappa_1)\right] \le \frac{f(\kappa_1)+f(\kappa_2)}{2} \tag{1.2}$$

with $\alpha>0$.

Sarikaya and Yıldırım also give the following Hermite-Hadamard type inequality for the Riemann-Liouville fractional integrals in [36].

Theorem 1.3. *Let $f:[\kappa_1,\kappa_2]\to\mathbb{R}$ be a positive function with $\kappa_1<\kappa_2$ and $f\in L_1[\kappa_1,\kappa_2]$. If f is a convex function on $[\kappa_1,\kappa_2]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{\kappa_1+\kappa_2}{2}\right) \le \frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(\kappa_2-\kappa_1\right)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)^+}f(\kappa_2)+J^{\alpha}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)^-}f(\kappa_1)\right] \le \frac{f(\kappa_1)+f(\kappa_2)}{2}. \tag{1.3}$$

Whereupon Sarikaya et al. obtain the Hermite-Hadamard inequality for Riemann-Lioville fractional integrals, many authors have studied to generalize this inequality and establish Hermite-Hadamard inequality other fractional integrals such as k-fractional integral, Hadamard fractional integrals, Katugampola frtactional integrals, Conformable fractional integrals, etc. For some of them, please see ([1–3, 7, 9, 12], [15]-[21], [25, 26, 28, 30, 38, 39], [41]-[46]).

2. New Generalized Fractional Integral Operators

In this section we summarize the generalized fractional integrals defined by Sarikaya and Ertuğral in [35].
Let's define a function $\varphi:[0,\infty)\to[0,\infty)$ satisfying the following conditions :

$$\int_0^1\frac{\varphi(\xi)}{\xi}d\xi<\infty.$$

We define the following left-sided and right-sided generalized fractional integral operators, respectively, as follows:

$${}_{\kappa_1^+}I_{\varphi}f(\kappa)=\int_{\kappa_1}^{\kappa}\frac{\varphi(\kappa-\xi)}{\kappa-\xi}f(\xi)d\xi,\quad \kappa>\kappa_1, \tag{2.1}$$

$${}_{\kappa_2^-}I_{\varphi}f(\kappa)=\int_{\kappa}^{\kappa_2}\frac{\varphi(\xi-\kappa)}{\xi-\kappa}f(\xi)d\xi,\quad \kappa<\kappa_2. \tag{2.2}$$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann-Liouville fractional integrals, k-Riemann-Liouville fractional integrals, Katugampola fractional integrals, conformable fractional integrals, Hadamard fractional integrals, etc. These important special cases of the integral operators (2.1) and (2.2) are mentioned below.

i) If we take $\varphi(\xi)=\xi$, the operator (2.1) and (2.2) reduce to the Riemann integral as follows:

$$I_{\kappa_1^+}f(\kappa)=\int_{\kappa_1}^{\kappa}f(\xi)d\xi,\quad \kappa>\kappa_1,$$

$$I_{\kappa_2^-}f(\kappa)=\int_{\kappa}^{\kappa_2}f(\xi)d\xi,\quad \kappa<\kappa_2.$$

ii) If we take $\varphi(\xi) = \frac{\xi^{\alpha}}{\Gamma(\alpha)}$, the operator (2.1) and (2.2) reduce to the Riemann-Liouville fractional integral as follows:

$$I^{\alpha}_{\kappa_1^+} f(\kappa) = \frac{1}{\Gamma(\alpha)} \int_{\kappa_1}^{\kappa} (\kappa-\xi)^{\alpha-1} f(\xi) d\xi, \quad \kappa > \kappa_1,$$

$$I^{\alpha}_{\kappa_2^-} f(\kappa) = \frac{1}{\Gamma(\alpha)} \int_{\kappa}^{\kappa_2} (\xi-\kappa)^{\alpha-1} f(\xi) d\xi, \quad \kappa < \kappa_2.$$

iii) If we take $\varphi(\xi) = \frac{1}{k\Gamma_k(\alpha)} \xi^{\frac{\alpha}{k}}$, the operator (2.1) and (2.2) reduce to the k-Riemann-Liouville fractional integral as follows:

$$I^{\alpha}_{\kappa_1^+,k} f(\kappa) = \frac{1}{k\Gamma_k(\alpha)} \int_{\kappa_1}^{\kappa} (\kappa-\xi)^{\frac{\alpha}{k}-1} f(\xi) d\xi, \quad \kappa > \kappa_1,$$

$$I^{\alpha}_{\kappa_2^-,k} f(\kappa) = \frac{1}{k\Gamma_k(\alpha)} \int_{\kappa}^{\kappa_2} (\xi-\kappa)^{\frac{\alpha}{k}-1} f(\xi) d\xi, \quad \kappa < \kappa_2$$

where

$$\Gamma_k(\alpha) = \int_0^{\infty} \xi^{\alpha-1} e^{-\frac{\xi^k}{k}} d\xi, \quad \mathcal{R}(\alpha) > 0$$

and

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1} \Gamma\left(\frac{\alpha}{k}\right), \quad \mathcal{R}(\alpha) > 0; k > 0$$

are given by Mubeen and Habibullah in [27].
iv) If we take $\varphi(\xi) = \xi(\kappa-\xi)^{\alpha-1}$, the operator (2.1) reduces to the conformable fractional operators as follows:

$$I^{\alpha}_{\kappa_1} f(\kappa) = \int_{\kappa_1}^{\kappa} \xi^{\alpha-1} f(\xi) d\xi = \int_{\kappa_1}^{\kappa} f(\xi) d_{\alpha}\xi, \quad \kappa > \kappa_1, \ \alpha \in (0,1)$$

is given by Khalil et al. in [22].
Sarikaya and Ertuğral also establish the following Hermite-Hadamard inequality for the generalized fractional integral operators:

Theorem 2.1. *Let $f : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a convex function on $[\kappa_1, \kappa_2]$ with $\kappa_1 < \kappa_2$, then the following inequalities for fractional integral operators hold*

$$f\left(\frac{\kappa_1+\kappa_2}{2}\right) \le \frac{1}{2\Psi(1)} \left[{}_{\kappa_1+}I_{\varphi} f(\kappa_2) + {}_{\kappa_2-} I_{\varphi} f(\kappa_1) \right] \le \frac{f(\kappa_1)+f(\kappa_2)}{2} \tag{2.3}$$

where the mapping $\Psi : [0,1] \to \mathbb{R}$ is defined by

$$\Psi(\kappa) = \int_0^{\kappa} \frac{\varphi((\kappa_2-\kappa_1)\xi)}{\xi} d\xi.$$

For more recent results related to generalized fractional integral inequalities see, ([4, 5, 10, 35]).
In this paper, we obtain some midpoint type inequality for the generalized fractional integrals.

3. Two New Equalities for Generalized Fractional Integrals

Firstly, we give the following lemmas which will be used frequently in next section.

