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The monotonicity and convexity of the period function for a class of symmetric Newtonian systems of degree 8

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Abstract

In this paper, we study the monotonicity and convexity of the period function associated with centers of a specific class of symmetric Newtonian systems of degree 8. In this regard, we prove that if the period annulus surrounds only one elementary center, then the corresponding period function is monotone; but, for the other cases, the period function has exactly one critical point. We also prove that in all cases, the period function is convex.

Keywords. Newtonian system, Period function, Monotonicity.
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1. Introduction

Let f be a real-valued analytic function on $\mathbb{R}$ and consider the Newtonian system

$$\begin{cases} \dot{x} = -y \\ \dot{y} = f(x), \end{cases} \tag{1.1}$$

with $f(0) = 0$ and $f'(0) > 0$. The Hamiltonian function associated with system (1.1) is of the form $H(x, y) = \frac{y^2}{2} + F(x)$, where $F(x) = \int_0^x f(s)ds$ is the potential function of system (1.1). Under these conditions, it is easy to verify that the origin is an elementary center of system (1.1) which is surrounded by a family of periodic orbits (period annulus) $\{\Gamma_a\}$, passing through the point $(0, a)$ for $a \in (0, c)$, where $c > 0$. We denote the projection of this period annulus on the x-axes by (c_l, c_r), where $c_l < 0 < c_r$ and $F(c_l) = F(c_r) = \frac{c^2}{2}$ (see Figure 1). Note that $xf(x) > 0$ for each $x \in (c_l, c_r) \setminus \{0\}$.

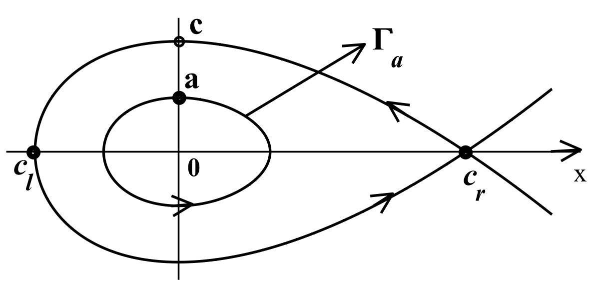

Figure 1. The location of Γ_a.

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The period function $T:(0,c)\to[0,\infty)$ is defined as the period of the periodic orbit Γ_a. By the first equation of system (1.1), T is given by

$$T(a)=-\oint_{\Gamma_a}\frac{dx}{y},\quad a\in(0,c), \tag{1.2}$$

where the orientation of the above Abelian integral is determined by the vector field (1.1) which is counter-clockwise. We recall that a critical point of T is a value $a^*\in(0,c)$ such that $T'(a^*)=0$ and in this case, $T(a^*)$ is named as a critical period. The study of the period function and finding the number of its critical periods are interesting problems which are closely related to the study of Abelian integrals, week Hilbert 16th problem and some nonlinear boundary value problems, for example, see [2, 9]. Most of the papers on this subject are devoted to finding some conditions to guarantee the monotonicity of period function, see for instance [3, 8, 11]. However, there are also very few papers dealing with the number of critical periods, see [1, 6, 7].

Schaaf in [8] considered a class of Hamiltonian systems of the form

$$\begin{cases}\dot{x}=-g(y)\\ \dot{y}=f(x),\end{cases} \tag{1.3}$$

and found some conditions on f and g to guarantee the monotonicity of the associated period function. System (1.1) is a special case of (1.3) by taking $g(y)=y$. So we recall his result for Newtonian system (1.1). In fact, he supposed the Newtonian system (1.1) satisfies one of the following assumptions:

(A) $(5(f'')^2-3f'f''')(x)>0$ if $f'(0)>0$, and $f(x)f''(x)<0$ if $f'(x)=0$;
(B) $(5(f'')^2-3f'f''')(x)<0$ if $f'(0)>0$, and $f(x)f''(x)<0$ if $f'(x)=0$;

and then he obtained a result implying the following theorem.

Theorem 1.1. *([8]) If condition (A) (resp. (B)) holds for $x\in(c_l,c_r)$, then $T'(a)>0$ (resp. $T'(a)<0$) for all $a\in(0,c)$.*

Li and Lu in [5] added the following assumptions to the Newtonian system (1.1),

(C) If there exists $x\in(c_l,0)$ with $f'(x)<0$, then $(3f'(2Fff'-f^2)-2Fff'')(c_l)>0$, and if there exists $x\in(0,c_r)$ such that $f'(x)<0$, then $(3f'(2Ff'-f^2)-2Fff'')(c_r)>0$;
(D) If there exists $x\in(c_l,0)$ with $f'(x)<0$, then $(3f'(2Ff'-f^2)-2Fff'')(c_l)<0$, and if there exists $x\in(0,c_r)$ such that $f'(x)<0$, then $(3f'(2Ff'-f^2)-2Fff'')(c_r)<0$;

and proved the following theorem.

Theorem 1.2. *([5]) If assumption (A) (resp. (B)) is fulfilled for $x\in(c_l,c_r)$, and assumption (C) (resp. (D)) holds, then $T''(a)>0$ (resp. $T''(a)<0$) for $a\in(0,c)$.*

Moreover, they considered a class of Newtonian system of the form

$$\begin{cases}\dot{x}=-y\\ \dot{y}=x(x-\alpha)(x-\beta)(x-1),\end{cases} \tag{1.4}$$

where $0\le\alpha\le\beta\le1$, and proved the following theorem.

Theorem 1.3. *([5]) The period function associated with each period annulus of the system (1.4) is monotonically increasing if the period annulus surrounds only one elementary center. Otherwise, the period function has exactly one critical point.*

In this paper, we will consider a class of symmetric Newtonian systems of the form

$$\begin{cases}\dot{x}=-y\\ \dot{y}=-x(x^2-\alpha^2)(x^2-\beta^2)(x^2-1),\end{cases} \tag{1.5}$$

with hyperelliptic Hamiltonian function $H(x,y)=\frac{y^2}{2}+F(x)$, where

$$F(x)=-\frac{1}{8}x^8+\frac{1}{6}\left(\alpha^2+\beta^2+1\right)x^6-\frac{1}{4}\left(\alpha^2\beta^2+\alpha^2+\beta^2\right)x^4+\frac{1}{2}\alpha^2\beta^2x^2, \tag{1.6}$$

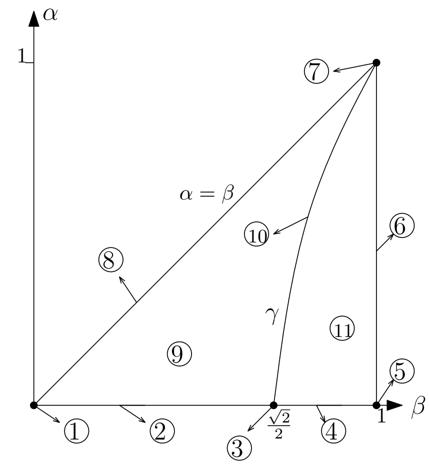

FIGURE 2. Bifurcation diagram of system (1.5).

and $0 \leq \alpha \leq \beta \leq 1$. Following Li and Lu's idea in [5], our main goal is to prove Theorem 1.3 for the system (1.5). In this way, we give the bifurcation diagram and all topologically different phase portraits of the system (1.5) in section 2. Then, in section 3, we investigate the monotonicity and convexity of $T(a)$ when $\{\Gamma_a\}$ is a period annulus surrounding a unique elementary center of system (1.5). Finally, in section 4 we discuss on convexity and the number of critical periods of $T(a)$ when the period annulus $\{\Gamma_a\}$ surrounds more than one equilibrium point of the system (1.5) counted with multiplicities.

2. Bifurcation diagram and phase portraits of system (1.5)

In this section, the bifurcation diagram and all topologically different phase portraits of the Newtonian system (1.5) will be considered. Let us denote the equilibrium points of system (1.5) by $p_0 = (0,0)$, $p_{\pm\beta} = (\pm\beta, 0)$, $p_{\pm\alpha} = (\pm\alpha, 0)$ and $p_{\pm 1} = (\pm 1, 0)$. Also, denote the corresponding critical energy levels of H by

$$\begin{aligned}
h_0 &= H(p_0) = 0,\\
h_\alpha &= H(p_{\pm\alpha}) = \frac{1}{24}\,\alpha^4\left(\alpha^4 - 2\,\alpha^2\beta^2 - 2\,\alpha^2 + 6\,\beta^2\right),\\
h_\beta &= H(p_{\pm\beta}) = \frac{1}{24}\,\beta^4\left(\beta^4 - 2\,\alpha^2\beta^2 - 2\,\beta^2 + 6\,\alpha^2\right),\\
h_1 &= H(p_{\pm 1}) = \frac{1}{24}(6\,\alpha^2\beta^2 - 2\,\alpha^2 - 2\,\beta^2 + 1).
\end{aligned}$$

Note that the global structure of the phase portrait of system (1.5) will change either there exists a critical point $\bar{x}$ which is degenerate i.e., $F''(\bar{x}) = 0$ or at least two maximum values of F are equal. The first happens when $\alpha = 0$, $\beta = 1$ or $\alpha = \beta$ and the later holds if $h_\alpha = h_\beta$ or equivalently $2\beta^2 - \alpha^2 = 1$. Therefore, as shown in Figure 2, the bifurcation diagram of system (1.5) contains the boundary of the triangle $T = \{(\alpha, \beta)|\ 0 \leq \alpha \leq \beta \leq 1\}$ and the curve

$$\gamma := \{(\alpha, \beta)|0 \leq \alpha \leq \beta \leq 1, \beta = \sqrt{\frac{\alpha^2 + 1}{2}}\}.$$

According to this bifurcation diagram, system (1.5) has 11 topologically different phase portraits, illustrated in $(1) - (11)$ of Figure 3, that we classify them respectively as follows:

(1): If $\alpha = \beta = 0$, then p_0 is a nilpotent centre of order two and $p_{\pm 1}$ are hyperbolic saddles. The corresponding energy levels are $h_0 = h_\alpha = h_\beta = 0$ and $h_1 = \frac{1}{24}$ (see Figure 3 (1)).

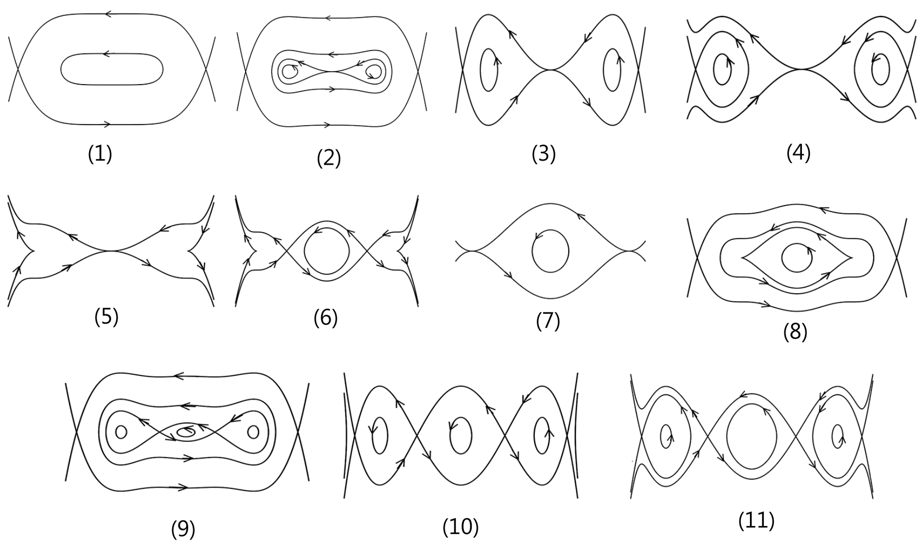

FIGURE 3. Classification of phase portraits of system (1.5).

(2): If $\alpha = 0$ and $0 < \beta < \frac{\sqrt{2}}{2}$, then p_0 is a nilpotent saddle of order one, $p_{\pm\beta}$ are elementary centers and $p_{\pm 1}$ are hyperbolic saddles. The corresponding energy levels are $h_0 = h_\alpha = 0$, $h_\beta = \frac{1}{24}\beta^6\left(\beta^2 - 1\right)$ and $h_1 = \frac{1}{24}(1 - 2\beta^2)$. So $h_\beta < h_0 < h_1$ (see Figure 3 (2)).

(3): If $\alpha = 0$ and $\beta = \frac{\sqrt{2}}{2}$, then p_0 is a nilpotent saddle of order one, $p_{\pm\beta}$ are elementary centers and $p_{\pm 1}$ are hyperbolic saddles. The corresponding energy levels are $h_\beta = -\frac{1}{128}$ and $h_0 = h_\alpha = h_1 = 0$ (see Figure 3 (3)).

(4): If $\alpha = 0$ and $\frac{\sqrt{2}}{2} < \beta < 1$, then p_0 is a nilpotent saddle of order one, $p_{\pm\beta}$ are elementary centers and $p_{\pm 1}$ are hyperbolic saddles. The corresponding energy levels are $h_0 = h_\alpha = 0$, $h_\beta = \frac{1}{24}\beta^6\left(\beta^2 - 1\right)$ and $h_1 = \frac{1}{24}(1 - 2\beta^2)$. So $h_\beta < h_1 < h_0$ (see Fig. 3 (4)).

