Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 9, No. 4, 2021, pp. 1198-1213
DOI:10.22034/cmde.2020.40112.1749

An approximation to the solution of one-dimensional hyperbolic telegraph equation based on the collocation of quadratic B-spline functions

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Abstract In this work, collocation method based on B-spline functions is used to obtained a numerical solution for one-dimensional hyperbolic telegraph equation. The proposed method is consists of two main steps. As first step, by using finite difference scheme for time variable, partial differential equation is converted to an ordinary differential equation by space variable. In the next step, for solving this equation collocation method is used. In the analysis section of the proposed method, the convergence of the method is studied. Also, some numerical results are given to demonstrate the validity and applicability of the presented technique. The L_∞, L_2 and Root-Mean-Square(RMS) in the solutions show the efficiency of the method computationally.

Keywords. Quadratic B-spline, One-dimensional hyperbolic telegraph equation, Collocation method, Convergence analysis.

2010 Mathematics Subject Classification. 65N35, 65D07, 65N22.

1. Introduction

We study the second-order linear hyperbolic telegraph equation

$$u_{tt} + 2\alpha u_t + \beta^2 u = u_{xx} + f(x,t)\ ,\ a \le x \le b\ ,\quad 0 \le t \le T, \tag{1.1}$$

with the initial conditions

$$\begin{cases} u(x,0) = f_0(x), & a \le x \le b, \\ u_t(x,0) = f_1(x), & a \le x \le b, \end{cases} \tag{1.2}$$

Received: 31 May 2020 ; Accepted: 06 December 2020.
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and boundary conditions

$$\begin{cases} u(a,t) = g_0(t), & 0 \le t \le T, \\ \\ u(b,t) = g_1(t), & 0 \le t \le T, \end{cases} \tag{1.3}$$

where α and β are constants.
Telegraph equation and wave equation are typical examples of hyperbolic partial differential equations. Telegraph equation is a linear differential equation which describes the voltage and current on an electrical transmission line with distance and time. In fact, the telegraph equation is commonly used in signal analysis for transmission and propagation of electrical signals and also has applications in the other fields, see, for example [9, 10, 18]. In the past several years, many different methods have been used to estimate the solution of the one-dimensional hyperbolic telegraph equation, including dual reciprocity boundary integral equation(DRBIE) method [3], alternating group explicit method [5], unconditionally stable difference schemes [6], variational iteration method [1]. In the proposed method, time variable discretization and collocation method for space variable have been used. This method is used in various papers on partial differential equations as [7, 8, 20, 21]. Also different methods based on cubic B-spline have been studied in [12, 14, 17]. The proposed method in comparison with the cubic B-spline method, despite the use of a function with a lower degree, due to use of midpoints as collocation points and the structure of the method has an order of convergence equal to the cubic B-spline method. On the other hand, compared to the cubic B-spline method, the proposed method has a lower computational values and therefore has a higher speed. In this work, numerical solution of the one-dimensional hyperbolic telegraph equation by using the quadratic B-spline collocation scheme is proposed. The collocation method together with B-spline approximations represents a cost-effective approach since it only requires the evaluation of the unknown parameters at the grid points.

The organization of this article is as follows: In section 2, quadratic B-spline collocation scheme is explained. In section 3, we present a finite difference approximation to discretize the Eq. (1.1) in time variable. The uniform convergence of the quadratic B-spline method is given in section 4. In section 5, some numerical illustrations and results are presented to demonstrate the efficiency of our proposed method. Note that we have computed the numerical results by Mathematica-8 programming.

2. Quadratic B-spline collocation method

To solve the Eq. (1.1) by collocation method with quintic B-splines as basis functions, we define the approximation for $u(x,t)$ as

$$U(x,t) = \sum_{i=-1}^{N} c_i(t) B_i(x), \tag{2.1}$$

where $c_i(t)$ are time-dependent quantities to be determined from the boundary conditions and collocation form of the differential equations. Also $B_i(x)$ are the quadratic

B-spline basis functions at knots, given by

$$B_i(z) = \frac{1}{h^2} \begin{cases} (z - z_{i-1})^2, & z \in [z_{i-1}, z_i), \\ 2h^2 - (z_{i+1} - z)^2 - (z - z_i)^2, & z \in [z_i, z_{i+1}), \\ (z_{i+2} - z)^2, & z \in [z_{i+1}, z_{i+2}), \\ 0, & otherwise. \end{cases}$$

The above formula is based on the relationship introduced in [11, 16]. Also the reader can see more information about the effect of mesh knots on the quadratic B-spline functions in the [2].