Lemma 3.1. *Let $f:[\kappa_1,\kappa_2]\to\mathbb{R}$ be a differentiable mapping on (κ_1,κ_2) with $\kappa_1<\kappa_2$. If $f'\in L[\kappa_1,\kappa_2]$, then we have the following equality for generalized fractional integrals*

$$
\begin{aligned}
&\frac{1}{\Lambda(0)+\Delta(0)}\left[{}_{\kappa_2^-}I_\varphi f(\kappa_1+\kappa_2-\kappa)+{}_{\kappa_1^+}I_\varphi f(\kappa_1+\kappa_2-\kappa)\right]-f(\kappa_1+\kappa_2-\kappa)\\
&\qquad=\frac{(\kappa-\kappa_1)}{\Lambda(0)+\Delta(0)}\int_0^1 \Lambda(\xi)f'\left(\xi\kappa_2+(1-\xi)\left(\kappa_1+\kappa_2-\kappa\right)\right)d\xi\\
&\qquad\quad-\frac{(\kappa_2-\kappa)}{\Lambda(0)+\Delta(0)}\int_0^1 \Delta(\xi)f'\xi\kappa_1+(1-\xi)\left(\kappa_1+\kappa_2-\kappa\right)d\xi
\end{aligned}
\tag{3.1}
$$

with all $\kappa\in[\kappa_1,\kappa_2]$, where Λ and Δ are defined by

$$
\begin{aligned}
\Lambda(\xi) &= \int_\xi^1 \frac{\varphi((\kappa-\kappa_1)u)}{u}du\\
\Delta(\xi) &= \int_\xi^1 \frac{\varphi((\kappa_2-\kappa)u)}{u}du.
\end{aligned}
\tag{3.2}
$$

Proof. Integrating by parts, we have

$$
\begin{aligned}
I_1&=\int_0^1 \Lambda(\xi)f'\left(\xi\kappa_2+(1-\xi)\left(\kappa_1+\kappa_2-\kappa\right)\right)d\xi\\
&=\left[\Lambda(\xi)\frac{f(t\kappa_2+(1-\xi)(\kappa_1+\kappa_2-\kappa))}{\kappa-\kappa_1}\Bigg|_0^1+\int_0^1\frac{f\left(t\kappa_2+(1-\xi)\left(\kappa_1+\kappa_2-\kappa\right)\right)}{\kappa-\kappa_1}\frac{\varphi((\kappa-\kappa_1)\xi)}{\xi}d\xi\right].
\end{aligned}
$$

Using the change of veriable $tb+(1-\xi)\left(\kappa_1+\kappa_2-\kappa\right)=\varsigma$, we get,

$$
\begin{aligned}
I_1&=-\Lambda(0)\frac{f(\kappa_1+\kappa_2-\kappa)}{\kappa-\kappa_1}+\frac{1}{\kappa-\kappa_1}\int_{\kappa_1+\kappa_2-\kappa}^{\kappa_2} f(\varsigma)\frac{\varphi(\varsigma-(\kappa_1+\kappa_2-\kappa))}{\varsigma-(\kappa_1+\kappa_2-\kappa)}d\varsigma\\
&=-\Lambda(0)\frac{f(\kappa_1+\kappa_2-\kappa)}{\kappa-\kappa_1}+\frac{1}{\kappa-\kappa_1}\,{}_{\kappa_2^-}I_\varphi f(\kappa_1+\kappa_2-\kappa).
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
I_2&=\int_0^1 \Delta(\xi)f'\left(\xi\kappa_1+(1-\xi)\left(\kappa_1+\kappa_2-\kappa\right)\right)d\xi\\
&=\Delta(0)\frac{f(\kappa_1+\kappa_2-\kappa)}{\kappa_2-\kappa}-\frac{1}{\kappa_2-\kappa}\,{}_{\kappa_1^+}I_\varphi f(\kappa_1+\kappa_2-\kappa).
\end{aligned}
$$

If we multiply I_1 and I_2 by $\frac{\kappa-\kappa_1}{\Lambda(0)+\Delta(0)}$ and $-\frac{\kappa_2-\kappa}{\Lambda(0)+\Delta(0)}$, respectively, then we get

$$
\frac{\kappa-\kappa_1}{\Lambda(0)+\Delta(0)}I_1-\frac{\kappa_2-\kappa}{\Lambda(0)+\Delta(0)}I_2=\frac{1}{\Lambda(0)+\Delta(0)}\left[{}_{\kappa_2^-}I_\varphi f(\kappa_1+\kappa_2-\kappa)+{}_{\kappa_1^+}I_\varphi f(\kappa_1+\kappa_2-\kappa)\right]-f(\kappa_1+\kappa_2-\kappa)
$$

which completes the proof. □

Lemma 3.2. *The assumptions of Lemma 3.1 hold. Then we have the following equality for generalized fractional integrals*

$$\begin{aligned}\frac{1}{\Psi(1)+\Theta(1)}\Big[{}_{(\kappa_1+\kappa_2-\kappa)^-}I_\varphi f(\kappa_2)+{}_{(\kappa_1+\kappa_2-\kappa)^+}I_\varphi f(\kappa_1)\Big]-f(\kappa_1+\kappa_2-\kappa)\\ =\frac{(\kappa-\kappa_1)}{\Psi(1)+\Theta(1)}\int_0^1\Psi(\xi)f'\left(\xi\left(\kappa_1+\kappa_2-\kappa\right)+(1-\xi)\kappa_2\right)d\xi\\ +\frac{(\kappa_2-\kappa)}{\Psi(1)+\Theta(1)}\int_0^1\Theta(\xi)f'\left(\xi\left(\kappa_1+\kappa_2-\kappa\right)+(1-\xi)\kappa_1\right)d\xi\end{aligned}\tag{3.3}$$

with all $\kappa\in[\kappa_1,\kappa_2]$, where Ψ and Θ are defined by

$$\begin{aligned}\Psi(\xi)&=\int_0^\xi\frac{\varphi((\kappa-\kappa_1)u)}{u}du\\ \Theta(\xi)&=\int_0^\xi\frac{\varphi((\kappa_2-\kappa)u)}{u}du.\end{aligned}\tag{3.4}$$

Proof. Integrating by parts, we have

$$\begin{aligned}I_3&=\int_0^1\Psi(\xi)f'(\xi(\kappa_1+\kappa_2-\kappa)+(1-\xi)\kappa_2)d\xi\\ &=\Psi(\xi)\frac{f(\xi(\kappa_1+\kappa_2-\kappa)+(1-\xi)\kappa_2)}{\kappa_1-\kappa}|_0^1-\int_0^1\frac{f(\xi(\kappa_1+\kappa_2-\kappa)+(1-\xi)\kappa_2)}{\kappa_1-\kappa}\frac{\varphi((\kappa-\kappa_1)\xi)}{\xi}d\xi.\end{aligned}$$

Using change of veriable $\xi\left(\kappa_1+\kappa_2-\kappa\right)+(1-\xi)\kappa_2=\tau$, we get,

$$\begin{aligned}I_3&=-\Psi(1)\frac{f(\kappa_1+\kappa_2-\kappa)}{\kappa-\kappa_1}+\frac{1}{\kappa-\kappa_1}\int_{\kappa_1+\kappa_2-\kappa}^{\kappa_2}f(\tau)\frac{\varphi(\kappa_2-\tau)}{\kappa_2-\tau}d\tau\\ &=-\Psi(1)\frac{f(\kappa_1+\kappa_2-\kappa)}{\kappa-\kappa_1}+\frac{1}{\kappa-\kappa_1}\,{}_{(\kappa_1+\kappa_2-\kappa)^-}I_\varphi f(\kappa_2).\end{aligned}$$