(5): If $\alpha = 0$ and $\beta = 1$, then p_0 is a nilpotent saddle of order one and $p_{\pm 1}$ are cusps of order one with energy levels $h_0 = h_\alpha = 0$ and $h_\beta = h_1 = -\frac{1}{24}$ (see Fig. 3 (5)).

(6): If $0 < \alpha < 1$ and $\beta = 1$, then p_0 is an elementary center, $p_{\pm\alpha}$ are hyperbolic saddles and $p_{\pm 1}$ are cusps of order one. The corresponding energy levels are $h_0 = 0$, $h_\alpha = \frac{1}{24}\,\alpha^4\left(\alpha^4 - 4\,\alpha^2 + 6\right)$ and $h_\beta = h_1 = \frac{1}{24}(4\alpha^2 - 1)$. So $h_0 < h_\alpha < h_1$ (see Figure 3 (6)).

(7): If $\alpha = \beta = 1$, then p_0 is an elementary center and $p_{\pm 1}$ are nilpotent saddles of order one with energy levels $h_0 = 0$ and $h_\alpha = h_\beta = h_1 = \frac{1}{8}$ (see Figure 3 (7)).

(8): If $0 < \alpha = \beta < 1$, then p_0 is an elementary center, $p_{\pm\alpha}$ are cusps of order one and $p_{\pm 1}$ are hyperbolic saddles, with energy levels $h_0 = 0$, $h_\alpha = h_\beta = \frac{1}{24}\alpha^6\left(4 - \alpha^2\right)$ and $h_1 = \frac{1}{24}(6\,\alpha^4 - 4\,\alpha^2 + 1)$. So $h_0 < h_\alpha < h_1$ (see Figure 3 (8)).

(9): If $\alpha < \beta < \sqrt{\frac{\alpha^2+1}{2}}$ and $0 < \alpha < 1$, then p_0 and $p_{\pm\beta}$ are elementary centers and $p_{\pm\alpha}$ and $p_{\pm 1}$ are hyperbolic saddles. Also, $h_0 < h_\alpha$, $h_\beta < h_\alpha$ and $h_\alpha < h_1$ (see Figure 3 (9)).

(10): If $\beta = \sqrt{\frac{\alpha^2+1}{2}}$ and $0 < \alpha < 1$, then p_0 and $p_{\pm\beta}$ are elementary centers and $p_{\pm\alpha}$ and $p_{\pm 1}$ are hyperbolic saddles. Also, $h_0 < h_\alpha$, $h_\beta < h_\alpha$ and $h_\alpha = h_1$ (see Figure 3 (10)).

(11): If $\sqrt{\frac{\alpha^2+1}{2}} < \beta < 1$ and $0 < \alpha < 1$, then p_0 and $p_{\pm\beta}$ are elementary centers and $p_{\pm\alpha}$ and $p_{\pm 1}$ are hyperbolic saddles. Also, $h_0 < h_\alpha$, $h_\beta < h_1$ and $h_1 < h_\alpha$ (see Figure 3 (11)).

3. Period annulus surrounding a unique elementary center

This section will consider the monotonicity and convexity of the period functions associated with those period annuli of the system (1.5) which surround a unique elementary center. To tackle this, we consider two cases separately. First, we treat those period annulus which surrounding the unique elementary center at the origin, and then we consider those period annulus which surrounding one of the elementary centers at $p_{\pm\beta}$.

3.1. **Period annulus surrounding only the elementary center p_0.** This subsection is devoted to period annulus surrounding only the elementary center p_0. As shown in Figure 3, this happens in cases $(6)-(11)$, when $0<\alpha\leq\beta\leq 1$. Note that in these cases we have $f(x)=-x(x^2-\alpha^2)(x^2-\beta^2)(x^2-1)$, $f(0)=0$, $f'(0)=\alpha^2\beta^2>0$ and $xf(x)=-x^2(x^2-\alpha^2)(x^2-\beta^2)(x^2-1)>0$ for $x\in(-\alpha,\alpha)\setminus\{0\}$. So we have the following lemma.

Lemma 3.1. *For $f(x)=-x(x^2-\alpha^2)(x^2-\beta^2)(x^2-1)$ and $F(x)=\int_0^x f(s)ds$, assumption (A) holds for $x\in(-\alpha,\alpha)$ and assumption (C) is true for $c_l=-\alpha$ and $c_r=\alpha$.*

Proof. We set

$$\begin{aligned}A(x;\alpha,\beta):=&5(f''(x))^2-3f'(x)f'''(x)\\ =&4410x^{10}+a_4x^8+a_3x^6+a_2x^4+a_1x^2+a_0,\end{aligned}\tag{3.1}$$

where

$$\begin{aligned}a_4&=-3990(\alpha^2+\beta^2+1),\\ a_3&=1100(\alpha^4+\beta^4+1)+2704(\alpha^2\beta^2+\alpha^2+\beta^2),\\ a_2&=-570(\alpha^4+\beta^4+\alpha^4\beta^2+\alpha^2\beta^4+\alpha^2+\beta^2)-1080\,\alpha^2\beta^2,\\ a_1&=126(\alpha^4+\beta^4+\alpha^4\beta^4)+72(\alpha^4\beta^2+\alpha^2\beta^4+\alpha^2\beta^2),\\ a_0&=18(\alpha^4\beta^4+\alpha^4\beta^2+\alpha^2\beta^4).\end{aligned}$$

Now, we will prove that if $(\alpha,\beta)\in\Pi=\{0<\alpha\leq\beta\leq 1\}$, then $A(x;\alpha,\beta)>0$ for $x\in(-\alpha,\alpha)$. A direct computation gives $A(-\alpha;\alpha,\beta)=A(\alpha;\alpha,\beta)=16\alpha^2\bar{A}(\alpha,\beta)$, where

$$\bar{A}(\alpha,\beta)=95\,\alpha^8-116\,\alpha^6\beta^2+41\,\alpha^4\beta^4-116\,\alpha^6+106\,\alpha^4\beta^2-30\,\alpha^2\beta^4+41\,\alpha^4-30\,\alpha^2\beta^2+9\,\beta^4,\quad(\alpha,\beta)\in\Pi.$$

To prove $\bar{A}(\alpha,\beta)>0$ for $(\alpha,\beta)\in\Pi$, we note that

$$\begin{aligned}&\bar{A}(\alpha,1)=(95\,\alpha^4-42\,\alpha^2+9)\left(\alpha^2-1\right)^2>0, &&0<\alpha<1,\\ &\bar{A}(\alpha,\alpha)=20\,\alpha^4\left(\alpha^2-1\right)^2>0, &&0<\alpha<1,\\ &\bar{A}(\frac{1}{2},\beta)=\frac{1}{16}(65\,\beta^4-43\,\beta^2+\frac{287}{16})>0, &&0<\beta<1.\end{aligned}$$

Thus, if there exists $\bar{\alpha}\in(0,1)$ such that $\bar{A}(\bar{\alpha},\beta)$ has roots in the interval $(\bar{\alpha},1)$, then the number of these roots (counted with multiplicities) should be even. By continuously moving α from $\bar{\alpha}$ to $\frac{1}{2}$, one can find $\hat{\alpha}$ such that $\bar{A}(\hat{\alpha},\beta)$ has a multiple zero in $(\hat{\alpha},1)$. So the equations

$$\bar{A}(\alpha,\beta)=0,\qquad\frac{\partial\bar{A}}{\partial\beta}(\alpha,\beta)=0,$$

should have a solution in $\Pi\backslash\{\beta=1\}$. Computing the resultant between $\bar{A}(\alpha,\beta)$ and $\frac{\partial\bar{A}}{\partial\beta}(\alpha,\beta)$ with respect to α, we get

$$Res(\bar{A},\frac{\partial\bar{A}}{\partial\beta},\alpha)=13931406950400\,\beta^{16}\left(6667\,\beta^4-132\,\beta^2+15568\right)^2\left(\beta^2-1\right)^4,$$

which doesn't have any solution for $\beta\in(0,1)$. Hence

$$A(-\alpha;\alpha,\beta)=A(\alpha;\alpha,\beta)>0.\tag{3.2}$$

Now by setting $\alpha = \frac{1}{4}$ and $\beta = \frac{1}{2}$ in $A(x;\alpha,\beta)$, we get

$$A(x;\frac{1}{4},\frac{1}{2}) = 4410\,x^{10} - \frac{41895}{8}\,x^8 + \frac{131859}{64}\,x^6 - \frac{120645}{512}\,x^4 + \frac{20223}{2048}\,x^2 + \frac{189}{2048}.$$

By applying Sturm's theorem to $A(x;\frac{1}{4},\frac{1}{2})$ and straightforward calculation with Maple, it follows that the above expression is non-vanishing for all $x \in \mathbb{R}$. Hence,

$$A(x;\frac{1}{4},\frac{1}{2}) > 0. \tag{3.3}$$

Since A is continuous, it follows from (3.2) and (3.3) that if there exists $(\alpha_1,\beta_1) \in \Pi$ such that $A(x;\alpha_1,\beta_1)$ has a root in $(-\alpha_1,\alpha_1)$, then varying (α,β) from $(\frac{1}{4},\frac{1}{2})$ to (α_1,β_1) in Π continuously, one may find a point $(\alpha_2,\beta_2) \in \Pi$ which $A(x;\alpha_2,\beta_2)$ has a multiple root in $(-\alpha_2,\alpha_2)$, i.e. the equations

$$A(x;\alpha,\beta) = 0, \qquad \frac{\partial A}{\partial x}(x;\alpha,\beta) = 0, \tag{3.4}$$

have a solution in $(-\alpha_2,\alpha_2)$. By computing the resultant with respect to x between $A(x;\alpha,\beta)$ and $\frac{\partial A}{\partial x}(x;\alpha,\beta)$, we obtain

$$Res(A, \frac{\partial A}{\partial x}, x) = C\alpha^2\beta^2(\alpha^2\beta^2 + \alpha^2 + \beta^2)[P_1(\alpha,\beta)P_2(\alpha,\beta)]^2, \tag{3.5}$$

where $C = 10791437225693943863181312000$ and

$$\begin{aligned}
P_1(\alpha,\beta) =& 225\,\alpha^8\beta^4 - 306\,\alpha^6\beta^6 + 225\,\alpha^4\beta^8 - 50\,\alpha^8\beta^2 - 78\,\alpha^6\beta^4 + 225\,\beta^8 - 306\,\alpha^6 \\
&+ 225\,\alpha^8 - 78\,\alpha^6\beta^2 + 186\,\alpha^4\beta^4 - 78\,\alpha^2\beta^6 - 78\,\alpha^4\beta^6 - 50\,\alpha^2\beta^8 - 78\,\alpha^4\beta^2 - 78\,\alpha^2\beta^4 \\
&- 306\,\beta^6 + 225\,\alpha^4 - 50\,\alpha^2\beta^2 + 225\,\beta^4, \\
P_2(\alpha,\beta) =& 124416359375\,\alpha^{20}\beta^8 - 135340051875\,\alpha^{18}\beta^{10} + 334804385685\,\alpha^{16}\beta^{12} - 129618282626\,\alpha^{14}\beta^{14} \\
&+ 334804385685\,\alpha^{12}\beta^{16} - 135340051875\,\alpha^{10}\beta^{18} + 124416359375\,\alpha^8\beta^{20} + 408049812500\,\alpha^{20}\beta^6 \\
&- 266804794375\,\alpha^{18}\beta^8 + 641553565485\,\alpha^{16}\beta^{10} + 334476148678\,\alpha^{14}\beta^{12} + 334476148678\,\alpha^{12}\beta^{14} \\
&+ 641553565485\,\alpha^{10}\beta^{16} - 266804794375\,\alpha^8\beta^{18} + 408049812500\,\alpha^6\beta^{20} + 961575656250\,\alpha^{20}\beta^4 \\
&- 272865873750\,\alpha^{18}\beta^6 + 1591928735775\,\alpha^{16}\beta^8 + 849153788154\,\alpha^{14}\beta^{10} + 2651495392518\,\alpha^{12}\beta^{12}
\end{aligned}$$