The interval $[a, b]$ is partitioned into a mesh of uniform length $h = z_{j+1} - z_j$, by the knots z_j where $j = 0, 1, 2, \ldots, N$ such that $a = z_0 < z_1 \ldots z_{N-1} < z_N$ and $z_j = z_0 + jh$. Also the numerical solution $U(x, t)$ is given at mid knots $x_i = \frac{(z_{i+1} + z_i)}{2}$. The values of $B_i(x)$ and its first and second derivatives at the mid knots points are given in Table 1 and the values of $B_i(x)$ at the knots given in Table 2.

TABLE 1. B_i, $B_i^{'}$, $B_i^{''}$ *at mid points.*

x	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
B_i	0	$\frac{1}{4}$	$\frac{3}{2}$	$\frac{1}{4}$	0
$hB_i^{'}$	0	1	0	-1	0
$h^2B_i^{''}$	0	2	-4	2	0

TABLE 2. B_i *at node points.*

z	z_{i-2}	z_{i-1}	z_i	z_{i+1}	z_{i+2}
B_i	0	0	1	1	0

Using approximate function (2.1) and Table 1, we have

$$U_i = \frac{1}{4}c_{i-1} + \frac{3}{2}c_i + \frac{1}{4}c_{i+1}, \tag{2.2}$$

$$hU_i^{'} = -c_{i-1} + c_{i+1}, \tag{2.3}$$

$$h^2U_i^{''} = 2c_{i-1} - 4c_i + 2c_{i+1}. \tag{2.4}$$

3. Construction of the method

We discretize the time derivatives of the Eq. (1.1) using a finite-difference formula. Using the finite difference method, we can write

$$\frac{u^{n+1}-2u^n+u^{n-1}}{\Delta t^2}+2\alpha\frac{u^{n+1}-u^{n-1}}{2\Delta t}+\beta^2\frac{u^{n+1}+u^{n-1}}{2} = \frac{u_{xx}^{n+1}+u_{xx}^{n-1}}{2}+f(x_n,t_n). \tag{3.1}$$

Rearranging the term and simplifying, we get

$$\left(1+\alpha\Delta t+\frac{\beta^2\Delta t^2}{2}\right)u^{n+1}-\frac{\Delta t^2}{2}u_{xx}^{n+1}=\Psi^n(x), \tag{3.2}$$

where

$$\Psi^n(x)=2u^n+\left(\alpha\Delta t-\frac{\beta^2\Delta t^2}{2}-1\right)u^{n-1}+\frac{\Delta t^2}{2}u_{xx}^{n-1}+\Delta t^2 f(x_n,t_n). \tag{3.3}$$

Substituting the approximate solution U for u and putting the values of the mid values U, its derivatives using Eqs. (2.2)-(2.4) at the knots in Eq. (3.2) yields the following difference equation with the variables c_i , $i=-1,0,\dots,N$,

$$\acute{a}c_{i-1}^{n+1}+\acute{b}c_i^{n+1}+\acute{a}c_{i+1}^{n+1}=h^2\Psi^n(x_i)\ ,\ i=0,1,...,N-1, \tag{3.4}$$

where

$$\acute{a}=\frac{h^2}{4}\left(1+\alpha\Delta t+\frac{\beta^2\Delta t^2}{2}\right)-\Delta t^2,\ \acute{b}=\frac{3h^2}{2}\left(1+\alpha\Delta t+\frac{\beta^2\Delta t^2}{2}\right)+2\Delta t^2. \tag{3.5}$$