Similarly, we get

$$\begin{aligned}I_4&=\int_0^1\Theta(\xi)f'\left(\xi\left(\kappa_1+\kappa_2-\kappa\right)+(1-\xi)\kappa_1\right)d\xi\\ &=\Theta(1)\frac{f(\kappa_1+\kappa_2-\kappa)}{\kappa_2-\kappa}-\frac{1}{\kappa_2-\kappa}\,{}_{(\kappa_1+\kappa_2-\kappa)^+}I_\varphi f(\kappa_1).\end{aligned}$$

If we multiply I_3 and I_4 by $\frac{\kappa-\kappa_1}{\Psi(1)+\Theta(1)}$ and $-\frac{\kappa_2-\kappa}{\Psi(1)+\Theta(1)}$, respectively, then we get,

$$\frac{\kappa-\kappa_1}{\Psi(1)+\Theta(1)}I_3-\frac{\kappa_2-\kappa}{\Psi(1)+\Theta(1)}I_4=\frac{1}{\Psi(1)+\Theta(1)}\left[{}_{(\kappa_1+\kappa_2-\kappa)^-}I_\varphi f(\kappa_2)+{}_{(\kappa_1+\kappa_2-\kappa)^+}I_\varphi f(\kappa_1)\right]-f(\kappa_1+\kappa_2-\kappa)$$

which completes the proof. □

4. Midpoint Type Inequalities for Generalized Fractional Integrals

In this section, utilizing the identities obtained previous section, we establish some midpoint type inequalities for generalized fractional integrals.

Theorem 4.1. *$f:[\kappa_1,\kappa_2]\to\mathbb{R}$ be a differentiable mapping on (κ_1,κ_2) with $0\le\kappa_1<\kappa_2$. If $|f'|$ is convex on $[\kappa_1,\kappa_2]$, then we have the following inequality for generalized fractional integrals*

$$\left|\frac{1}{\Lambda(0)+\Delta(0)}\times\left[{}_{\kappa_2^-}I_\varphi f(\kappa_1+\kappa_2-\kappa)+{}_{\kappa_1^+}I_\varphi f(\kappa_1+\kappa_2-\kappa)\right]-f(\kappa_1+\kappa_2-\kappa)\right| \tag{4.1}$$

$$\le\frac{1}{\Lambda(0)+\Delta(0)}\Bigg\{(\kappa-\kappa_1)\left|f'(\kappa_2)\right|\int_0^1\Lambda(\xi)\xi d\xi+(\kappa_2-\kappa)\left|f'(\kappa_1)\right|\int_0^1\Delta(\xi)\xi d\xi$$

$$+\left|f'(\kappa_1+\kappa_2-\kappa)\right|\int_0^1(1-\xi)\left[(\kappa-\kappa_1)\Lambda(\xi)+(\kappa_2-\kappa)\Delta(\xi)\right]d\xi\Bigg\} \tag{4.2}$$

with all $\kappa\in[\kappa_1,\kappa_2]$, where Λ and Δ are defined as in (3.2).

Proof. By the Lemma 3.1, we have

$$\begin{aligned}
&\left|\frac{1}{\Lambda(0)+\Delta(0)}\times\left[{}_{\kappa_2^-}I_\varphi f(\kappa_1+\kappa_2-\kappa)+{}_{\kappa_1^+}I_\varphi f(\kappa_1+\kappa_2-\kappa)\right]-f(\kappa_1+\kappa_2-\kappa)\right|\\
&\le\frac{(\kappa-\kappa_1)}{\Lambda(0)+\Delta(0)}\left|\int_0^1\Lambda(\xi)f'\left(\xi\kappa_2+(1-\xi)\left(\kappa_1+\kappa_2-\kappa\right)\right)d\xi\right|\\
&+\frac{(\kappa_2-\kappa)}{\Lambda(0)+\Delta(0)}\left|\int_0^1\Delta(\xi)f'\left(\xi\kappa_1+(1-\xi)\left(\kappa_1+\kappa_2-\kappa\right)\right)d\xi\right|\\
&\le\frac{(\kappa-\kappa_1)}{\Lambda(0)+\Delta(0)}\int_0^1\left|\Lambda(\xi)f'\left(\xi\kappa_2+(1-\xi)\left(\kappa_1+\kappa_2-\kappa\right)\right)\right|d\xi\\
&+\frac{(\kappa_2-\kappa)}{\Lambda(0)+\Delta(0)}\int_0^1\left|\Delta(\xi)f'\left(\xi\kappa_1+(1-\xi)\left(\kappa_1+\kappa_2-\kappa\right)\right)\right|d\xi.
\end{aligned}$$

Using convexity of $|f'|$, we obtain

$$\begin{aligned}
&\left|\frac{1}{\Lambda(0)+\Delta(0)}\times\left[{}_{\kappa_2^-}I_{\varphi}f(\kappa_1+\kappa_2-\kappa)+{}_{\kappa_1^+}I_{\varphi}f(\kappa_1+\kappa_2-\kappa)\right]-f(\kappa_1+\kappa_2-\kappa)\right| \\
&\leq \frac{(\kappa-\kappa_1)}{\Lambda(0)+\Delta(0)}\left\{\int_0^1 \Lambda(\xi)\xi\left|f'(\kappa_2)\right|+(1-\xi)\left|f'\left(\kappa_1+\kappa_2-\kappa\right)\right|d\xi\right. \\
&\left.+\frac{(\kappa_2-\kappa)}{\Lambda(0)+\Delta(0)}\int_0^1 \Delta(\xi)\xi\left|f'(\kappa_1)\right|+(1-\xi)\left|f'\left(\kappa_1+\kappa_2-\kappa\right)\right|d\xi\right\} \\
&=\frac{1}{\Lambda(0)+\Delta(0)}\times\left\{(\kappa-\kappa_1)\left|f'(\kappa_2)\right|\int_0^1\Lambda(\xi)\xi d\xi+(\kappa_2-\kappa)\left|f'(\kappa_1)\right|\int_0^1\Delta(\xi)\xi d\xi\right. \\
&\left.+\left|f'\left(\kappa_1+\kappa_2-\kappa\right)\right|\int_0^1(1-\xi)\left[\left(\kappa-\kappa_1\right)\Lambda(\xi)+(\kappa_2-\kappa)\Delta(\xi)\right]d\xi\right\}.
\end{aligned}$$

This completes the proof. □

Corollary 4.2. *Under assumption of Theorem 4.1 with $\varphi(\xi)=\xi$, we have the following inequality*

$$\begin{aligned}
\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}f(\xi)d\xi-f(\kappa_1+\kappa_2-\kappa)\right| &\leq \frac{1}{\kappa_2-\kappa_1}\left\{\frac{1}{6}\left[(\kappa-\kappa_1)^2\left|f^{'}(\kappa_2)\right|+(\kappa_2-\kappa)^2\left|f^{'}(\kappa_1)\right|\right]\right. \\
&\left.+\frac{1}{3}\left|f^{'}\left(\kappa_1+\kappa_2-\kappa\right)\right|\left[(\kappa-\kappa_1)^2+(\kappa_2-\kappa)^2\right]\right\}.
\end{aligned} \tag{4.3}$$

Remark 4.3. If we choose $\kappa=\frac{\kappa_1+\kappa_2}{2}$ in Corollary 4.2, then we have the following inequality

$$\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}f(\xi)d\xi-f\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|\leq\frac{\kappa_2-\kappa_1}{8}\left[\left|f^{'}(\kappa_1)\right|+\left|f^{'}(\kappa_2)\right|\right]$$

which is proved by Kirmaci in [24].