$$
\begin{aligned}
&+ 849153788154\,\alpha^{10}\beta^{14} + 1591928735775\,\alpha^{8}\beta^{16} - 272865873750\,\alpha^{6}\beta^{18} + 961575656250\,\alpha^{4}\beta^{20}\\
&+ 408049812500\,\alpha^{20}\beta^{2} - 272865873750\,\alpha^{18}\beta^{4} - 364990138050\,\alpha^{16}\beta^{6} - 377773237310\,\alpha^{14}\beta^{8}\\
&+ 298859364930\,\alpha^{12}\beta^{10} + 298859364930\,\alpha^{10}\beta^{12} - 377773237310\,\alpha^{8}\beta^{14} - 364990138050\,\alpha^{6}\beta^{16}\\
&- 272865873750\,\alpha^{4}\beta^{18} + 408049812500\,\alpha^{2}\beta^{20} + 124416359375\,\alpha^{20} - 266804794375\,\alpha^{18}\beta^{2}\\
&+ 1591928735775\,\alpha^{16}\beta^{4} - 377773237310\,\alpha^{14}\beta^{6} + 6534531610810\,\alpha^{12}\beta^{8} + 1442974207290\,\alpha^{10}\beta^{10}\\
&+ 6534531610810\,\alpha^{8}\beta^{12} - 377773237310\,\alpha^{6}\beta^{14} + 1591928735775\,\alpha^{4}\beta^{16} - 266804794375\,\alpha^{2}\beta^{18}\\
&+ 124416359375\,\beta^{20} + 641553565485\,\alpha^{16}\beta^{2} + 849153788154\,\alpha^{14}\beta^{4} + 298859364930\,\alpha^{12}\beta^{6}\\
&+ 1442974207290\,\alpha^{10}\beta^{8} + 1442974207290\,\alpha^{8}\beta^{10} + 298859364930\,\alpha^{6}\beta^{12} + 849153788154\,\alpha^{4}\beta^{14}\\
&+ 641553565485\,\alpha^{2}\beta^{16} - 135340051875\,\beta^{18} + 334804385685\,\alpha^{16} + 334476148678\,\alpha^{14}\beta^{2}\\
&+ 2651495392518\,\alpha^{12}\beta^{4} + 298859364930\,\alpha^{10}\beta^{6} + 6534531610810\,\alpha^{8}\beta^{8} + 298859364930\,\alpha^{6}\beta^{10}\\
&+ 2651495392518\,\alpha^{4}\beta^{12} + 334476148678\,\alpha^{2}\beta^{14} + 334804385685\,\beta^{16} + 334476148678\,\alpha^{12}\beta^{2}\\
&+ 849153788154\,\alpha^{10}\beta^{4} - 377773237310\,\alpha^{8}\beta^{6} - 377773237310\,\alpha^{6}\beta^{8} + 849153788154\,\alpha^{4}\beta^{10}\\
&+ 334476148678\,\alpha^{2}\beta^{12} - 129618282626\,\beta^{14} + 334804385685\,\alpha^{12} + 641553565485\,\alpha^{10}\beta^{2}\\
&+ 1591928735775\,\alpha^{8}\beta^{4} - 364990138050\,\alpha^{6}\beta^{6} + 1591928735775\,\alpha^{4}\beta^{8} + 641553565485\,\alpha^{2}\beta^{10}\\
&+ 334804385685\,\beta^{12} - 135340051875\,\alpha^{10} - 266804794375\,\alpha^{8}\beta^{2} - 272865873750\,\alpha^{6}\beta^{4}\\
&- 272865873750\,\alpha^{4}\beta^{6} - 266804794375\,\alpha^{2}\beta^{8} - 135340051875\,\beta^{10} + 124416359375\,\alpha^{8}\\
&+ 408049812500\,\alpha^{6}\beta^{2} + 961575656250\,\alpha^{4}\beta^{4} + 408049812500\,\alpha^{2}\beta^{6} - 135340051875\,\alpha^{18}\\
&- 129618282626\,\alpha^{14} + 124416359375\,\beta^{8},
\end{aligned}
$$

Now we want to show that $P_1P_2 \neq 0$ for $(\alpha,\beta) \in \Pi$. Indeed, we have

$$
\begin{aligned}
&P_1(\alpha,1) = 16\left(25\,\alpha^4 + 2\,\alpha^2 + 9\right)\left(\alpha^2 - 1\right)^2 > 0, && 0 < \alpha < 1,\\
&P_1(\alpha,\alpha) = 16\,\alpha^4\left(9\,\alpha^4 + 2\,\alpha^2 + 25\right)\left(\alpha^2 - 1\right)^2 > 0, && 0 < \alpha < 1,\\
&P_1(\frac{1}{2},\beta) = \frac{55497}{256}\beta^4 - \frac{10725}{32}\beta^6 + \frac{3625}{16}\beta^8 - \frac{2405}{128}\beta^2 + \frac{2601}{256}, && 0 < \beta \le 1.
\end{aligned}
$$

Applying Sturm's Theorem, one concludes that $P_1(\frac{1}{2},\beta) \neq 0$ for $0 < \beta \le 1$ and so it is positive. A similar argument shows that if there exists $\bar{\alpha} \in (0,1)$ such that $P_1(\bar{\alpha},\beta)$ has roots on the interval $(\bar{\alpha},1)$, then the equations

$$
P_1(\alpha,\beta) = 0, \qquad \frac{\partial P_1}{\partial \beta}(\alpha,\beta) = 0,
$$

should have a solution in $\Pi\backslash\{\beta = 1\}$. Computing the resultant with respect to α between $P_1(\alpha,\beta)$ and $\frac{\partial P_1}{\partial \beta}(\alpha,\beta)$, we obtain

$$
Res(P_1, \frac{\partial P_1}{\partial \beta}, \alpha) = D\beta^{16}\left(\beta^2 - 1\right)^4 [p_0(\beta)]^6,
$$

where $D = 16906881566693365288796160000$ and

$$
p_0(\beta) = \left(125\,\beta^{12} - 150\,\beta^{10} + 323\,\beta^8 - 84\,\beta^6 + 323\,\beta^4 - 150\beta^2 + 125\right).
$$

Sturm's Theorem implies that $p_0(\beta)$ doesn't have any solution for $\beta \in (0,1)$. Hence, $P_1(\alpha,\beta) \neq 0$ for $(\alpha,\beta) \in \Pi$. We also have

$$\begin{aligned}
P_2(\alpha,1) =& 2026508000000\,\alpha^{20} - 1350021440000\,\alpha^{18} + 4771583235840\,\alpha^{16} + 1352476833792\,\alpha^{14}\\
&+ 13773802194432\,\alpha^{12} + 6194401747968\,\alpha^{10} + 16655571555840\,\alpha^{8} - 617440143360\,\alpha^{6}\\
&+ 8911080005376\,\alpha^{4} + 2234549464576\,\alpha^{2} + 518143103744, \qquad 0<\alpha<1,\\
P_2(\alpha,\alpha) =& 256\,\alpha^{8}\left(2023996499\,\alpha^{20} + 8728708846\,\alpha^{18} + 34808906271\,\alpha^{16} - 2411875560\,\alpha^{14}\right.\\
&+ 65060826390\,\alpha^{12} + 24196881828\,\alpha^{10} + 53803914822\,\alpha^{8} + 5283112632\,\alpha^{6}\\
&\left.+ 18638997015\,\alpha^{4} - 5273521250\,\alpha^{2} + 7916046875\right), \qquad 0<\alpha<1,\\
P_2(\frac{1}{2},\beta) =& \frac{1}{2^{20}}\left(307640738240000000\,\beta^{20} - 235440199280000000\,\beta^{18}\right.\\
&+ 624859105476000000\,\beta^{16} + 626505759600000\,\beta^{14}\\
&+ 645190692553890000\,\beta^{12} + 94340347409122500\,\beta^{10}\\
&+ 188574175067299875\,\beta^{8} + 81909753788118300\,\beta^{6}\\
&+ 66695730948144210\,\beta^{4} + 6366244596556360\,\beta^{2}\\
&\left.+ 453375473850131\right), \qquad 0<\beta\leq 1.
\end{aligned}$$

By applying Sturm's Theorem to each above expressions, we see that $P_2(\alpha,1)$, $P_2(\alpha,\alpha)$ and $P_2(\frac{1}{2},\beta)$ are positive on their corresponding intervals. Again, if there exists $\bar{\alpha} \in (0,1)$ such that $P_2(\bar{\alpha},\beta)$ has roots on the interval $(\bar{\alpha},1)$, then the equations

$$P_2(\alpha,\beta)=0, \qquad \frac{\partial P_2}{\partial \beta}(\alpha,\beta)=0,$$

must have a solution in $\Pi\backslash\{\beta=1\}$. Note that

$$Res(P_2,\frac{\partial P_2}{\partial \beta},\alpha) = E\beta^{68}(p_1(\beta))^2(p_2(\beta))^4(p_3(\beta))^6(p_4(\beta))^6,$$

where E is a constant and p_1, p_2, p_3, and p_4 are polynomials of degree 36, 48, 12 and 36, respectively. It follows from Sturm's Theorem that $p_i(\beta) \neq 0$ for all $\beta \in (0,1]$ and $i=1,\cdots,4$. Therefore $P_2(\alpha,\beta) \neq 0$ for $(\alpha,\beta) \in \Pi$. So $A(x;\alpha,\beta) > 0$ for $(\alpha,\beta) \in \Pi$ and $x \in (-\alpha,\alpha)$ and the first part of the assumption (A) is verified. We also have

$$f'(x) = -7\,x^6 + 5\left(\alpha^2+\beta^2+1\right)x^4 - 3\left(\alpha^2\beta^2+\alpha^2+\beta^2\right)x^2 + \alpha^2\beta^2, \tag{3.6}$$

which implies that

$$\begin{aligned}
f'(-1) &= f'(1) = -2(\alpha^2-1)(\beta^2-1) < 0,\\
f'(-\beta) &= f'(\beta) = 2\beta^2(\alpha^2-\beta^2)(\beta^2-1) > 0,\\
f'(-\alpha) &= f'(\alpha) = -2\alpha^2(\alpha^2-\beta^2)(\alpha^2-1) < 0,\\
f'(0) &= \alpha^2\beta^2 > 0.
\end{aligned}$$

Since f' is a polynomial of degree six, $f'(x)$ has exactly a simple positive root $x_1 \in (0,\alpha)$. It follows that f' vanishes at $-x_1 \in (-\alpha,0)$ and $f(x_1) = -f(-x_1) < 0$, because f' is even. Also, it is easy to see that x_1 is a local maximum of f. This yields that $f(x_1)f''(x_1) = f(-x_1)f''(-x_1) < 0$, which implies the second part of the assumption (A). Moreover, $f(\pm\alpha)=0$ and $(3f'(2Ff'-f^2)-2Fff'')(\pm\alpha) = 6(Ff'^2)(\pm\alpha) > 0$. So assumption (C) is verified. □

By putting Lemma 3.1 and Theorems 1.1 and 1.2 together, the following theorem can be concluded.

Theorem 3.2. *The period function associated with the period annulus surrounding only the elementary center p_0 of system (1.5) is convex and monotonically increasing.*

3.2. **Period annulus surrounding only one of the elementary centers $p_{\pm\beta}$.** First, we note that in cases $(2)-(4)$ and $(9)-(11)$ of the phase portraits (shown in Figure 3), there are two period annulus which one of them surrounds the elementary center $p_{-\beta}$ and the other surrounds the elementary center p_{β}. In these cases, we have $0 \le \alpha < \beta < 1$, see Fig 3. By the symmetry property of system (1.5), we only consider the period annulus surrounding the elementary center p_{β}. We bring p_{β} to the origin by the change of variable $x - \beta = t$. Consequently, the potential function $F(x)$ in (1.6) transforms to

$$\tilde{F}(t) = \frac{1}{24} t^2 (2\,\beta + t)^2 (6\,\alpha^2\beta^2 + 8\,\alpha^2 t\beta + 4\,\alpha^2 t^2 - 6\,\beta^4 - 16\,\beta^3 t - 20\,\beta^2 t^2 - 12\,\beta\, t^3 - 3\,t^4 - 6\,\alpha^2 + 6\,\beta^2 + 8\,t\beta + 4\,t^2).$$

Therefore,

$$\tilde{f}(t) = \tilde{F}'(t) = -(t+\beta)((t+\beta)^2 - \alpha^2)((t+\beta)^2 - \beta^2)((t+\beta)^2 - 1), \tag{3.7}$$

and in this case the projection of period annulus on the t-axes is $(\alpha - \beta, 1 - \beta)$. It is easy to check that $\tilde{f}(0) = 0$, $\tilde{f}'(0) > 0$ and $t\tilde{f}(t) > 0$ for $t \in (\alpha - \beta, 1 - \beta)\backslash\{0\}$. Now, we have the following lemma.