The system (3.4) consists of N linear equations in $N+2$ unknowns $\{c_{-1},c_0\ ,...,c_{N-1}\ ,c_N\}$. To obtain a unique solution, we must use the boundary conditions. From the boundary conditions and Table 2, we can write

$$c_{-1}^{n+1}=g_0(t_{n+1})-c_0^{n+1}, \tag{3.6}$$

$$c_N^{n+1}=g_1(t_{n+1})-c_{N-1}^{n+1}, \tag{3.7}$$

by putting $i=0,N-1$ in (3.4) and using Eqs. (3.5)-(3.7) we get the results as

$$(\acute{b}-\acute{a})c_0^{n+1}+\acute{a}c_1^{n+1}=h^2\Big(\Psi(x_0)-\frac{\acute{a}g_0(t_{n+1})}{h^2}\Big), \tag{3.8}$$

$$\acute{a}c_{N-2}^{n+1}+(\acute{b}-\acute{a})c_{N-1}^{n+1}=h^2\Big(\Psi(x_{N-1})-\frac{\acute{a}g_1(t_{n+1})}{h^2}\Big). \tag{3.9}$$

Associating (3.8) and (3.9) with (3.4), we obtain an N×N system of equations in the following form

$$AC=h^2Q, \tag{3.10}$$

where

$$A = \begin{pmatrix} \acute{b}-\acute{a} & \acute{a} & 0 & \dots & 0 \\ \acute{a} & \acute{b} & \acute{a} & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots & \\ 0 & \dots & \acute{a} & \acute{b} & \acute{a} \\ 0 & \dots & 0 & \acute{a} & \acute{b}-\acute{a} \end{pmatrix}, \tag{3.11}$$

$$C = \left(c_0^{n+1}, \dots, c_{N-2}^{n+1}, c_{N-1}^{n+1}\right)^T,$$

$$Q = \left(\Psi^n(x_0) - \frac{\acute{a} g_0(t_{n+1})}{h^2}, \Psi^n(x_1), \dots, \Psi^n(x_{N-2}), \Psi^n(x_{N-1}) - \frac{\acute{a} g_1(t_{n+1})}{h^2}\right)^T.$$

This system can be solved by the Thomas algorithm. To start any computation, it is necessary to know $u(x, \Delta t)$. By Taylor series, we can write

$$u(x, \Delta t) = f_0(x) + \Delta t f_1(x) + \frac{\Delta t^2}{2}\Big(f(x,0) - \beta^2 f_0(x) - 2\alpha f_1(x) + \frac{d^2 f_0(x)}{dx^2}\Big) + O(\Delta t^3). \tag{3.12}$$

Based on the discussion in this section, we can say that $|\acute{b}| > 2|\acute{a}|$. Then for matrix A, we have $|a_{i,i}| = |\acute{b}| > 2|\acute{a}| = \sum_{j\neq i} a_{i,j}$ for all i so matrix A is a strictly diagonally dominant matrix. The summary of the proposed method can be written as the following algorithm.

Proposed method algorithm

1- Consider $\alpha, \beta, a, b, N, \Delta t$ as input value;
2- Define $h = (b-a)/n, \{z_i\} = \{a + ih\}, \{x_i\} = \{(z_{i+1} + z_i)/2\}$;
3- Define A as (3.11);
4- If $t = \Delta t$, consider $u(x,t)$ as (3.12);
5- Else , by using Eq. (3.10) obtain C;
6- Consider $u(x,t) = \sum_{i=-1}^{N} c_i B_i(x)$;

4. Convergence analysis

Now we discuss the convergence of the collocation method has been given in Section 3.

Theorem 4.1. *The time discretization process (3.1) is of the first order convergence.*

Proof. Suppose that $\varepsilon_i = u(t_i) - u^i$ be the local truncation error for (3.1) at the ith level of time where u^i the approximate solution and $u(t_i)$ be the exact solution of the problem (1.1). By using the truncation error in temporal direction, we can write

$$\mid \varepsilon_i \mid \leq \varrho_i \Delta t^2 \ , \ i \geq 2. \tag{4.1}$$

Also for $i = 1$ with help of (3.12), we have

$$\mid \varepsilon_1 \mid \leq \varrho_1 \Delta t^3. \tag{4.2}$$

To continue we assume that E_{n+1} be the global error in time discretizing process and $\varrho = max\{\varrho_1, ..., \varrho_n\}$. We can write the following global error estimate at n+1 level
$E_{n+1} = \sum_{i=1}^{n} \varepsilon_i, \ (\Delta t \leq T/n)$,
with the help of (4.1) and (4.2) we can write

$$\mid E_{n+1} \mid = \mid \sum_{i=1}^{n} \varepsilon_i \mid \leq \varrho_1 \Delta t^3 + \sum_{i=2}^{n} \varrho_i \Delta t^2 \leq n\varrho \Delta t^2 \leq n\varrho \frac{T}{n} \Delta t = \rho \Delta t,$$