Corollary 4.4. *Under assumption of Theorem 4.1 with $\varphi(\xi)=\frac{\xi^\alpha}{\Gamma(\alpha)}$, we have the following inequality*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{(\kappa_2-\kappa)^\alpha+(\kappa-\kappa_1)^\alpha}\times\left[J_{\kappa_2^-}^{\alpha}f(\kappa_1+\kappa_2-\kappa)+J_{\kappa_1^+}^{\alpha}f(\kappa_1+\kappa_2-\kappa)\right]-f(\kappa_1+\kappa_2-\kappa)\right| \\
&\leq\frac{1}{(\kappa_2-\kappa)^\alpha+(\kappa-\kappa_1)^\alpha}\frac{\alpha}{2(\alpha+2)}\left\{(\kappa-\kappa_1)^{\alpha+1}|f^{'}(\kappa_2)|+(\kappa_2-\kappa)^{\alpha+1}|f^{'}(\kappa_1)|\right. \\
&\left.+|f^{'}(\kappa_1+\kappa_2-\kappa)|\frac{\alpha+3}{\alpha+1}[(\kappa_2-\kappa^{\alpha+1}+(\kappa-\kappa_1)^{\alpha+1}]\right\}
\end{aligned} \tag{4.4}$$

Remark 4.5. If we choose $\kappa=\frac{\kappa_1+\kappa_2}{2}$ in Corollary 4.4, then we have the following inequality

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left[J_{\kappa_2^-}^{\alpha}f\left(\frac{\kappa_1+\kappa_2}{2}\right)+J_{\kappa_1^+}^{\alpha}f\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]-f\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|\leq\frac{(\kappa_2-\kappa_1)\alpha}{4(\alpha+1)}\left[|f'(\kappa_1)|+|f'(\kappa_2)|\right]$$

which is given by Budak and Agarwal in [9, Corollary 2 for $q=1$].

Corollary 4.6. *Under assumption of Theorem 4.1 with* $\varphi(\xi)=\frac{\xi^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, *we have the following inequality*

$$\begin{aligned}&\left|\frac{\Gamma_k(\alpha+k)}{(\kappa_2-\kappa)^{\frac{\alpha}{k}}+(\kappa-\kappa_1)^{\frac{\alpha}{k}}}\times\left[J^{\alpha}_{\kappa_2^-,k}f(\kappa_1+\kappa_2-\kappa)+J^{\alpha}_{\kappa_1^+,k}f(\kappa_1+\kappa_2-\kappa)\right]-f(\kappa_1+\kappa_2-\kappa)\right|\\&\leq\frac{1}{(\kappa_2-\kappa)^{\alpha/k}+(\kappa-\kappa_1)^{\alpha/k}}\\&\times\left\{(\kappa-\kappa_1)^{\frac{\alpha}{k}+1}\frac{\alpha}{2(\alpha+2k)}\left|f^{'}(\kappa_2)\right|+(\kappa_2-\kappa)^{\frac{\alpha}{k}+1}\frac{\alpha}{2(\alpha+2k)}\left|f^{'}(\kappa_1)\right|\right.\\&\left.+\left|f^{'}(\kappa_1+\kappa_2-\kappa)\right|\frac{\alpha(\alpha+3k)}{2(\alpha+k)(\alpha+2k)}\left[(\kappa_2-\kappa)^{\frac{\alpha}{k}+1}+(\kappa-\kappa_1)^{\frac{\alpha}{k}+1}\right]\right\}.\end{aligned}$$

Remark 4.7. Under assumption of Corollary 4.6 with $\kappa=\frac{\kappa_1+\kappa_2}{2}$, we have the following inequality

$$\left|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\kappa_2^-,k}f\left(\frac{\kappa_1+\kappa_2}{2}\right)+J^{\alpha}_{\kappa_1^+,k}f\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]-f\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|\leq\frac{(\kappa_2-\kappa_1)\alpha}{4(\alpha+k)}\left[|f'(\kappa_1)|+|f'(\kappa_2)|\right].$$

Theorem 4.8. $f:[\kappa_1,\kappa_2]\to\mathbb{R}$ *be a differentiable mapping on* (κ_1,κ_2) *with* $0\leq\kappa_1<\kappa_2$. *If* $|f'|^q$ *is convex on* $[\kappa_1,\kappa_2]$ *for some fixed* $q>1$, *then we have the following inequality for generalized fractional integrals*

$$\begin{aligned}&\left|\frac{1}{\Lambda(0)+\Delta(0)}\times\left[{}_{\kappa_2^-}I_{\varphi}f(\kappa_1+\kappa_2-\kappa)+{}_{\kappa_1^+}I_{\varphi}f(\kappa_1+\kappa_2-\kappa)\right]-f(\kappa_1+\kappa_2-\kappa)\right|\\&\leq\frac{\kappa-\kappa_1}{\Lambda(0)+\Delta(0)}\left(\int_0^1\Lambda(\xi)^p d\xi\right)^{\frac{1}{p}}\left(\frac{|f'(\kappa_1+\kappa_2-\kappa)|^q+|f'(\kappa_2)|^q}{2}\right)^{\frac{1}{q}}\\&+\frac{\kappa_2-\kappa}{\Lambda(0)+\Delta(0)}\left(\int_0^1\Delta(\xi)^p d\xi\right)^{\frac{1}{p}}\left(\frac{|f'(\kappa_1+\kappa_2-\kappa)|^q+|f'(\kappa_1)|^q}{2}\right)^{\frac{1}{q}}\end{aligned}$$

with all $\kappa\in[\kappa_1,\kappa_2]$, *where* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By the Lemma 3.1, we have

$$\begin{aligned}&\left|\frac{1}{\Lambda(0)+\Delta(0)}\times\left[{}_{\kappa_2^-}I_{\varphi}f(\kappa_1+\kappa_2-\kappa)+{}_{\kappa_1^+}I_{\varphi}f(\kappa_1+\kappa_2-\kappa)\right]-f(\kappa_1+\kappa_2-\kappa)\right|\\&\leq\frac{\kappa-\kappa_1}{\Lambda(0)+\Delta(0)}\int_0^1\Lambda(\xi)\left|f'\left(\xi\kappa_2+(1-\xi)(\kappa_1+\kappa_2-\kappa)\right)\right|d\xi\\&+\frac{\kappa_2-\kappa}{\Lambda(0)+\Delta(0)}\int_0^1\Delta(\xi)\left|f'\left(\xi\kappa_1+(1-\xi)(\kappa_1+\kappa_2-\kappa)\right)\right|d\xi.\end{aligned}$$