Lemma 3.3. *For $f = \tilde{f}$ and $F = \tilde{F}$ assumption (A) holds in the interval $(\alpha - \beta, 1 - \beta)$ and assumption (C) is true for $c_1 = \alpha - \beta$ and $c_2 = 1 - \beta$.*

Proof. We set

$$\begin{aligned} B(t;\alpha,\beta) :=& 5(\tilde{f}''(t))^2 - 3\tilde{f}'(t)\tilde{f}'''(t) \\ =& 4410 t^{10} + 44100\beta t^9 + b_8 t^8 + b_7 t^7 + b_6 t^6 + b_5 t^5 + b_4 t^4 + b_3 t^3 + b_2 t^2 + b_1 t + b_0, \end{aligned}$$

where,

$$\begin{aligned}
b_8 =& 210(926\beta^2 - 19\alpha^2 - 19),\\
b_7 =& 1680(296\beta^3 - 19\alpha^2\beta - 19\beta),\\
b_6 =& (1100\,\alpha^4 - 109016\,\alpha^2\beta^2 + 815480\,\beta^4 + 2704\,\alpha^2 - 109016\,\beta^2 + 1100),\\
b_5 =& 24\beta(275\,\alpha^4 - 8634\,\alpha^2\beta^2 + 37270\,\beta^4 + 676\,\alpha^2 - 8634\,\beta^2 + 275),\\
b_4 =& (15930\,\alpha^4\beta^2 - 239310\,\alpha^2\beta^4 + 663300\,\beta^6 - 570\,\alpha^4 + 39480\,\alpha^2\beta^2\\
&- 239310\,\beta^4 - 570\,\alpha^2 + 15930\,\beta^2),\\
b_3 =& 40\,\beta(493\,\alpha^4\beta^2 - 4291\,\alpha^2\beta^4 + 8194\,\beta^6 - 57\,\alpha^4 + 1244\,\alpha^2\beta^2\\
&- 4291\,\beta^4 - 57\,\alpha^2 + 493\,\beta^2),\\
b_2 =& (13206\,\alpha^4\beta^4 - 74580\,\alpha^2\beta^6 + 103230\,\beta^8 - 3348\,\alpha^4\beta^2 + 34152\,\alpha^2\beta^4\\
&- 74580\,\beta^6 + 126\,\alpha^4 - 3348\,\alpha^2\beta^2 + 13206\,\beta^4),\\
b_1 =& 12\beta(381\,\alpha^4\beta^4 - 1498\,\beta^6\alpha^2 + 1565\,\beta^8 - 178\,\alpha^4\beta^2 + 1004\,\alpha^2\beta^4\\
&- 1498\,\beta^6 + 21\,\alpha^4 - 178\,\alpha^2\beta^2 + 381\,\beta^4),\\
b_0 =& 16\beta^2(41\,\alpha^4\beta^4 - 116\,\alpha^2\beta^6 + 95\,\beta^8 - 30\,\alpha^4\beta^2 + 106\,\alpha^2\beta^4\\
&- 116\,\beta^6 + 9\,\alpha^4 - 30\,\alpha^2\beta^2 + 41\,\beta^4).
\end{aligned}$$

Now, we prove that if $(\alpha,\beta) \in \Omega = \{0 \le \alpha < \beta < 1\}$, then $B(t;\alpha,\beta) > 0$ for $t \in I = (\alpha - \beta, 1 - \beta)$. Let

$$\begin{aligned} \bar{B}(\alpha,\beta) := B(\alpha - \beta;\alpha,\beta) = 16\alpha^2(95\,\alpha^8 - 116(\beta^2+1)\alpha^6 \\ + (41\beta^4 + 106\beta^2 + 41)\alpha^4 - 30\beta^2(\beta^2+1)\alpha^2 + 9\beta^4), \quad (\alpha,\beta) \in \Omega. \end{aligned}$$

To show that $\bar{B}(\alpha,\beta) > 0$ for $(\alpha,\beta) \in \Omega$, note that

$$\bar{B}(0,\beta) = 9\beta^4 > 0, \qquad 0 < \beta < 1,$$
$$\bar{B}(\beta,\beta) = 20\,\beta^4(\beta-1)^2(\beta+1)^2 > 0, \qquad 0 < \beta < 1,$$
$$\bar{B}(\alpha,\frac{1}{2}) = 95\,\alpha^8 - 145\,\alpha^6 + \frac{1121\,\alpha^4}{16} - \frac{75\,\alpha^2}{8} + \frac{9}{16} > 0, \qquad 0 \le \alpha < 1.$$

Thus, if there exists a value $\hat{\beta} \in (0,1)$ such that $\bar{B}(\alpha,\hat{\beta})$ has roots for $\alpha \in (0,\hat{\beta})$, then the number of these roots (counted with multiplicities) is even. By continuously moving β from $\hat{\beta}$ to $\frac{1}{2}$, we can find $\bar{\beta}$ such that $\bar{B}(\alpha,\bar{\beta})$ has a multiple zero on $(0,\bar{\beta})$. Hence the equations

$$\bar{B}(\alpha,\beta) = 0, \qquad \frac{\partial \bar{B}}{\partial \alpha}(\alpha,\beta) = 0, \tag{3.8}$$

have a solution in $\Omega \setminus \{\alpha = 0\}$. By computing the resultant between $\bar{B}(\alpha,\beta)$ and $\frac{\partial \bar{B}}{\partial \alpha}$ with respect to β, we obtain

$$\begin{aligned} Res(\bar{B},\frac{\partial \bar{B}}{\partial \alpha},\beta) =& 120736\alpha^{12}(229805\,\alpha^{12} - 507680\,\alpha^{10} + 590056\,\alpha^8 \\ & - 415728\,\alpha^6 + 185727\,\alpha^4 - 49644\,\alpha^2 + 5904)^2(\alpha^2-1)^4. \end{aligned}$$

Sturm's Theorem implies that the above expression has no zero for $\alpha \in (0,1)$. We set

$$\begin{aligned} \tilde{B}(\alpha,\beta) :=& B(1-\beta;\alpha,\beta) = 144\,\alpha^4\beta^4 - 480\,\alpha^4\beta^2 - 480\,\alpha^2\beta^4 + 656\,\alpha^4 \\ & + 1696\,\alpha^2\beta^2 + 656\beta^4 - 1856\,\alpha^2 - 1856\,\beta^2 + 1520, \quad (\alpha,\beta) \in \Omega. \end{aligned}$$

To prove the inequality $\tilde{B}(\alpha,\beta) > 0$ for $(\alpha,\beta) \in \Omega$, observe that

$$\tilde{B}(0,\beta) = 656\,\beta^4 - 1856\,\beta^2 + 1520 > 0, \qquad 0 < \beta < 1,$$
$$\tilde{B}(\beta,\beta) = 144\,\beta^8 - 960\,\beta^6 + 3008\,\beta^4 - 3712\,\beta^2 + 1520 > 0, \quad 0 < \beta < 1,$$
$$\tilde{B}(\alpha,\frac{1}{2}) = 545\,\alpha^4 - 1462\,\alpha^2 + 1097 > 0, \qquad 0 \le \alpha < 1.$$

Again, if there exists $\hat{\beta} \in (0,1)$ such that $\tilde{B}(\alpha,\hat{\beta})$ has zero for $\alpha \in (0,\hat{\beta})$, then the number of these zeros is even (counted with multiplicities). By continuously moving β from $\hat{\beta}$ to $\frac{1}{2}$, we can find $\bar{\beta}$ such that $\tilde{B}(\alpha,\bar{\beta})$ has a multiple zero on $(0,\bar{\beta})$. Hence, the equations

$$\tilde{B}(\alpha,\beta) = 0, \qquad \frac{\partial \tilde{B}}{\partial \alpha}(\alpha,\beta) = 0, \tag{3.9}$$

have a solution in Ω. Computing the resultant between $\tilde{B}(\alpha,\beta)$ and $\frac{\partial \tilde{B}}{\partial \alpha}(\alpha,\beta)$ with respect to β, we obtain

$$Res(\tilde{B},\frac{\partial \tilde{B}}{\partial \alpha},\beta) = 180(113\,\alpha^4 - 466\,\alpha^2 + 973)(\alpha^2-1)^2, \tag{3.10}$$

which has no solution for $\alpha \in (0,1)$. So we have proved that

$$B(\alpha-\beta;\alpha,\beta) > 0, \quad B(1-\beta;\alpha,\beta) > 0, \quad (\alpha,\beta) \in \Omega. \tag{3.11}$$

Now, by setting $\alpha = \frac{1}{2}$ and $\beta = \frac{3}{4}$ in $B(t;\alpha,\beta)$, we get

$$\begin{aligned} B(t;\frac{1}{2},\frac{3}{4}) =& 4410\,t^{10} + 33075\,t^9 + \frac{417585}{4}\,t^8 + 179865\,t^7 + \frac{5862907}{32}\,t^6 \\ & + \frac{7122663}{64}\,t^5 + \frac{39217755}{1024}\,t^4 + \frac{6543405}{1024}t^3 + \frac{9930663}{32768}\,t^2 \\ & + \frac{553581}{65536}\,t + \frac{684855}{65536} > 0, \qquad t \in (-\frac{1}{4},\frac{1}{4}). \end{aligned} \tag{3.12}$$

By continuity of B, it follows from (3.11) and (3.12) that if there exists $(\alpha_1, \beta_1) \in \Omega$ such that $B(t; \alpha_1, \beta_1)$ has a root in $t \in (\alpha_1 - \beta_1, 1 - \beta_1)$, then changing (α, β) from $(\frac{1}{2}, \frac{3}{4})$ to (α_1, β_1) in Ω continuously, one may find $(\alpha_2, \beta_2) \in \Omega$ in which $B(t; \alpha_2, \beta_2)$ has a multiple root in $(\alpha_2 - \beta_2, 1 - \beta_2)$, i.e. the equations

$$B(t; \alpha, \beta) = 0, \qquad \frac{\partial B}{\partial t}(t; \alpha, \beta) = 0, \tag{3.13}$$

have a solution in $(\alpha_2 - \beta_2, 1 - \beta_2)$. Computing the resultant with respect to t between $B(t; \alpha, \beta)$ and $\frac{\partial B}{\partial t}(t; \alpha, \beta)$, we get

$$Res(B, \frac{\partial B}{\partial t}, t) = C\alpha^2\beta^2(\alpha^2\beta^2 + \alpha^2 + \beta^2)P_1(\alpha, \beta)P_2(\alpha, \beta), \tag{3.14}$$

where the above expression is the same as in (3.5). Hence, $B(t; \alpha, \beta) > 0$ for $(\alpha, \beta) \in \Omega$ and $t \in I$. Accordingly, the first part of the assumption (A) is true. In what follows, we verify the second part of the condition (A). We have

$$\begin{aligned} f'(t) = &-7\,t^6 - 42\,\beta\,t^5 + \left(5\,\alpha^2 - 100\,\beta^2 + 5\right)t^4 + \left(20\,\alpha^2\beta - 120\,\beta^3 + 20\,\beta\right)t^3 \\ &+ \left(27\,\alpha^2\beta^2 - 75\,\beta^4 - 3\,\alpha^2 + 27\,\beta^2\right)t^2 + \left(14\,\alpha^2\beta^3 - 22\,\beta^5 - 6\,\alpha^2\beta + 14\,\beta^3\right)t \\ &+ 2\,\alpha^2\beta^4 - 2\,\beta^6 - 2\,\alpha^2\beta^2 + 2\,\beta^4, \end{aligned}$$

and we get

$$\begin{aligned} f'(-1-\beta) &= -2\left(\beta^2 - 1\right)\left(\alpha^2 - 1\right) < 0, \\ f'(-2\beta) &= 2\beta^2\left(\beta^2 - 1\right)\left(\alpha^2 - \beta^2\right) > 0, \\ f'(-\alpha-\beta) &= -2\alpha^2\left(\beta^2 - 1\right)\left(\alpha^2 - \beta^2\right) < 0, \\ f'(-\beta) &= \alpha^2\beta^2 > 0, \\ f'(\alpha-\beta) &= -2\alpha^2\left(\beta^2 - 1\right)\left(\alpha^2 - \beta^2\right) < 0, \\ f'(0) &= 2\beta^2\left(\beta^2 - 1\right)\left(\alpha^2 - \beta^2\right) > 0, \\ f'(1-\beta) &= -2\left(\beta^2 - 1\right)\left(\alpha^2 - 1\right) < 0, \end{aligned}$$

Since $f'(t)$ is a polynomial of degree six, it has exactly one simple root $t_1 \in (\alpha - \beta, 0)$ and one simple root $t_2 \in (0, 1 - \beta)$. It is clear that $f(t_1) < 0$ and $f(t_2) > 0$. Also, one can check that t_1 is a local minimum, and t_2 is a local maximum. So we have $f''(t_1) > 0$ and $f''(t_2) < 0$. This implies the second part of the condition (A). Moreover, since $f(\alpha - \beta) = f(1 - \beta) = 0$, the second part of assumption (C) holds. This completes the proof. □

Finally, considering Lemma 3.3 and Theorems 1.2, 1.3 and symmetry property of system (1.5), we conclude the following theorem.

Theorem 3.4. *The period functions associated with those period annulus of system* (1.5) *that surround only one of the elementary centers $p_{\pm\beta}$ are monotonically increasing and convex.*

4. Period annulus surrounding five singularities

This section treats the period functions associated with those period annulus of the system (1.5) which surrounding five singularities counted with multiplicities. As shown in Figure 3, this happens in cases (1), (2), (8) and (9), which correspond to region $S = S_1 \cup \gamma_1 \cup \gamma_2 \cup \{(0, 0)\}$ in the (α, β)-plane, where

$$\gamma_1 = \{(\alpha, \beta) : 0 < \alpha = \beta < 1\}, \quad \gamma_2 = \{\alpha = 0, 0 < \beta < \frac{\sqrt{2}}{2}\},$$

and S_1 is the open curved triangle region bounded by line segments γ_1, γ_2 and curve $\gamma = \{(\alpha, \beta) | 0 \leq \alpha \leq \beta \leq 1, \beta = \sqrt{\frac{\alpha^2+1}{2}}\}$, see Figure 4. Thus, if Γ_h is a closed orbit surrounding five singularities counted with multiplicities, then we have the following four cases.