where $\rho = \varrho T$. Hence this proves the theorem. □

We assume that $\widehat{u}(x)$ be the exact solution of Eq. (3.2) with boundary conditions (1.2) and (1.3), also $S(x) = \sum_{i=-1}^{N} c_i(t) B_i(x)$ be the B-spline approximation to $\widehat{u}(x)$. Due to round off errors in computations, we assume that $S^*(x) = \sum_{i=-1}^{N} c_i^*(t) B_i(x)$ be the computed spline approximation to S. In order to derive a bound for $\| \widehat{u}(x) - S(x) \|_\infty$, we need to estimate the $\| \widehat{u}(x) - S^*(x) \|_\infty$ and $\| S^*(x) - S(x) \|_\infty$. Now we substitute $S^*(x)$ in Eq. (3.10) and get the following result

$$AC^* = h^2 Q^*, \tag{4.3}$$

where

$$C^* = \left(c_0^{*n+1}, \ldots, c_{N-2}^{*n+1}, c_{N-1}^{*n+1}\right)^T,$$

$$Q^* = \Big(\Psi^{n*}(x_0) - \frac{\acute{a} g_0(t_{n+1})}{h^2}, \Psi^{n*}(x_1), \ldots, \Psi^{n*}(x_{N-2}), \Psi^{n*}(x_{N-1}) - \frac{\acute{a} g_1(t_{n+1})}{h^2}\Big)^T.$$

Considering (3.10) and (4.3), we can write

$$A(C^* - C) = h^2(Q^* - Q). \tag{4.4}$$

For our purpose, we need the following Theorem.

Theorem 4.2. *Consider Δ as an equally spaced partition of interval $[a, b]$ given by $\Delta = \{a = x_0 < x_1, \ldots < x_N = b\}$, $h = |x_j - x_{j-1}|$. Besides, assume that $f(x) \in C^4[a, b]$ such that $\forall x \in [a, b], |f^{(4)}(x)| \leq L$ and $S(x)$ be the unique spline* *interpolation function of $f(x)$ at the knots $\big(x_j, f(x_j)\big)$. Then, there exists a constant λ_j such that:*

$$\| f^{(j)} - S^{(j)} \| \leq \lambda_j L h^{4-j} \ , \ j = 0, 1, 2, 3.$$

Proof. For the proof see [11, 15, 19]. □

By using (3.3), we get the result as

$$|\Psi^*(x_i) - \Psi(x_i)| \leq |\ (1 + \alpha\Delta t + \frac{\beta^2 \Delta t^2}{2})(S^*(x_i) - S(x_i))\ | \\ + \frac{\Delta t^2}{2}\ |\ (S^{*''}(x_i) - S^{''}(x_i))\ | \ . \tag{4.5}$$

From the Eq. (4.5) and using Theorem 4.2, we can get

$$\| Q^* - Q \| \leq |\ 1 + \alpha\Delta t + \frac{\beta^2 \Delta t^2}{2}\ |\ \lambda_0 L h^4 + \frac{\Delta t^2}{2} \lambda_2 L h^2, \tag{4.6}$$

and thus

$$\| Q^* - Q \| \leq M_1 h^2,$$

where $M_1 = (1 + \alpha\Delta t + \frac{\beta^2 \Delta t^2}{2})\lambda_0 L h^2 + \frac{\Delta t^2}{2}\lambda_2 L$.

Lemma 4.3. *For the B-splines $\{B_{-1}, \cdots, B_N\}$, we have the following inequality:*

$$|\sum_{i=-1}^{N} B_i(x)| \leq \frac{7}{2}, \quad (a \leq x \leq b). \tag{4.7}$$

Proof. From the real analysis we have $|\sum_{i=-1}^{N} B_i(x)| \leq \sum_{i=-1}^{N} |B_i(x)|$,

if $x = x_i$, $i = 1, \ldots, N$, then, we have

$|\sum_{i=-1}^{N} B_i(x)| = 2 \leq \frac{7}{2}$,

and if $x_{i-1} \leq x \leq x_i$, then, we can write

$|\sum_{i=-1}^{N} B_i(x)| \leq | B_{i-2}(x) | + | B_{i-1}(x) | + | B_i(x) | + | B_{i+1}(x) |$
$\leq \frac{1}{4} + \frac{3}{2} + \frac{3}{2} + \frac{1}{4} \leq \frac{7}{2}$. □