Using the Hölder's inequality and the convexity of $|f'|^q$, we obtain

$$\left|\frac{1}{\Lambda(0)+\Delta(0)} \times \left[{}_{\kappa_2^-}I_\varphi f(\kappa_1+\kappa_2-\kappa) + {}_{\kappa_1^+}I_\varphi f(\kappa_1+\kappa_2-\kappa)\right] - f(\kappa_1+\kappa_2-\kappa)\right|$$

$$\leq \frac{\kappa-\kappa_1}{\Lambda(0)+\Delta(0)} \left(\int_0^1 (\Lambda(\xi))^p \, d\xi\right)^{\frac{1}{p}} \times \left(\int_0^1 |f'(\xi\kappa_2+(1-\xi)(\kappa_1+\kappa_2-\kappa))|^q \, d\xi\right)^{\frac{1}{q}}$$

$$+ \frac{\kappa_2-\kappa}{\Lambda(0)+\Delta(0)} \left(\int_0^1 (\Delta(\xi))^p \, d\xi\right)^{\frac{1}{p}} \times \left(\int_0^1 |f'(\xi\kappa_1+(1-\xi)(\kappa_1+\kappa_2-\kappa))|^q \, d\xi\right)^{\frac{1}{q}}$$

$$\leq \frac{\kappa-\kappa_1}{\Lambda(0)+\Delta(0)} \left(\int_0^1 (\Lambda(\xi))^p \, d\xi\right)^{\frac{1}{p}} \times \left(\int_0^1 \left[\xi |f'(\kappa_2)|^q + (1-\xi)|f'(\kappa_1+\kappa_2-\kappa)|^q\right] d\xi\right)^{\frac{1}{q}}$$

$$+ \frac{\kappa_2-\kappa}{\Lambda(0)+\Delta(0)} \left(\int_0^1 (\Delta(\xi))^p \, d\xi\right)^{\frac{1}{p}} \times \left(\int_0^1 \left[\xi |f'(\kappa_1)|^q + (1-\xi)|f'(\kappa_1+\kappa_2-\kappa)|^q\right] d\xi\right)^{\frac{1}{q}}$$

$$= \frac{\kappa-\kappa_1}{\Lambda(0)+\Delta(0)} \left(\int_0^1 (\Lambda(\xi))^p \, d\xi\right)^{\frac{1}{p}} \left(\frac{|f'(\kappa_1+\kappa_2-\kappa)|^q + |f'(\kappa_2)|^q}{2}\right)^{\frac{1}{q}}$$

$$+ \frac{\kappa_2-\kappa}{\Lambda(0)+\Delta(0)} \left(\int_0^1 (\Delta(\xi))^p \, d\xi\right)^{\frac{1}{p}} \left(\frac{|f'(\kappa_1+\kappa_2-\kappa)|^q + |f'(\kappa_1)|^q}{2}\right)^{\frac{1}{q}}.$$

The proof is completed. □

Corollary 4.9. *Under assumption of Theorem 4.8 with $\varphi(\xi)=\xi$, we have the following inequality*

$$\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} f(\xi)d\xi - f(\kappa_1+\kappa_2-\kappa)\right| \leq \frac{1}{(\kappa_2-\kappa_1)(p+1)^{\frac{1}{p}}} \left\{(\kappa-\kappa_1)^2 \left(\frac{|f'(\kappa_1+\kappa_2-\kappa)|^q + |f'(\kappa_2)|^q}{2}\right)^{\frac{1}{q}} + (\kappa_2-\kappa)^2 \left(\frac{|f'(\kappa_1+\kappa_2-\kappa)|^q + |f'(\kappa_1)|^q}{2}\right)^{\frac{1}{q}}\right\}.$$

Remark 4.10. Under assumption of Corollary 4.9 with $\kappa=\frac{\kappa_1+\kappa_2}{2}$, we have the following inequality

$$\begin{aligned}\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} f(\xi)d\xi - f\left(\frac{\kappa_1+\kappa_2}{2}\right)\right| &\leq \frac{\kappa_2-\kappa_1}{4(p+1)^{\frac{1}{p}}}\left[\left(\frac{|f'(\kappa_1)|^q+3|f'(\kappa_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(\kappa_2)|^q+3|f'(\kappa_1)|^q}{4}\right)^{\frac{1}{q}}\right] \\ &\leq \frac{\kappa_2-\kappa_1}{4}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[|f'(\kappa_1)|+|f'(\kappa_2)|\right].\end{aligned}$$

Corollary 4.11. *Under assumption of Theorem 4.8 with $\varphi(\xi)=\frac{\xi^\alpha}{\Gamma(\alpha)}$, we have the following inequality*

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{(\kappa_2-\kappa)^\alpha+(\kappa-\kappa_1)^\alpha}\times\left[J^\alpha_{\kappa_2^-}f(\kappa_1+\kappa_2-\kappa)+J^\alpha_{\kappa_1+}f(\kappa_1+\kappa_2-\kappa)\right]-f(\kappa_1+\kappa_2-\kappa)\right| \\ &\leq \frac{1}{(\kappa_2-\kappa)^\alpha+(\kappa-\kappa_1)^\alpha}\left(\frac{\alpha p}{\alpha p+1}\right)^{\frac{1}{p}}\times\left\{(\kappa-\kappa_1)^{\alpha+1}\left(\frac{|f'(\kappa_1+\kappa_2-\kappa)|^q+|f'(\kappa_2)|^q}{2}\right)^{\frac{1}{q}}\right. \\ &\left.+(\kappa_2-\kappa)^{\alpha+1}\left(\frac{|f'(\kappa_1+\kappa_2-\kappa)|^q+|f'(\kappa_1)|^q}{2}\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Remark 4.12. Under assumption of Corollary 4.11 with $\kappa=\frac{\kappa_1+\kappa_2}{2}$, we have the following inequality

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left[J^\alpha_{\kappa_2^-}f\left(\frac{\kappa_1+\kappa_2}{2}\right)+J^\alpha_{\kappa_1+}f\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]-f\left(\frac{\kappa_1+\kappa_2}{2}\right)\right| \\ &\leq \frac{\kappa_2-\kappa_1}{4}\left(\frac{\alpha p}{\alpha p+1}\right)^{\frac{1}{p}}\times\left[\left(\frac{|f'(\kappa_1)|^q+3|f'(\kappa_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(\kappa_2)|^q+3|f'(\kappa_1)|^q}{4}\right)^{\frac{1}{q}}\right] \\ &\leq \frac{\kappa_2-\kappa_1}{4}\left(\frac{4\alpha p}{\alpha p+1}\right)^{\frac{1}{p}}\left[|f'(\kappa_1)|+|f'(\kappa_2)|\right].\end{aligned}$$