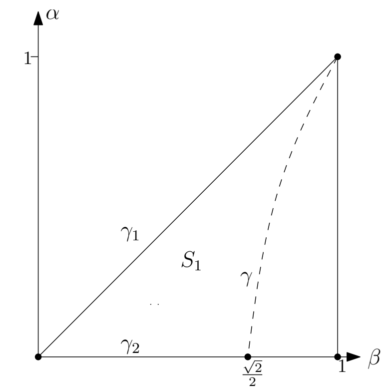

FIGURE 4. The region S which contains an annulus surrounding five singularities.

(1) In case (9), we have $(\alpha, \beta) \in S_1$ and system (1.5) has three elementary centers at p_0 and $p_{\pm\beta}$, and four hyperbolic saddles at $p_{\pm\alpha}$ and $p_{\pm 1}$. There is a double eight-figure loop through the saddles $p_{\pm\alpha}$ that surrounds all three elementary centers. In this case, Γ_h surrounds the double eight-figure loop (see Figure 3(9)). As shown in Figure 4, the upper half of Γ_h has three local maximum at $(\pm\beta, a)$ and $(0, b)$. and two local minimum at $(\pm\alpha, d)$, where $a, b, d > 0$.
(2) In case (8), we have $(\alpha, \beta) \in \gamma_1$ and in this case, centers $p_{\pm\beta}$ and saddles $p_{\pm\alpha}$ come together and make an eye-figure loop surrounding the origin as an elementary center (see Figure 3(8)). therefore, Γ_h surrounds this eye-figure loop and it's upper half has one maximum point at $(0, b)$ and two inflection points. (In fact, one of these inflection points is produced by merging maximum point (β, a) with the minimum point (α, d) and the other is produced by merging maximum point $(-\beta, a)$ with the minimum point $(-\alpha, d)$).
(3) In case (2), we have $(\alpha, \beta) \in \gamma_2$ and system (1.5) has two saddles at $p_{\pm 1}$, two centers at $p_{\pm\beta}$ and a nilpotent saddle at the origin. Note that this nilpotent saddle is produced by merging $p_{\pm\alpha}$ and $(0, 0)$. Thus, there is an eight-figure loop connecting to the origin and surrounding centers $p_{\pm\beta}$. In this case, the orbit Γ_h surrounds this eight-figure loop (See Figure 3 (2)). The upper half of Γ_h has two maximum point at $(\pm\beta, a)$ and one minimum at $(0, b)$.
(4) In case (1), we have $(\alpha, \beta) = (0, 0)$ and all five singularities p_0, $p_{\pm\alpha}$ and $p_{\pm\beta}$ of system (1.5) come together to make a degenerate center at p_0. So, Γ_h surrounds this degenerate center and has a unique maximum at $(0, b)$ (see Figure 3 (1)).

In all the above cases, along the orbit Γ_h, we have

$$H(x, y) = \frac{y^2}{2} + F(x) = h = \frac{a^2}{2} + F(\pm\beta) = \frac{d^2}{2} + F(\pm\alpha) = \frac{b^2}{2}. \tag{4.1}$$

Since system (1.5) is symmetric with respect to both x-axes and y-axes, we can write $\frac{T(h)}{4} = T_1(h) + T_2(h)$, where

$$T_1(h) = \int_{s_h}^{-\beta} \frac{dx}{y}, \qquad T_2(h) = \int_{-\beta}^{0} \frac{dx}{y},$$

and s_h is $x-$coordinate of the intersection point of Γ_h and x-axes in negative section, see Fig 5. From equation $\frac{y^2}{2} + F(x) = h$ in (4.1), we treat y as a function of x and h i.e. $y = y(x, h)$ for $x \in (s_h, 0)$. First, we investigate the behavior of $T_2(h)$. If $(\alpha, \beta) = (0, 0)$, then $T_2(h) \equiv 0$. Otherwise, since $y = y(x, h) > 0$ for $x \in [-\beta, 0]$ and $\frac{\partial y}{\partial h} = \frac{1}{y}$ along the orbit Γ_h, we get

$$T_2'(h) = -\int_{-\beta}^{0} \frac{dx}{y^3} < 0, \qquad T_2''(h) = \int_{-\beta}^{0} \frac{dx}{y^5} > 0. \tag{4.2}$$

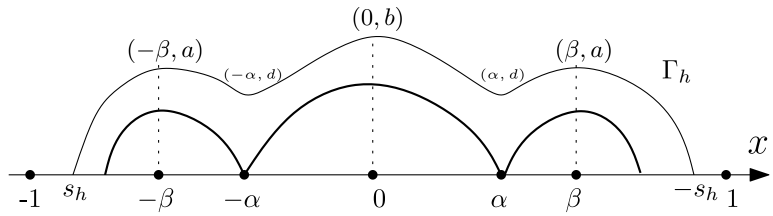

FIGURE 5. Half of the period annulus surrounding 5 singularities.

Now, we need to study the behavior of $T_1(h)$. By setting $t = x + \beta$ in $f(x)$, we bring equilibrium point $p_{-\beta}$ of system (1.5) to the origin. Therefore, function $f(x)$ and its potential function transform to

$$f_1(t) = -(t-\beta)\left(-\alpha^2 + (t-\beta)^2\right)\left(-\beta^2 + (t-\beta)^2\right)\left((t-\beta)^2 - 1\right), \tag{4.3}$$

$$\begin{aligned} F_1(t) =& \frac{1}{24} t^2 (2\beta - t)^2 (6\alpha^2\beta^2 - 8\alpha^2\beta t + 4\alpha^2 t^2 - 6\beta^4 + 16\beta^3 t \\ &- 20\beta^2 t^2 + 12\beta t^3 - 3t^4 - 6\alpha^2 + 6\beta^2 - 8t\beta + 4t^2). \end{aligned} \tag{4.4}$$

Inspired by the idea of the proof of Theorem 2.1 in [5], we use the change of variables

$$(sign\ f_1(x))\sqrt{F_1(x)} = r\cos\theta, \qquad y = \sqrt{2}r\sin\theta, \tag{4.5}$$

that map the level curve $\frac{y^2}{2} + F_1(x) = \frac{a^2}{2}$ to the circle $r = \frac{a}{\sqrt{2}}$. Using (4.5), we get $F_1(x) = r^2\cos^2\theta$ and

$$f_1(x)dx = -2r^2\cos\theta\sin\theta d\theta = -ay\cos\theta d\theta, \tag{4.6}$$

$$f_1(x)\frac{\partial x}{\partial r} = 2r\cos^2\theta = \frac{2F_1(x)}{r}. \tag{4.7}$$

On account of $x = x(\theta, a)$ and using (4.7), we get

$$\frac{\partial x}{\partial a} = \frac{\partial x}{\partial r}\frac{\partial r}{\partial a} = \frac{2F_1(x)}{af_1(x)}. \tag{4.8}$$

Now we substitute (4.6) in $T_1(h)$ to get

$$T_1(h) = a\int_{\frac{\pi}{2}}^{\pi} \frac{\cos\theta}{f_1(x)} d\theta, \tag{4.9}$$

where a depends on h as stated in (4.1) and $\frac{da}{dh} = \frac{1}{a}$. Taking (4.8) into account, we get

$$\begin{aligned} T_1'(h) &= \frac{\partial T_1}{\partial a}\frac{da}{dh} = \frac{1}{a}\int_{\frac{\pi}{2}}^{\pi} \phi(x)\cos\theta d\theta, \\ T_1''(h) &= \frac{1}{a^3}\int_{\frac{\pi}{2}}^{\pi} \psi(x)\cos\theta d\theta, \end{aligned} \tag{4.10}$$

where

$$\phi(x) = \frac{f_1^2 - 2F_1 f_1'}{f_1^3}(x), \quad \psi(x) = \frac{2F_1\phi' - f_1\phi}{f_1}(x). \tag{4.11}$$

The following lemma shows that the period function $T(h)$ has at least one critical point in the interval $(F(-\alpha), F(-1))$ for every $(\alpha, \beta) \in S$.

Lemma 4.1. *For $(\alpha, \beta) \in S$, we have*

$$\lim_{h \searrow F(-\alpha)} T(h) = +\infty, \qquad \lim_{h \nearrow F(-1)} T(h) = +\infty. \tag{4.12}$$

Proof. From [4, pp. 56-58,] we recall that if Γ_h tends to a saddle connection, then $T(h)$ tends in a monotone way to $+\infty$. Taking this into account and noting that in all cases, if $h \nearrow F(-1)$ then Γ_h tends to a saddle connection, which implies that $\lim_{h\nearrow F(-1)} T(h) = +\infty$. Moreover, if $(\alpha, \beta) \in S_1$ or $(\alpha, \beta) \in \gamma_2$ and $h \searrow F(-\alpha)$ then Γ_h tends to a saddle connection and $T(h)$ tends to $+\infty$. We treat other cases separately as follows. If $(\alpha, \beta) \in \gamma_1$ and $h \searrow F(-\alpha)$ then Γ_h tends to the eye-figure loop. Hence, according to (4.9) and (4.5), for $h \approx F(-\alpha) = F(-\beta)$, we have

$$\begin{aligned} T_1(h) = a\int_{\frac{\pi}{2}}^{\pi} \frac{\cos\theta}{f_1(x)} d\theta &= \int_{\frac{\pi}{2}}^{\pi} \frac{(sign\ f_1(x))\sqrt{2F_1(x)}}{f_1(x)} d\theta \\ &\approx \int_{\frac{\pi}{2}}^{\pi} \frac{d\theta}{\sqrt{6\alpha^3(1-\alpha^2)}|x|^{\frac{1}{2}}}, \end{aligned}$$

which goes to infinity when $h \searrow F(-\alpha)$, because $x \to 0$. If $(\alpha, \beta) = (0, 0)$ and $h \searrow F(-\alpha)$ then Γ_h tends to nilpotent center $p_0 = (0, 0)$. Similarly, for $h \approx 0$, we have

$$T_1(h) = a\int_{\frac{\pi}{2}}^{\pi} \frac{\cos\theta}{f_1(x)} d\theta \approx \int_{\frac{\pi}{2}}^{\pi} \frac{d\theta}{\sqrt{6}x^2},$$

which goes to infinity when $h \searrow F(-\alpha) = 0$, because $x \to 0$. □

Taking Lemma 4.1 into account, the next lemma proves that the critical point of $T(h)$ is unique.

Lemma 4.2. *If $h \in (F(-\alpha), F(-1))$ and $(\alpha, \beta) \in S$, then $T''(h) > 0$.*

Proof. Since, $\theta \in (\frac{\pi}{2}, \pi)$, then $\cos\theta < 0$. If we show that $\psi(x) < 0$ for $\beta - 1 < x < 0$, then from (4.10), we get $T_1''(h) > 0$. Hence, on account of (4.2), we get $T''(h) > 0$. Therefore, in what follow we show that $\psi(x) < 0$ for $\beta - 1 < x < 0$. By substituting F_1 and f_1 from (4.3) and (4.4) in (4.11), we get

$$\psi(x) = \frac{M(x;\alpha,\beta)}{144\Big((x-\beta)(\alpha^2-(x-\beta)^2)((\beta-x)^2-1)\Big)^5}, \tag{4.13}$$

where $M(x;\alpha,\beta) = \sum_{n=0}^{18} m_n(\alpha,\beta)x^n$ and coefficients $m_n(\alpha,\beta)$ are as follows:

$$\begin{aligned} m_{18} =& 297, \\ m_{17} =& -5346\,\beta, \\ m_{16} =& -1236\,\alpha^2 + 45690\,\beta^2 - 1236, \\ m_{15} =& 19776\,\alpha^2\beta - 246336\,\beta^3 + 19776\,\beta, \\ m_{14} =& 1815\,\alpha^4 - 149196\,\alpha^2\beta^2 + 938790\,\beta^4 + 5304\,\alpha^2 - 149196\,\beta^2 + 1815, \\ m_{13} =& -25410\,\alpha^4\beta + 704424\,\alpha^2\beta^3 - 2685396\,\beta^5 - 74256\,\alpha^2\beta + 704424\,\beta^3 - 25410\,\beta, \\ m_{12} =& -1120\,\alpha^6 + 166320\,\alpha^4\beta^2 - 2329176\,\alpha^2\beta^4 + 5974688\,\beta^6 - 7920\,\alpha^4 \\ &+ 484896\,\alpha^2\beta^2 - 2329176\,\beta^4 - 7920\,\alpha^2 + 166320\,\beta^2 - 1120, \end{aligned}$$