The matrix A in Eq. (3.10) is a tridiagonal matrix and strictly diagonally dominant matrix and thus from theory of matrices we can say that the matrix A is nonsingular. Hence from (4.4) we can write

$$(C^* - C) = h^2 A^{-1}(Q^* - Q). \tag{4.8}$$

By taking the infinity norm from (4.8) and applying (4.6), we get

$$\| C^* - C \| \leq h^2 \| A^{-1} \| \| Q^* - Q \| \leq M_1 h^4 \| A^{-1} \| . \tag{4.9}$$

Let η_i, $(1 \leq i \leq N)$ be the summation of the ith row of the matrix A, therefore we get

$$\eta_1 = \eta_N = \frac{3h^2}{2}\big(1 + \alpha\Delta t + \frac{\beta^2 \Delta t^2}{2}\big) + 2\Delta t^2,$$

$$\eta_i = 2h^2\big(1 + \alpha\Delta t + \frac{\beta^2\Delta t^2}{2}\big) \quad , i = 2(1)N-1.$$

From the theory of matrices, we know that

$$\sum_{i=1}^{N} a_{ki}^{-1} \eta_i = 1,$$

where a_{ki}^{-1} are the elements of A^{-1}. As a result, we can write

$$\|A^{-1}\| = \sum_{i=1}^{N} |a_{ki}^{-1}| \leq \frac{1}{\min_{1\leq i \leq N} \eta_i} = \frac{1}{h^2 \eta_\iota}, \tag{4.10}$$

where l is some index between 1 and N. Following result is obtained by substituting (4.10) into (4.9),

$$\| C^* - C \| \leq \frac{M_1 h^4}{h^2 \eta_\iota} \leq M_2 h^2, \tag{4.11}$$

where $M_2 = \frac{M_1}{\eta_\iota}$ is constant.

Theorem 4.4. *Let $\hat{u}(x)$ be the exact solution of Eq. (3.1) and let $S(x)$ be the B-spline approximation to $\hat{u}(x)$ then*

$$\| \hat{u}(x) - S(x) \| \leq \varpi h^2.$$

where ϖ is constant.

Proof. Considering the B-spline collocation approximation and the computed spline approximation, we can write:

$$S^*(x) - S(x) = \sum_{i=-1}^{N} (c_i^* - c_i) B_i(x),$$

thus taking norm and using (4.7) and (4.11), we obtain

$$\begin{aligned} \| S^*(x) - S(x) \| = \| \sum_{i=-1}^{N} (c_i^* - c_i) B_i(x) \| &\leq | \sum_{i=-1}^{N} B_i(x) | \| C^* - C \| \\ &\leq \frac{7 M_2 h^2}{2}. \end{aligned} \tag{4.12}$$

TABLE 3. *$L\infty$, L_2 and RMS error for Example 5.1 at different times.*

Time	1	2	3	4	5
L_∞	4.280E-006	4.627E-006	2.279E-006	5.277E-007	7.732E-007
L_2	5.610E-005	7.753E-005	3.961E-005	6.998E-006	1.323E-005
RMS	1.984E-006	2.741E-006	1.401E-006	2.474E-007	4.677E-007

Also from Theorem 4.2, we can write

$$| \widehat{u} - S^*(x) | \leq \lambda_0 L h^4, \tag{4.13}$$

and therefore with helping (4.12) and (4.13), we get

$$\| \widehat{u} - S(x) \| \leq \| \widehat{u} - S^*(x) \| + \| S^*(x) - S(x) \| \leq \lambda_0 L h^4 + \frac{7M_2 h^2}{2} = \varpi h^2,$$

where $\varpi = \lambda_0 L h^2 + \frac{7M_2}{2}$. □

From the above discussions, we can say that, if $u(x,t)$ be the exact solution of (1.1) and $U(x,t)$ be the numerical approximation by our methods, then we can write

$$\| u(x,t) - U(x,t) \| \leq \Upsilon(h^2 + \Delta t),$$

where Υ is a constant.