Corollary 4.13. *Under assumption of Theorem 4.8 with $\varphi(\xi)=\frac{\xi^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, we have the following inequality*

$$\begin{aligned}&\left|\frac{\Gamma_k(\alpha+k)}{(\kappa_2-\kappa)^{\frac{\alpha}{k}}+(\kappa-\kappa_1)^{\frac{\alpha}{k}}}\times\left[J^\alpha_{\kappa_2^-,k}f(\kappa_1+\kappa_2-\kappa)+J^\alpha_{\kappa_1^+,k}f(\kappa_1+\kappa_2-\kappa)\right]-f(\kappa_1+\kappa_2-\kappa)\right| \\ &\leq \frac{1}{(\kappa_2-\kappa)^{\frac{\alpha}{k}}+(\kappa-\kappa_1)^{\frac{\alpha}{k}}}\left(\frac{\alpha p}{\alpha p+k}\right)^{\frac{1}{p}}\times\left\{(\kappa-\kappa_1)^{\frac{\alpha}{k}+1}\left[\frac{|f'(\kappa_1+\kappa_2-\kappa)|^q+|f'(\kappa_2)|^q}{2}\right]^{\frac{1}{q}}\right. \\ &\left.+(\kappa_2-\kappa)^{\frac{\alpha}{k}+1}\left[\frac{|f'(\kappa_1)|^q+|f'(\kappa_1+\kappa_2-\kappa)|^q}{2}\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

Remark 4.14. Under assumption of Corollary 4.13 with $\kappa=\frac{\kappa_1+\kappa_2}{2}$, we have the following inequality

$$\left|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^\alpha_{\kappa_2^-,k}f\left(\frac{\kappa_1+\kappa_2}{2}\right)+J^\alpha_{\kappa_1^+,k}f\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]-f\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|\leq\frac{(\kappa_2-\kappa_1)}{4}\left(\frac{4\alpha p}{\alpha p+k}\right)^{\frac{1}{p}}\left[|f'(\kappa_1)|+|f'(\kappa_2)|\right].$$

Theorem 4.15. *$f : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a differentiable mapping on (κ_1, κ_2) with $0 \le \kappa_1 < \kappa_2$. If $|f'|$ is convex on $[\kappa_1, \kappa_2]$, then we have the following inequality for generalized fractional integrals*

$$\begin{aligned}
\left|\frac{1}{\Psi(1)+\Theta(1)}\right. \quad &\times \quad \left. \left[{}_{(\kappa_1+\kappa_2-\kappa)^-}I_\varphi f(\kappa_2) + {}_{(\kappa_1+\kappa_2-\kappa)^+}I_\varphi f(\kappa_1)\right] - f(\kappa_1+\kappa_2-\kappa)\right| \\
&\le \quad \frac{1}{\Psi(1)+\Theta(1)} \times \left\{ (\kappa-\kappa_1)\left|f^{'}(\kappa_2)\right| \int_0^1 (1-\xi)\Psi(\xi)d\xi + (\kappa_2-\kappa)\left|f^{'}(\kappa_1)\right| \int_0^1 (1-\xi)\Theta(\xi)d\xi \right. \\
&\quad \left. + \left|f^{'}(\kappa_1+\kappa_2-\kappa)\right| \int_0^1 \xi\left[(\kappa-\kappa_1)\Psi(\xi)+(\kappa_2-\kappa)\Theta(\xi)\right]d\xi \right\} \qquad (4.5)
\end{aligned}$$

with all $\kappa \in [\kappa_1, \kappa_2]$, where Ψ and Θ are defined as in (3.4).

Proof. By the Lemma 3.2, we have

$$\begin{aligned}
&\left|\frac{1}{\Psi(1)+\Theta(1)} \times \left[{}_{(\kappa_1+\kappa_2-\kappa)^-}I_\varphi f(\kappa_2) + {}_{(\kappa_1+\kappa_2-\kappa)^+}I_\varphi f(\kappa_1)\right] - f(\kappa_1+\kappa_2-\kappa)\right| \\
&\le \frac{(\kappa-\kappa_1)}{\Psi(1)+\Theta(1)} \left| \int_0^1 \Psi(\xi) f'\left(\xi\left(\kappa_1+\kappa_2-\kappa\right)+(1-\xi)\kappa_2\right)d\xi \right| \\
&+ \frac{(\kappa_2-\kappa)}{\Psi(1)+\Theta(1)} \left| \int_0^1 \Theta(\xi) f'\left(\xi\left(\kappa_1+\kappa_2-\kappa\right)+(1-\xi)\kappa_1\right)d\xi \right| \\
&\le \frac{(\kappa-\kappa_1)}{\Psi(1)+\Theta(1)} \int_0^1 \left|\Psi(\xi) f'\left(\xi\left(\kappa_1+\kappa_2-\kappa\right)+(1-\xi)\kappa_2\right)\right| d\xi \\
&+ \frac{(\kappa_2-\kappa)}{\Psi(1)+\Theta(1)} \int_0^1 \left|\Theta(\xi) f'\left(\xi\left(\kappa_1+\kappa_2-\kappa\right)+(1-\xi)\kappa_2\right)\right| d\xi.
\end{aligned}$$

Using the convexity of $|f'|$, we obtain

$$\begin{aligned}
&\left|\frac{1}{\Psi(1)+\Theta(1)} \times \left[{}_{(\kappa_1+\kappa_2-\kappa)^-}I_\varphi f(\kappa_2) + {}_{(\kappa_1+\kappa_2-\kappa)^+}I_\varphi f(\kappa_1)\right] - f(\kappa_1+\kappa_2-\kappa)\right| \\
&\le \frac{(\kappa-\kappa_1)}{\Psi(1)+\Theta(1)} \left\{ \int_0^1 \Psi(\xi)\left[\xi\left|f'\left(\kappa_1+\kappa_2-\kappa\right)\right| + (1-\xi)\left|f'(\kappa_2)\right|\right]d\xi \right. \\
&+ \left. \frac{(\kappa_2-\kappa)}{\Psi(1)+\Theta(1)} \int_0^1 \Theta(\xi)\left[\xi\left|f'\left(\kappa_1+\kappa_2-\kappa\right)\right| + (1-\xi)\left|f'(\kappa_1)\right|\right]d\xi \right\} \\
&= \frac{1}{\Psi(1)+\Theta(1)} \times \left\{ (\kappa-\kappa_1)\left|f'(\kappa_2)\right| \int_0^1 (1-\xi)\Psi(\xi)d\xi + (\kappa_2-\kappa)\left|f'(\kappa_1)\right| \int_0^1 (1-\xi)\Theta(\xi)d\xi \right. \\
&+ \left. \left|f'\left(\kappa_1+\kappa_2-\kappa\right)\right| \int_0^1 \xi\left[(\kappa-\kappa_1)\Psi(\xi)+(\kappa_2-\kappa)\Theta(\xi)\right]d\xi \right\}.
\end{aligned}$$

This completes the proof. □

Remark 4.16. If we choose $\varphi(\xi) = \xi$ in Theorem 4.15, then the inequality (4.5) reduces to the inequality (4.3).