$$
\begin{aligned}
m_{11} =&13440\,\alpha^6\beta - 674520\,\alpha^4\beta^3 + 5716992\,\alpha^2\beta^5 - 10569840\,\beta^7 + 95040\,\alpha^4\beta \\
&- 1957440\,\alpha^2\beta^3 + 5716992\,\beta^5 + 95040\,\alpha^2\beta - 674520\,\beta^3 + 13440\,\beta, \\
m_{10} =&256\,\alpha^8 - 74620\,\alpha^6\beta^2 + 1892487\,\alpha^4\beta^4 - 10770128\,\alpha^2\beta^6 + 15067541\,\beta^8 \\
&+ 4900\,\alpha^6 - 524496\,\alpha^4\beta^2 + 5455332\,\alpha^2\beta^4 - 10770128\,\beta^6 + 11817\,\alpha^4 \\
&- 524496\,\alpha^2\beta^2 + 1892487\,\beta^4 + 4900\,\alpha^2 - 74620\,\beta^2 + 256, \\
m_9 =&- 2560\,\alpha^8\beta + 253400\,\alpha^6\beta^3 - 3882150\,\alpha^4\beta^5 + 15873632\,\alpha^2\beta^7 \\
&- 17425810\,\beta^9 - 49000\,\alpha^6\beta + 1760160\,\alpha^4\beta^3 - 11096808\,\alpha^2\beta^5 \\
&+ 15873632\,\beta^7 - 118170\,\alpha^4\beta + 1760160\,\alpha^2\beta^3 - 3882150\,\beta^5 \\
&- 49000\,\alpha^2\beta + 253400\,\beta^3 - 2560\,\beta, \\
m_8 =&11680\,\alpha^8\beta^2 - 585352\,\alpha^6\beta^4 + 5996262\,\alpha^4\beta^6 - 18477476\,\alpha^2\beta^8 \\
&+ 16371206\,\beta^{10} - 1120\,\alpha^8 + 221080\,\alpha^6\beta^2 - 3997500\,\alpha^4\beta^4 \\
&+ 16972808\,\alpha^2\beta^6 - 18477476\,\beta^8 - 7152\,\alpha^6 + 530982\,\alpha^4\beta^2 \\
&- 3997500\,\alpha^2\beta^4 + 5996262\,\beta^6 - 7152\,\alpha^4 + 221080\,\alpha^2\beta^2 \\
&- 585352\,\beta^4 - 1120\,\alpha^2 + 11680\,\beta^2, \\
m_7 =&- 32000\,\alpha^8\beta^3 + 966656\,\alpha^6\beta^5 - 7070496\,\alpha^4\beta^7 + 17014816\,\alpha^2\beta^9 \\
&- 12443008\,\beta^{11} + 8960\,\alpha^8\beta - 592640\,\alpha^6\beta^3 + 6463200\,\alpha^4\beta^5 \\
&- 19799680\,\alpha^2\beta^7 + 17014816\,\beta^9 + 57216\,\alpha^6\beta - 1411776\,\alpha^4\beta^3 \\
&+ 6463200\,\alpha^2\beta^5 - 7070496\,\beta^7 + 57216\,\alpha^4\beta - 592640\,\alpha^2\beta^3 \\
&+ 966656\,\beta^5 + 8960\,\alpha^2\beta - 32000\,\beta^3, \\
m_6 =&58076\,\alpha^8\beta^4 - 1166116\,\alpha^6\beta^6 + 6377844\,\alpha^4\beta^8 - 12321196\,\alpha^2\beta^{10} \\
&+ 7574000\,\beta^{12} - 31400\,\alpha^8\beta^2 + 1043156\,\alpha^6\beta^4 - 7610124\,\alpha^4\beta^6 \\
&+ 17659468\,\alpha^2\beta^8 - 12321196\,\beta^{10} + 1596\,\alpha^8 - 198636\,\alpha^6\beta^2 \\
&+ 2456040\,\alpha^4\beta^4 - 7610124\,\alpha^2\beta^6 + 6377844\,\beta^8 + 4092\,\alpha^6 \\
&- 198636\,\alpha^4\beta^2 + 1043156\,\alpha^2\beta^4 - 1166116\,\beta^6 + 1596\,\alpha^4 \\
&- 31400\,\alpha^2\beta^2 + 58076\,\beta^4, \\
m_5 =&- 72488\,\alpha^8\beta^5 + 1030232\,\alpha^6\beta^7 - 4361304\,\alpha^4\beta^9 + 6914632\,\alpha^2\beta^{11} \\
&- 3626944\,\beta^{13} + 62960\,\alpha^8\beta^3 - 1254776\,\alpha^6\beta^5 + 6550536\,\alpha^4\beta^7 \\
&- 11938120\,\alpha^2\beta^9 + 6914632\,\beta^{11} - 9576\,\alpha^8\beta + 390792\,\alpha^6\beta^3 \\
&- 2912400\,\alpha^4\beta^5 + 6550536\,\alpha^2\beta^7 - 4361304\,\beta^9 - 24552\,\alpha^6\beta \\
&+ 390792\,\alpha^4\beta^3 - 1254776\,\alpha^2\beta^5 + 1030232\,\beta^7 - 9576\,\alpha^4\beta + 62960\,\alpha^2\beta^3 - 72488\,\beta^5,
\end{aligned}
$$

$$
\begin{aligned}
m_4 =& 62416\,\alpha^8\beta^6 - 656656\,\alpha^6\beta^8 + 2211024\,\alpha^4\beta^{10} - 2929136\,\alpha^2\beta^{12} \\
&+ 1327520\,\beta^{14} - 78464\,\alpha^8\beta^4 + 1037680\,\alpha^6\beta^6 - 4066320\,\alpha^4\beta^8 \\
&+ 5984272\,\alpha^2\beta^{10} - 2929136\,\beta^{12} + 23376\,\alpha^8\beta^2 - 474576\,\alpha^6\beta^4 \\
&+ 2371392\,\alpha^4\beta^6 - 4066320\,\alpha^2\beta^8 + 2211024\,\beta^{10} - 864\,\alpha^8 \\
&+ 59856\,\alpha^6\beta^2 - 474576\,\alpha^4\beta^4 + 1037680\,\alpha^2\beta^6 - 656656\,\beta^8 \\
&- 864\,\alpha^6 + 23376\,\alpha^4\beta^2 - 78464\,\alpha^2\beta^4 + 62416\,\beta^6, \\
m_3 =& - 36064\,\alpha^8\beta^7 + 290976\,\alpha^6\beta^9 - 797088\,\alpha^4\beta^{11} + 895712\,\alpha^2\beta^{13} \\
&- 354432\,\beta^{15} + 61376\,\alpha^8\beta^5 - 576160\,\alpha^6\beta^7 + 1757280\,\alpha^4\beta^9 \\
&- 2134624\,\alpha^2\beta^{11} + 895712\,\beta^{13} - 29664\,\alpha^8\beta^3 + 361056\,\alpha^6\beta^5 \\
&- 1296960\,\alpha^4\beta^7 + 1757280\,\alpha^2\beta^9 - 797088\,\beta^{11} + 3456\,\alpha^8\beta \\
&- 75744\,\alpha^6\beta^3 + 361056\,\alpha^4\beta^5 - 576160\,\alpha^2\beta^7 + 290976\,\beta^9 \\
&+ 3456\,\alpha^6\beta - 29664\,\alpha^4\beta^3 + 61376\,\alpha^2\beta^5 - 36064\,\beta^7, \\
m_2 =& 13036\,\alpha^8\beta^8 - 83272\,\alpha^6\beta^{10} + 189360\,\alpha^4\beta^{12} - 182840\,\alpha^2\beta^{14} \\
&+ 63716\,\beta^{16} - 28696\,\alpha^8\beta^6 + 201472\,\alpha^6\beta^8 - 492528\,\alpha^4\beta^{10} \\
&+ 502592\,\alpha^2\beta^{12} - 182840\,\beta^{14} + 20256\,\alpha^8\beta^4 - 165072\,\alpha^6\beta^6 \\
&+ 447984\,\alpha^4\beta^8 - 492528\,\alpha^2\beta^{10} + 189360\,\beta^{12} - 4776\,\alpha^8\beta^2 \\
&+ 51648\,\alpha^6\beta^4 - 165072\,\alpha^4\beta^6 + 201472\,\alpha^2\beta^8 - 83272\,\beta^{10} \\
&+ 180\,\alpha^8 - 4776\,\alpha^6\beta^2 + 20256\,\alpha^4\beta^4 - 28696\,\alpha^2\beta^6 + 13036\,\beta^8, \\
m_1 =& - 2520\,\alpha^8\beta^9 + 13200\,\alpha^6\beta^{11} - 25440\,\alpha^4\beta^{13} + 21360\,\alpha^2\beta^{15} \\
&- 6600\,\beta^{17} + 6960\,\alpha^8\beta^7 - 38400\,\alpha^6\beta^9 + 77280\,\alpha^4\beta^{11} \\
&- 67200\,\alpha^2\beta^{13} + 21360\,\beta^{15} - 6720\,\alpha^8\beta^5 + 39840\,\alpha^6\beta^7 \\
&- 84960\,\alpha^4\beta^9 + 77280\,\alpha^2\beta^{11} - 25440\,\beta^{13} + 2640\,\alpha^8\beta^3 \\
&- 17280\,\alpha^6\beta^5 + 39840\,\alpha^4\beta^7 - 38400\,\alpha^2\beta^9 + 13200\,\beta^{11} \\
&- 360\,\alpha^8\beta + 2640\,\alpha^6\beta^3 - 6720\,\alpha^4\beta^5 + 6960\,\alpha^2\beta^7 - 2520\,\beta^9, \\
m_0 =& 168\,\alpha^8\beta^{10} - 768\,\alpha^6\beta^{12} + 1296\,\alpha^4\beta^{14} - 960\,\alpha^2\beta^{16} + 264\,\beta^{18} \\
&- 576\,\alpha^8\beta^8 + 2688\,\alpha^6\beta^{10} - 4608\,\alpha^4\beta^{12} + 3456\,\alpha^2\beta^{14} - 960\,\beta^{16} \\
&+ 720\,\alpha^8\beta^6 - 3456\,\alpha^6\beta^8 + 6048\,\alpha^4\beta^{10} - 4608\,\alpha^2\beta^{12} + 1296\,\beta^{14} \\
&- 384\,\alpha^8\beta^4 + 1920\,\alpha^6\beta^6 - 3456\,\alpha^4\beta^8 + 2688\,\alpha^2\beta^{10} - 768\,\beta^{12} \\
&+ 72\,\alpha^8\beta^2 - 384\,\alpha^6\beta^4 + 720\,\alpha^4\beta^6 - 576\,\alpha^2\beta^8 + 168\,\beta^{10}.
\end{aligned}
$$

Since the denominator of equation (4.13) is negative for $x \in (\beta-1, 0)$, we have to show that $M(x;\alpha,\beta)$ is positive for $x \in (\beta-1, 0)$ and $(\alpha,\beta) \in S$.

First, we prove $M(x;\alpha,\beta)$ is positive on the boundary of S for $x \in (\beta-1,0)$.

(i) Let $(\alpha,\beta) = (0,0)$. Therefore, for $x \in (-1,0)$ we have

$$M(x;0,0) = x^{10}\left(297\,x^8 - 1236\,x^6 + 1815\,x^4 - 1120\,x^2 + 256\right) > 0.$$

(ii) Let $(\alpha,\beta) \in \gamma_1$. Since on γ_1 we have $\alpha = \beta$, then

$$M(x;\beta,\beta) = -x^3(2\beta - x)^3 N(x,\beta),$$

where $N(x,\beta)$ is a polynomial of degree 12 with respect to x. We assert that $N(x,\beta)>0$ for $x\in(\beta-1,0)$. In fact

$$\begin{aligned}
N(\beta-1,\beta) =&12\,\left(\beta^2-1\right)^4>0,\\
N(0,\beta) =&112\,\beta^4\left(\beta^2-1\right)^4>0,\\
N(x,\frac{1}{2}) =&297\,x^{12}-1782\,x^{11}-\frac{3735}{2}x^9+\frac{7281}{2}x^{10}+\frac{3447}{16}\,x^6+\frac{14427}{4}\,x^7-\frac{48015}{16}\,x^8+\frac{3645}{16}\,x^3+\frac{3465}{16}\,x^4\\
&-\frac{24183}{16}\,x^5-\frac{999}{32}\,x^2-\frac{351}{32}\,x+\frac{567}{256}>0,\quad x\in(-\frac{1}{2},0).
\end{aligned}$$

Now, if there exists some $\hat{\beta}\in(0,1)$ such that $N(x,\hat{\beta})$ has root in $(\hat{\beta}-1,0)$, then by continuity of N, if we change β from $\frac{1}{2}$ to $\hat{\beta}$, we should find $\bar{\beta}\in(0,1)$ in which $N(x,\bar{\beta})$ has a multiple root in $(\bar{\beta}-1,0)$. By computing the resultant with respect to x between $N(x,\beta)$ and $\frac{\partial N}{\partial x}(x,\beta)$ we have

$$\begin{aligned}
Res(N,\frac{\partial N}{\partial x},x) =&C_1\,\beta^{12}(\beta^2-4)^{10}(\beta^2-1)^{24}(19836494433\,\beta^{20}-218103939324\,\beta^{18}+992591597436\,\beta^{16}\\
&-2585249712004\,\beta^{14}+4864706813326\,\beta^{12}-7190845624644\,\beta^{10}+8243627754476\,\beta^8\\
&-7804991817404\,\beta^6+5806396549161\,\beta^4-2703302266624\,\beta^2+844228231168)^2,
\end{aligned}$$

where, $C_1=48870184990145263121\,2032$. Applying Sturm's Theorem, one concludes that the above expression doesn't have any solution for $\beta\in(0,1)$. Therefore, $M(x;\beta,\beta)$ is positive for $x\in(\beta-1,0)$.