Corollary 4.17. *Under assumption of Theorem 4.15 with $\varphi(\xi)=\frac{\xi^{\alpha}}{\Gamma(\alpha)}$, we have the following inequality*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{(\kappa_2-\kappa)^{\alpha}+(\kappa-\kappa_1)^{\alpha}}\times\left[J^{\alpha}_{(\kappa_1+\kappa_2-\kappa)^-}f(\kappa_2)+J^{\alpha}_{(\kappa_1+\kappa_2-\kappa)^+}f(\kappa_1)\right]-f(\kappa_1+\kappa_2-\kappa)\right|\\
&\qquad\leq\frac{1}{(\kappa_2-\kappa)^{\alpha}+(\kappa-\kappa_1)^{\alpha}}\frac{1}{\alpha+2}\times\left\{\frac{1}{(\alpha+1)}\left[(\kappa-\kappa_1)^{\alpha+1}\left|f'(\kappa_2)\right|+(\kappa_2-\kappa)^{\alpha+1}\left|f'(\kappa_1)\right|\right]\right.\\
&\qquad\left.+\left|f'(\kappa_1+\kappa_2-\kappa)\right|\left[(\kappa_2-\kappa)^{\alpha+1}+(\kappa-\kappa_1)^{\alpha+1}\right]\right\}.
\end{aligned}$$

Remark 4.18. If we choose $\kappa=\frac{\kappa_1+\kappa_2}{2}$ in Corollary 4.17, then we have the following inequality

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)^-}f(\kappa_2)+J^{\alpha}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)^+}f(\kappa_1)\right]-f\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|\leq\frac{(\kappa_2-\kappa_1)}{4(\alpha+1)}\left[|f'(\kappa_1)|+|f'(\kappa_2)|\right]$$

which is given by Sarikaya and Yildirim in [36].

Corollary 4.19. *Under assumption of Theorem 4.15 with $\varphi(\xi)=\frac{\xi^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, we have the following inequality*

$$\begin{aligned}
&\left|\frac{\Gamma_k(\alpha+k)}{(\kappa_2-\kappa)^{\frac{\alpha}{k}}+(\kappa-\kappa_1)^{\frac{\alpha}{k}}}\times\left[J^{\alpha}_{(\kappa_1+\kappa_2-\kappa)^-,k}f(\kappa_2)+J^{\alpha}_{(\kappa_1+\kappa_2-\kappa)^+,k}f(\kappa_1)\right]-f(\kappa_1+\kappa_2-\kappa)\right|\\
&\qquad\leq\frac{1}{(\kappa_2-\kappa)^{\alpha/k}+(\kappa-\kappa_1)^{\alpha/k}}\frac{k}{(\alpha+2k)}\\
&\qquad\times\left\{\frac{k}{(\alpha+k)}\left[(\kappa-\kappa_1)^{\frac{\alpha}{k}+1}\left|f'(\kappa_2)\right|+(\kappa_2-\kappa)^{\frac{\alpha}{k}+1}\left|f'(\kappa_1)\right|\right]\right.\\
&\qquad\left.+\left|f'(\kappa_1+\kappa_2-\kappa)\right|\left[(\kappa_2-\kappa)^{\frac{\alpha}{k}+1}+(\kappa-\kappa_1)^{\frac{\alpha}{k}+1}\right]\right\}.
\end{aligned}$$

Remark 4.20. If we choose $\kappa=\frac{\kappa_1+\kappa_2}{2}$ in Corollary 4.19, then we have the following inequality

$$\left|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)^-,k}f(\kappa_2)+J^{\alpha}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)^+,k}f(\kappa_1)\right]-f\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|\leq\frac{(\kappa_2-\kappa_1)k}{4(\alpha+k)}\left[|f'(\kappa_1)|+|f'(\kappa_2)|\right]$$

which is given by Farid et al. in [15].

Theorem 4.21. *$f:[\kappa_1,\kappa_2]\to\mathbb{R}$ be a differentiable mapping on (κ_1,κ_2) with $0\leq\kappa_1<\kappa_2$. If $|f'|^q$ is convex on $[\kappa_1,\kappa_2]$ for some fixed $q>1$, then we have the following inequality for generalized fractional integrals*

$$\begin{aligned}
&\left|\frac{1}{\Psi(1)+\Theta(1)}\left[{}_{(\kappa_1+\kappa_2-\kappa)^-}I_{\varphi}f(\kappa_2)+{}_{(\kappa_1+\kappa_2-\kappa)^+}I_{\varphi}f(\kappa_1)\right]-f(\kappa_1+\kappa_2-\kappa)\right| \qquad (4.6)\\
&\quad\leq\frac{\kappa-\kappa_1}{\Psi(1)+\Theta(1)}\left(\int_0^1\Psi(\xi)^p d\xi\right)^{\frac{1}{p}}\left(\frac{|f'(\kappa_1+\kappa_2-\kappa)|^q+|f'(\kappa_2)|^q}{2}\right)^{\frac{1}{q}}\\
&\quad+\frac{\kappa_2-\kappa}{\Psi(1)+\Theta(1)}\left(\int_0^1\Theta(\xi)^p d\xi\right)^{\frac{1}{p}}\left(\frac{|f'(\kappa_1+\kappa_2-\kappa)|^q+|f'(\kappa_1)|^q}{2}\right)^{\frac{1}{q}}
\end{aligned}$$

with all $\kappa\in[\kappa_1,\kappa_2]$, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By the Lemma 3.2, we have

$$\begin{aligned}
&\left|\frac{1}{\Psi(1)+\Theta(1)}\left[\, {}_{(\kappa_1+\kappa_2-\kappa)^-}I_\varphi f(\kappa_2) + {}_{(\kappa_1+\kappa_2-\kappa)^+}I_\varphi f(\kappa_1)\right] - f(\kappa_1+\kappa_2-\kappa)\right| \\
&\leq \frac{\kappa-\kappa_1}{\Psi(1)+\Theta(1)} \int_0^1 \Psi(\xi)\left|f'\left(\xi\left(\kappa_1+\kappa_2-\kappa\right)+(1-\xi)\kappa_2\right)\right| d\xi \\
&+ \frac{\kappa_2-\kappa}{\Psi(1)+\Theta(1)} \int_0^1 \Theta(\xi)\left|f'\left(\xi\left(\kappa_1+\kappa_2-\kappa\right)+(1-\xi)\kappa_1\right)\right| d\xi.
\end{aligned}$$