(iii) Let $(\alpha,\beta)\in\gamma_2$. We get

$$M(x;0,\beta)=(\beta-x)^4N_0(x,\beta),\quad 0<\beta<\frac{\sqrt{2}}{2},$$

where

$$\begin{aligned}
&N_0(\beta-1,\beta)=-12\,\left(\beta^2-1\right)^3\left(\beta^2+1\right)^2>0,\\
&N_0(0,\beta)=24\,\beta^6(11\beta^2-7)\left(\beta^2-1\right)^3>0,
\end{aligned}$$

$$\begin{aligned}
N_0(x,\frac{1}{2}) =&297\,x^{14}-2079\,x^{13}+5583\,x^{12}-6471\,x^{11}+\frac{13053}{16}\,x^{10}+\frac{91023}{16}\,x^9-\frac{166057}{32}\,x^8+\frac{769}{2}\,x^7\\
&+\frac{115339}{64}\,x^6-\frac{55477}{64}\,x^5-\frac{667}{8}\,x^4+\frac{8747}{64}\,x^3-\frac{8499}{1024}\,x^2-\frac{10557}{1024}\,x+\frac{1377}{2048}>0,\ \text{for } x\in(-\frac{1}{2},0).
\end{aligned}$$

Similarly, computing the resultant with respect to x between N_0 and $\frac{\partial N_0}{\partial x}$, yields

$$\begin{aligned}
Res(N_0,\frac{\partial N_0}{\partial x},x) =&C_2\beta^{30}\left(11\,\beta^2-7\right)^2\left(\beta^2-1\right)^{12}\left(\beta^2-2\right)^{14}\left(\beta^2+1\right)^{14}\\
&\times(142733221319586425\,\beta^{28}-1169294930918600096\,\beta^{26}\\
&+3372641185026329411\,\beta^{24}-2687577605763201772\,\beta^{22}\\
&-5626044685429051562\,\beta^{20}+12296167417091577440\,\beta^{18}\\
&-2575830200312501898\,\beta^{16}-12624745972881421272\,\beta^{14}\\
&+10360822225718623269\,\beta^{12}+3304488880785423584\,\beta^{10}\\
&-7028558807284193705\,\beta^8+1147021568627052836\,\beta^6\\
&+1522286540556519164\,\beta^4-260343816785020928\,\beta^2-166968082647482368)^2,
\end{aligned}$$

where $C_2=-136836517972406736739\,36896$. Again, Sturm's Theorem shows that the above expression doesn't have any solution for $\beta\in(0,\frac{\sqrt{2}}{2})$. Thus, $M(x;\beta,\beta)>0$ for $x\in(\beta-1,0)$.

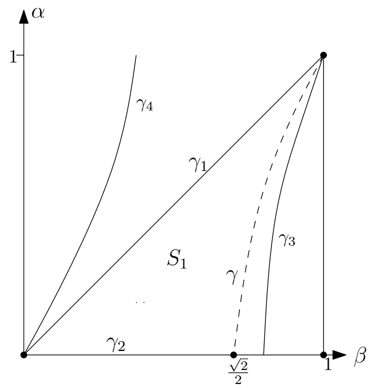

FIGURE 6. Relative position of curves γ_3 and γ_4 with respect to region S.

(iv) Let $(\alpha,\beta) \in \gamma = \{\alpha(\beta) = \sqrt{2\beta^2-1}, \frac{\sqrt{2}}{2} \le \beta \le 1\}$. Then, for $x \in (-\frac{1}{4}, 0)$, we get

$$M(\beta-1;\alpha(\beta),\beta) = -432(\beta^2-1)^7 > 0,$$
$$M(0;\alpha(\beta),\beta) = 72\beta^2(\beta^2-1)^8 > 0,$$

$$\begin{aligned} M(x;\alpha(\tfrac{3}{4}),\tfrac{3}{4}) =& 297\,x^{18} - \frac{8019}{2}\,x^{17} + \frac{194481}{8}\,x^{16} - 87237\,x^{15} + \frac{26256933}{128}\,x^{14} - \frac{84287385}{256}\,x^{13} + \frac{92455587}{256}\,x^{12} \\ & - \frac{262923327}{1024}\,x^{11} + \frac{6123011985}{65536}\,x^{10} + \frac{1844710497}{131072}\,x^{9} - \frac{18004476285}{524288}x^{8} + \frac{1007392707}{65536}\,x^{7} \\ & - \frac{2780130591}{2097152}\,x^{6} - \frac{5175000495}{4194304}\,x^{5} + \frac{864857007}{2097152}\,x^{4} - \frac{355316787}{16777216}\,x^{3} \\ & - \frac{5229615699}{1073741824}\,x^{2} - \frac{778248135}{2147483648}\,x + \frac{466948881}{8589934592} > 0. \end{aligned}$$

As before, by computing the resultant of $M(x;\alpha(\beta),\beta)$ and $\frac{\partial M}{\partial x}(x;\alpha(\beta),\beta)$ with respect to x, we obtain a polynomial on β, which has no solution for $\beta \in (\frac{\sqrt{2}}{2}, 1)$. Thus, $M(x;\alpha(\beta),\beta)$ is positive for $x \in (\beta-1, 0)$.

In (i)-(iv), we proved that $M(x;\alpha,\beta) > 0$ on ∂S. In what follows, we prove that $M(x;\alpha,\beta) > 0$ for $(\alpha,\beta) \in S_1$.

For $(\alpha,\beta) \in S_1$, we have

$$\begin{aligned} M(\beta-1;\alpha,\beta) =& -12(\beta^2-1)^3(\alpha^2-1)^2(2\alpha^2-\beta^2-1)^2 > 0, \\ M(0;\alpha,\beta) =& 24\beta^2(\beta^2-1)^3(\alpha^2-\beta^2)^3(7\,\alpha^2\beta^2 - 11\,\beta^4 - 3\,\alpha^2 + 7\,\beta^2). \end{aligned}$$

As illustrated in Figure 6, the branches of graph $7\,\alpha^2\beta^2 - 11\,\beta^4 - 3\,\alpha^2 + 7\,\beta^2$ which is denoted by γ_3 and γ_4 are located outside of S_1. Therefore, $M(0;\alpha,\beta) > 0$ for $(\alpha,\beta) \in S_1$. Also, note that

$$\begin{aligned} M(x;\tfrac{1}{4},\tfrac{1}{2}) =& 297\,x^{18} - 2673\,x^{17} + \frac{40437}{4}\,x^{16} - 20286\,x^{15} + \frac{5426631}{256}\,x^{14} \\ & - \frac{1391985}{256}\,x^{13} - \frac{1746513}{128}\,x^{12} + \frac{4399497}{256}\,x^{11} - \frac{6807969}{1024}\,x^{10} \\ & - \frac{2550159}{1024}\,x^{9} + \frac{29110095}{8192}\,x^{8} - \frac{2456307}{2048}\,x^{7} - \frac{27220293}{262144}\,x^{6} \\ & + \frac{46331919}{262144}\,x^{5} - \frac{8494551}{262144}\,x^{4} - \frac{1985067}{262144}\,x^{3} + \frac{15783579}{4194304}\,x^{2} \\ & - \frac{2066715}{4194304}\,x + \frac{137781}{8388608} > 0, \qquad \text{for } x \in (-\frac{1}{2}, 0). \end{aligned}$$

Hence, if there exist a point $(\hat{\alpha}, \hat{\beta}) \in S_1$ and $\hat{x} \in (\hat{\beta} - 1, 0)$ such that $M(\hat{x}; \hat{\alpha}, \hat{\beta}) \leq 0$, then moving (α, β) continuously from $(\hat{\alpha}, \hat{\beta})$ to $(\frac{1}{4}, \frac{1}{2})$ in S_1, we can find $(\tilde{\alpha}, \tilde{\beta}) \in S_1$ and $\tilde{x} \in (\tilde{\beta} - 1, 0)$ such that $(\tilde{x}; \tilde{\alpha}, \tilde{\beta})$ is a solution of system

$$M(x; \alpha, \beta) = 0, \qquad \frac{\partial M}{\partial x}(x; \alpha, \beta) = 0. \tag{4.14}$$

Now, we show that system (4.14) doesn't have any solution for $(\alpha, \beta) \in S_1$ and $x \in (\beta - 1, 0)$. Indeed,

$$\begin{aligned} Res(M, \frac{\partial M}{\partial x}, \alpha) =& C_3 x^{14} (\beta - x)^{32} (2\beta - x)^{14} ((\beta - x)^2 - 1)^{18} \\ & \times (p(x, \beta))^{14} (q(x, \beta))^2, \end{aligned}$$

where $C_3 = 2282521714753536$, $p(x, \beta) = 2\,\beta^2 - 2\,\beta\, x + x^2 - 2$ and $q(x, \beta)$ is a polynomial in x of degree 46. It is easy to check that $p(x, \beta) < 0$ for $x \in (\beta - 1, 0)$. Also, we have

$$\begin{aligned} q(\beta - 1, \beta) =& - 2457600\, \beta^{28} + 44236800\, \beta^{26} - 351436800\, \beta^{24} \\ & + 1661337600\, \beta^{22} - 5271552000\, \beta^{20} - 38051148800\, \beta^{18} \\ & - 20031897600\, \beta^{16} + 25303449600\, \beta^{14} - 24249139200\, \beta^{12} \\ & + 17571840000\, \beta^{10} - 9488793600\, \beta^{8} + 3706060800\, \beta^{6} \\ & - 990412800\, \beta^{4} + 162201600\, \beta^{2} - 12288000. \end{aligned}$$

By Sturm's Theorem it is easy to check that the above expression is non-vanishing on $(0, 1)$. On the other hand,

$$\begin{aligned} q(0, \beta) =& 1024\, \beta^{14} (2940784\, \beta^{32} - 42878528\, \beta^{30} + 291801664\, \beta^{28} \\ & - 1229816896\, \beta^{26} + 3591171584\, \beta^{24} - 7700300608\, \beta^{22} \\ & + 12534570048\, \beta^{20} - 15790302528\, \beta^{18} + 15546122592\, \beta^{16} \\ & - 11991947968\, \beta^{14} + 7216873664\, \beta^{12} - 3349458112\, \beta^{10} \\ & + 1174036864\, \beta^{8} - 300149696\, \beta^{6} + 3916259\, \beta^{4} \\ & - 5691840\, \beta^{2} + 284592). \end{aligned}$$

Numerical calculation shows that $q(0, \beta) = 0$ has a unique solution $\beta^* \approx 0.21666$ in $(0, 1)$. So, if $\beta < \beta^*$ (respectively, $\beta > \beta^*$), then the number of solutions of $q(x, \beta) = 0$ for $x \in (\beta - 1, 0)$ is odd (respectively, even). Simple calculations show that $q(x, 0.1)$ has a unique root at $x_1 \approx -0.85662$ for $x \in (-0.9, 0)$ and $q(x, 0.3)$ has exactly two roots at $x_2 \approx -0.66002$ and $x_3 = -0.02871$ for $x \in (-0.7, 0)$. After substituting $(x, \beta) = (x_1, 0.1)$ in (4.14) we obtain $\alpha_1 \approx 0.710787$. Also, for $(x, \beta) = (x_2, 0.3)$ the corresponding solution of system (4.14) is $\alpha_2 \approx 0.73837$. However, for $(x_3, 0.3)$, system (4.14) doesn't have any solution for α. Clearly, $(\alpha_1, 0.1)$ and $(\alpha_2, 0.3)$ are not located in S.

Computing the resultant of $q(x, \beta)$ and $\frac{\partial q}{\partial x}(x, \beta)$ with respect to x we get

$$Res(q, \frac{\partial q}{\partial x}, x) = \beta^{182} (\beta^2 - 2)^{15} (\beta^2 - 1)^{312} (\beta^2 - 5)^2 (3\beta^2 - 5)^2 \times L_1(\beta) (L_2(\beta) L_3(\beta))^2 (L_4(\beta))^4,$$

where $L_1(\beta) = 7\,\beta^8 - 2\,\beta^6 - 41\,\beta^4 - 144\,\beta^2 + 384$ and $L_2(\beta)$, $L_3(\beta)$ and $L_4(\beta)$ are polynomials in β of degrees 48, 48 and 144, respectively. Moreover, Sturm's Theorem implies that $L_i(\beta) \neq 0$ on $(0, 1)$ for $i = 1, \cdots, 4$. As a result, for $\beta \in (0, 1)$, each solution of $q(x, \beta) = 0$, named by $x = x(\beta)$, is simple. Accordingly, the equation $q(x, \beta) = 0$ has exactly one simple solution $x = x_0(\beta)$ in $(\beta - 1, 0)$ for $\beta \in (0, \beta^*)$ and exactly two simple solution $x = x_1(\beta)$ and $x = x_2(\beta)$ in $(\beta - 1, 0)$ for $\beta \in (\beta^*, 1)$. Now, we claim that $\frac{\partial M}{\partial \alpha}(x; \alpha, \beta) \neq 0$ along each solution of (4.14). Indeed

$$Res(M, \frac{\partial M}{\partial \alpha}, \alpha) = 782757789696 x^{12} (\beta - x)^{20} (2\beta - x)^{12} ((\beta - x)^2 - 1)^{12} \times (p(x, \beta))^8 (a_1(x, \beta))^2 a_2(x, \beta) (a_3(x, \beta))^2,$$

where $p(x, \beta)$ is the same as before and a_1, a_2 and a_3 are polynomials in x of degree 44, 14 and 10 respectively where their coefficients are polynomials with respect to β. To prove our claim, we only need to show that for $i = 1, 2, 3$,

$a_i(x,\beta)$ and $q(x,\beta)$ have no common roots satisfying

$$\beta - 1 < x < 0 < \beta < 1. \tag{4.15}$$

Note that

$$\begin{aligned}
Res(a_1(x,\beta), q(x,\beta), x) =& C_4 \beta^{224}(\beta^2-1)^{364}(\beta^2-2)^{36}(b_1(\beta))^2(c_1(\beta))^4,\\
Res(a_2(x,\beta), q(x,\beta), x) =& C_5 \beta^{84}(\beta^2-1)^{78}(\beta^2-2)^{12}(b_2(\beta))^2(c_2(\beta))^2,\\
Res(a_3(x,\beta), q(x,\beta), x) =& C_6 \beta^{28}(\beta^2-1)^{78}(b_3(\beta))^2(c_3(\beta))^2,
\end{aligned}$$

where C_4, C_5 and C_6 are constants and b_i and c_i for $i = 1, 2, 3$ are polynomials in β whose their degrees are $\deg b_1 = 188$, $\deg c_1 = 84$, $\deg b_2 = 126$, $\deg c_2 = 28$, $\deg b_3 = 88$ and $\deg c_3 = 14$.

By applying Sturm's Theorem, we obtain that $b_1(\beta) \neq 0$ and $c_1(\beta) \neq 0$ for $\beta \in (0,1)$. On the other hand $q(x,\sqrt{2}) = (x-\sqrt{2})^6 q_0(x)$, where

$$\begin{aligned}
q_0(x) = & \, 145101727232\,x^{40} + 224117784588896\,x^{38} + 49908701308457363\,x^{36}\\
& + 3814124509165001335\,x^{34} + 132751041164430232663\,x^{32}\\
& + 2419403027525678151621\,x^{30} + 25014610076278101504857\,x^{28}\\
& + 154083430545045467645631\,x^{26} + 582625357401025848750165\,x^{24}\\
& + 1375177444554761099340375\,x^{22} + 2039160591666824560578528\,x^{20}\\
& + 1894548440366475231840414\,x^{18} + 1090030210554766398228552\,x^{16}\\
& + 380044729700592264040560\,x^{14} + 77632295605537853982464\,x^{12}\\
& + 8842411953263012013056\,x^{10} + 522719079993355760640\,x^{8}\\
& + 14415340221165076480\,x^{6} + 156175787793981440\,x^{4}\\
& + 475184964481024\,x^{2} + 141436911616\\
& - \big(2902034544640\,x^{38} + 1391027777084704\,x^{36}\\
& + 172340476607542614\,x^{34} + 8684510563134831119\,x^{32}\\
& + 215956656413709305392\,x^{30} + 2941803892500654928715\,x^{28}\\
& + 23365626957475336073798\,x^{26} + 112447421356232019409947\,x^{24}\\
& + 335460882813568984703820\,x^{22} + 627447240627558376147245\,x^{20}\\
& + 737117128929392848744200\,x^{18} + 540034289999307609756366\,x^{16}\\
& + 242695723966658503819008\,x^{14} + 65099087290392926542352\,x^{12}\\
& + 10003375904403152610176\,x^{10} + 829702343867046840320\,x^{8}\\
& + 34051893053211258880\,x^{6} + 605200559041429504\,x^{4}\\
& + 3693758117855232\,x^{2} + 4507614949376\big)\, 2\sqrt{2}x.
\end{aligned}$$

Thus, by applying Descartes' rule of signs to $q_0(-x)$, we find that $q(x,\sqrt{2})$ doesn't have any negative roots. Therefore, $a_1(x,\beta)$ and $q(x,\beta)$ have no common roots satisfying (4.15). However, Sturm' Theorem implies that each one of $b_2(\beta)$, $c_2(\beta)$ and $b_3(\beta)$ has a unique root in $(0,1)$; consequently, the previous method defeats for these cases. To find out if there exist common roots of $a_i(x,\beta)$ $(i = 2,3)$ and $q(x,\beta)$ satisfying (4.15), we use Regular Chains Library in the computer algebra software package Maple to compute the intervals in which all common roots exist (for more details we refer to Appendix A of [10] or Maple help system). First, we consider the common roots of $a_2(x,\beta)$ and $q(x,\beta)$ as follows:

```
with(RegularChains):
```

```
with(ChainTools):
with(SemiAlgebraicSetTools):
sys1:=[a_2(x,beta),q(x,beta)]:
R:=PolynomialRing([x,beta]):
dec1:=Triangularize(sys1,R):
nops(dec1);
                                8
L:=map(Equations,dec1,R):
map(Dimension,dec1,R);
                    [0, 0, 0, 0, 0, 0, 0, 0]
for i from 1 to 8 do print(i,Equations(dec1[i],R)): end do:
```

1, [Length of output exceeds limit of 1000000]

$2, [p_{10}(\beta)x^2 + p_{11}(\beta)x + p_{12}(\beta), p_{13}(\beta)]$

$3, [-6\,\beta\,x^5 + x^6 - 40\,\beta\,x^3 + 30\,x^4 - 24\,\beta\,x + 60\,x^2 + 8, \beta^2 - 2]$

$4, [x, \beta + 1]$

$5, [x + 2, \beta + 1]$

$6, [x, \beta - 1]$

$7, [x - 2, \beta - 1]$

$8, [x, \beta]$

where p_{10}, p_{11}, p_{12} and p_{13} are polynomials in β of degree 26, 27, 26 and 28 respectively. Note that all the roots of regular chains $3, 4, \cdots, 8$ don't satisfy (4.15). Therefore we only consider first and second regular chains.

```
for i from 1 to 2 do C[i]:=Chain([L[i][2], L[i][1]],Empty(R), R):
                  RL[i]:=RealRootIsolate(C[i],R,'abserr'=1/10^5):
                  f[i]:=evalf(map(BoxValues,RL[i],R)):
 print(i,f[i]) end do;
```

$$
\begin{aligned}
1, [[&x = [0.5642776489, 0.5642852783], \beta = [-0.6506557835, -0.6506557835]],\\
&[x = [-1.865600586, -1.865592957], \beta = [-0.6506557835, -0.6506557835]],\\
&[x = [-0.5875797272, -0.5875740051], \beta = [-1.269586680, -1.269586680]],\\
&[x = [-1.951601028, -1.951595306], \beta = [-1.269586680, -1.269586680]],\\
&[x = [1.951595306, 1.951601028], \beta = [1.269586680, 1.269586680]],\\
&[x = [0.5875740051, 0.5875797272], \beta = [1.269586680, 1.269586680]],\\
&[x = [1.865592957, 1.865600586], \beta = [0.6506557835, 0.6506557835]],\\
&[x = [-0.5642852783, -0.5642776489], \beta = [0.6506557835, 0.6506557835]]],\\
2, [[&x = [-0.04209899902, -0.04209136963], \beta = [-0.8000901125, -0.8000901125]],\\
&[x = [-1.558090210, -1.558082581], \beta = [-0.8000901125, -0.8000901125]],\\
&[x = [1.558082581, 1.558090210], \beta = [0.8000901125, 0.8000901125]],\\
&[x = [0.04209136963, 0.04209899902], \beta = [0.8000901125, 0.8000901125]]].
\end{aligned}
$$

This Maple program shows that $a_2(x,\beta)$ and $q(x,\beta)$ have twelve pairs of common roots in the above twelve pairs of interval; but, no one satisfies (4.15).

Now, it is turn to consider the common roots of $a_3(x,\beta)$ and $q(x,\beta)$. So

```
with(RegularChains):
with(ChainTools):
with(SemiAlgebraicSetTools):
sys2:=[a_3(x,beta),q(x,beta)]:
R:= PolynomialRing([x,beta]):
dec2:=Triangularize(sys2,R):
nops(dec2);
                                 7
K:= map(Equations, dec2,R):
map(Dimension, dec2, R);
                      [0, 0, 0, 0, 0, 0, 0]
for i from 1 to 7 do print(i, Equations(dec2[i],R)): end do;
```

$$1, [p_{20}(\beta)x^2 + p_{21}(\beta)x + p_{22}(\beta), p_{23}(\beta)]$$
$$2, [p_{30}(\beta)x^2 + p_{31}(\beta)x + p_{32}(\beta), p_{33}(\beta)]$$
$$3, [x, \beta + 1]$$
$$4, [x + 2, \beta + 1]$$
$$5, [x, \beta - 1]$$
$$6, [x - 2, \beta - 1]$$
$$7, [x, \beta]$$

where p_{20}, p_{21}, p_{22}, p_{23}, p_{30}, p_{31}, p_{32}, p_{33} are polynomials in β of degree 86, 87, 86, 88, 12, 13, 12 and 14 respectively. It is clear that all the roots of regular chains $3, 4, \cdots, 7$ don't satisfy (4.15) and we only need to consider first and second regular chains. So

```
for i from 1 to 2 do C[i]:=Chain([K[i][2],K[i][1]],Empty(R),R):
                 RL[i]:=RealRootIsolate(C[i],R,'abserr'= 1/10^5):
                 f[i]:=evalf(map(BoxValues,RL[i],R)):
print(i,f[i]) end do;
```

$$1, [[x = [-0.6949234009, -0.6949157715], \beta = [-1.290633072, -1.290633072]],$$
$$[x = [-1.886344910, -1.886337280], \beta = [-1.290633072, -1.290633072]],$$
$$[x = [1.886337280, 1.886344910], \beta = [1.290633072, 1.290633072]],$$
$$[x = [0.6949157715, 0.6949234009], \beta = [1.290633072, 1.290633072]]]$$
$$2, [[x = [-0.04241943359, -0.04241180420], \beta = [-0.6557654040, -0.6557654040]],$$
$$[x = [-1.269119263, -1.269111633], \beta = [-0.6557654040, -0.6557654040]],$$
$$[x = [1.269111633, 1.269119263], \beta = [0.6557654040, 0.6557654040]],$$
$$[x = [0.04241180420, 0.04241943359], \beta = [0.6557654040, 0.6557654040]]]$$

This program shows that $a_3(x, \beta)$ and $q(x, \beta)$ have eight pairs of common roots in the above eight pairs of interval; but, no one satisfies (4.15).

In short, we proved that along each solution of (4.14), denoted by $x = x(\beta)$, we have $\frac{\partial M}{\partial \alpha}(x; \alpha, \beta) \neq 0$. Therefore, by the implicit function theorem, equation $M(x(\beta); \alpha, \beta) = 0$ determines a continuous function $\alpha = \alpha(\beta)$ such that $M(x(\beta); \alpha(\beta), \beta) = 0$, for β in some subinterval of $(0, 1)$.

Now we are in a position to prove that system (4.14) has no solution for $(\alpha, \beta) \in S_1$ and $x \in (\beta - 1, 0)$. The proof is by contradiction. Assume that there exist a point $(\tilde{\alpha}, \tilde{\beta}) \in S_1$ and $\tilde{x} \in (\tilde{\beta} - 1, 0)$ which $(\tilde{x}, \tilde{\alpha}, \tilde{\beta})$ satisfies (4.14). As a result of continuously changing β from $\tilde{\beta}$ to 0.1, the point $(\alpha(\beta), \beta)$ is also moving continuously from $(\tilde{\alpha}, \tilde{\beta})$ to $(\alpha_1, 0.1)$, where $\alpha_1 \approx 0.710787$. Accordingly, $(\alpha(\beta), \beta)$ takes a value on the boundary of S, which means $M(x; \alpha, \beta) = 0$ has a solution on the boundary of S for $x \in (\beta - 1, 0)$. This is a contradiction, because we proved $M(x; \alpha, \beta) > 0$ on the boundary of S for $x \in (\beta - 1, 0)$. □

Combining Lemmas 4.1 and 4.2, we have:

Theorem 4.3. *Period functions associated with those period annulus of system* (1.5) *which surround five singularities counted with multiplicities, are convex and have exactly one critical point.*

5. Conclusion

In this paper, we considered the monotonicity and convexity of the period function associated with centers of symmetric Newtonian system (1.5). First we discussed about the bifurcation diagram and all topologically different phase portraits of the Newtonian system (1.5). Then using Theorem 1.1, Theorem 1.2 and some computatinal methods, we proved that if the period annulus surrounds only one elementary center, then the corresponding period function is monotone; but, for the other cases, the period function has exactly one critical point. We also proved that in all cases, the period function is convex.

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