Using the Hölder's inequality and the convexity of $|f'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{1}{\Psi(1)+\Theta(1)}\left[\, {}_{(\kappa_1+\kappa_2-\kappa)^-}I_\varphi f(\kappa_2) + {}_{(\kappa_1+\kappa_2-\kappa)^+}I_\varphi f(\kappa_1)\right] - f(\kappa_1+\kappa_2-\kappa)\right| \\
&\leq \frac{\kappa-\kappa_1}{\Psi(1)+\Theta(1)}\left(\int_0^1 (\Psi(\xi))^p\, d\xi\right)^{\frac{1}{p}} \times \left(\int_0^1 \left|f'\left(\xi(\kappa_1+\kappa_2-\kappa)+(1-\xi)\,\kappa_2\right)\right|^q d\xi\right)^{\frac{1}{q}} \\
&+ \frac{\kappa_2-\kappa}{\Psi(1)+\Theta(1)}\left(\int_0^1 (\Theta(\xi))^p\, d\xi\right)^{\frac{1}{p}} \times \left(\int_0^1 \left|f'\left(\xi\left(\kappa_1+\kappa_2-\kappa\right)+(1-\xi)\kappa_1\right)\right|^q\right)^{\frac{1}{q}} \\
&\leq \frac{\kappa-\kappa_1}{\Psi(1)+\Theta(1)}\left(\int_0^1 (\Psi(\xi))^p\, d\xi\right)^{\frac{1}{p}} \times \left(\int_0^1 \left[(1-\xi)\left|f'(\kappa_2)\right|^q + \xi\left|f'\left(\kappa_1+\kappa_2-\kappa\right)\right|^q\right] d\xi\right)^{\frac{1}{q}} \\
&+ \frac{\kappa_2-\kappa}{\Psi(1)+\Theta(1)}\left(\int_0^1 (\Theta(\xi))^p\, d\xi\right)^{\frac{1}{p}} \times \left(\int_0^1 \left[|1-\xi) f'(\kappa_1)|^q + (\xi\left|f'\left(\kappa_1+\kappa_2-\kappa\right)\right|^q\right] d\xi\right)^{\frac{1}{q}} \\
&= \frac{\kappa-\kappa_1}{\Psi(1)+\Theta(1)}\left(\int_0^1 (\Psi(\xi))^p\, d\xi\right)^{\frac{1}{p}} \left(\frac{\left|f'\left(\kappa_1+\kappa_2-\kappa\right)\right|^q + \left|f'(\kappa_2)\right|^q}{2}\right)^{\frac{1}{q}} \\
&+ \frac{\kappa_2-\kappa}{\Psi(1)+\Theta(1)}\left(\int_0^1 (\Theta(\xi))^p\, d\xi\right)^{\frac{1}{p}} \left(\frac{\left|f'\left(\kappa_1+\kappa_2-\kappa\right)\right|^q + \left|f'(\kappa_1)\right|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

The proof is completed. □

Remark 4.22. If we choose $\varphi(\xi) = \xi$ in Theorem 4.21, then inequality (4.7) reduces to the inequality (4.5).

Corollary 4.23. *Under assumption of Theorem 4.21 with $\varphi(\xi)=\frac{\xi^{\alpha}}{\Gamma(\alpha)}$, we have the following inequality*

$$
\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{(\kappa_2-\kappa)^{\alpha}+(\kappa-\kappa_1)^{\alpha}}\times[J^{\alpha}_{(\kappa_1+\kappa_2-\kappa)-}f(\kappa_2)+J^{\alpha}_{(\kappa_1+\kappa_2-\kappa)+}f(\kappa_1)]-f(\kappa_1+\kappa_2-\kappa)\right|\\
&\leq\frac{1}{(\kappa_2-\kappa)^{\alpha}+(\kappa-\kappa_1)^{\alpha}}(\frac{1}{\alpha p+1})^{\frac{1}{p}}\times\left\{(\kappa-\kappa_1)^{\alpha+1}(\frac{|f'(\kappa_1+\kappa_2-\kappa)|^q+|f'(\kappa_2)|^q}{2})^{\frac{1}{q}}\right.\\
&\left.+(\kappa_2-\kappa)^{\alpha+1}(\frac{|f'(\kappa_1+\kappa_2-\kappa)|^q+|f'(\kappa_1)|^q}{2})^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Remark 4.24. If we choose $\kappa=\frac{\kappa_1+\kappa_2}{2}$ in Corollary 4.23, then we have the following inequality

$$
\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-}f(\kappa_2)+J^{\alpha}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+}f(\kappa_1)\right]-f\left(\frac{\kappa_1+\kappa_2}{2}\right)\right| \qquad (4.7)\\
&\leq\frac{\kappa_2-\kappa_1}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\times\left\{\left(\frac{|f'(\kappa_1)|^q+3|f'(\kappa_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f'(\kappa_1)|^q+|f'(\kappa_2)|^q}{4}\right)^{\frac{1}{q}}\right\}\\
&\leq\frac{\kappa_2-\kappa_1}{4}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}[|f'(\kappa_1)|+|f'(\kappa_2)|]
\end{aligned}
$$

which is proved by Sarikaya and Yildirim in [36].

Corollary 4.25. *Under assumption of Theorem 4.21 with $\varphi(\xi)=\frac{\xi^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, we have the following inequality*

$$
\begin{aligned}
&\left|\frac{\Gamma_k(\alpha+k)}{(\kappa_2-\kappa)^{\frac{\alpha}{k}}+(\kappa-\kappa_1)^{\frac{\alpha}{k}}}\times\left[J^{\alpha}_{(\kappa_1+\kappa_2-\kappa)-,k}f(\kappa_2)+J^{\alpha}_{(\kappa_1+\kappa_2-\kappa)+,k}f(\kappa_1)\right]-f(\kappa_1+\kappa_2-\kappa)\right|\\
&\leq\frac{1}{(\kappa_2-\kappa)^{\frac{\alpha}{k}}+(\kappa-\kappa_1)^{\frac{\alpha}{k}}}\left(\frac{k}{\alpha p+k}\right)^{\frac{1}{p}}\times\left\{(\kappa-\kappa_1)^{\frac{\alpha}{k}+1}\left[\frac{|f'(\kappa_1+\kappa_2-\kappa)|^q+|f'(\kappa_2)|^q}{2}\right]^{\frac{1}{q}}\right.\\
&\left.+(\kappa_2-\kappa)^{\frac{\alpha}{k}+1}\left[\frac{|f'(\kappa_1)|^q+|f'(\kappa_1+\kappa_2-\kappa)|^q}{2}\right]^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Remark 4.26. If we choose $\kappa=\frac{\kappa_1+\kappa_2}{2}$ in Corollary 4.25, then we have the following inequality

$$
\begin{aligned}
&\left|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)^-,k}f(\kappa_2)+J^{\alpha}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)^+,k}f(\kappa_1)\right]-f\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|\\
&\leq\frac{\kappa_2-\kappa_1}{4}\left(\frac{k}{\alpha p+k}\right)^{\frac{1}{p}}\times\left\{\left(\frac{|f'(\kappa_1)|^q+3|f'(\kappa_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f'(\kappa_1)|^q+|f'(\kappa_2)|^q}{4}\right)^{\frac{1}{q}}\right\}\\
&\leq\frac{\kappa_2-\kappa_1}{4}\left(\frac{4k}{\alpha p+k}\right)^{\frac{1}{p}}[|f'(\kappa_1)|+|f'(\kappa_2)|].
\end{aligned}
$$

which is given by Farid et al. in [15].

5. Conclusions

In this study, by obtained fractional equalities, we establish some midpoint type inequalities via generalized fractional integrals for functions whose derivatives in absolute values are convex. In the future works, authors can prove similar inequalities by using the different kinds of convexities.

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