5. Numerical examples

In this section, we obtain the numerical solutions of the one-dimensional hyperbolic telegraph equation for four problems. To show the efficiency of the present method for our problem in comparison with the exact solution, we report L_∞, L_2 and RMS error by the following formulae

$$L_\infty = \max_i | U(x_i,t) - u(x_i,t) |, \;\; L_2 = (\sum_i | U(x_i,t) - u(x_i,t) |^2)^{\frac{1}{2}},$$

$$RMS = \frac{(\sum_i | U(x_i,t) - u(x_i,t) |^2)^{\frac{1}{2}}}{\sqrt{N}},$$

where U is numerical solution and u denotes analytical solution.

Example 5.1. *We consider the second-order hyperbolic telegraph equation with $\alpha = \frac{1}{2}$ and $\beta = 1$, in the interval $0 \leq x \leq 4$, the analytical solution is given in [13] as $u(x,t) = \exp(x-t)$. In this case $f(x,t) = 0$. The initial conditions and boundary conditions are taken from the exact solution. The L_∞, L_2 and Root-Mean-Square (RMS) of errors are obtained in Table 3 for $t = 1, 2, 3, 4,$ and 5 with $\Delta t = 0.001$ and $h = 0.005$. Table 4 gives a comparisons between numerical and analytical solutions for different partitions. Figure 1 shows that the solution obtained by our method is close to the exact solution.*

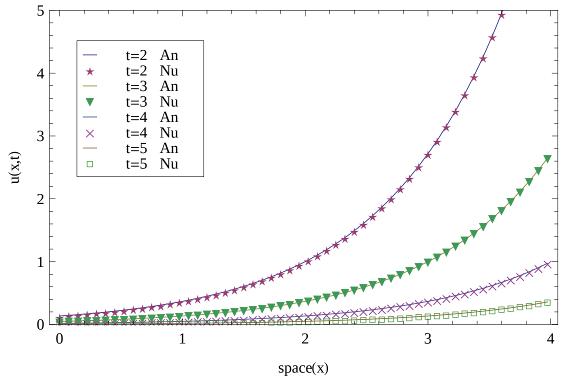

FIGURE 1. Comparisons between numerical and exact solution of Example 5.1 for different values of time with $\Delta t = 0.001$ and $h = 0.005$.

TABLE 4. Numerical results for Example 5.1.

Partitions	$\Delta t = 0.01, h = 0.05$		$\Delta t = 0.01, h = 0.01$		$\Delta t = 0.001, h = 0.01$	
Time	RMS	$L\infty$	RMS	$L\infty$	RMS	$L\infty$
0.5	3.997E-005	1.353E-004	9.393E-005	2.255E-004	4.637E-006	1.087E-005
1	1.911E-004	4.167E-004	1.042E-004	2.299E-004	1.119E-005	2.243E-005
1.5	2.743E-004	5.142E-004	8.177E-005	1.766E-004	1.377E-005	2.456E-005
2	2.716E-004	4.602E-004	5.287E-005	1.214E-004	1.259E-005	2.067E-005
2.5	2.157E-004	3.476E-004	2.986E-005	7.477E-005	9.382E-006	1.489E-005
3	1.407E-004	2.293E-004	1.984E-005	4.228E-005	5.682E-006	9.345E-006
3.5	7.158E-005	1.321E-004	1.962E-005	3.083E-005	2.528E-006	4.957E-006
4	2.542E-005	5.503E-005	2.127E-005	3.263E-005	1.004E-006	1.531E-006
4.5	3.379E-005	5.027E-005	1.788E-005	3.368E-005	1.949E-006	2.773E-006
5	4.628E-005	7.655E-005	8.149E-006	1.877E-005	2.164E-006	3.592E-006

Example 5.2. *In this example, we consider the hyperbolic telegraph (1.1) with* $\alpha = 4$ *and* $\beta = 2$*, in the interval* $0 \leq x \leq 2\pi$*, the exact solution is given in* [3] *as* $u(x,t) = \exp(-t)\sin(x)$*. In this case* $f(x,t) = (2 - 2\alpha + \beta^2)\exp(-t)\sin(x)$*. The initial conditions and boundary conditions are taken from the exact solution. In order to compare the solutions with* [3]*, we have taken* $\Delta t = 0.01$*,* $h = 0.05$*,* $h = 0.02$ *and* $h = 0.01$*. Table 5 shows a comparison between the RMS error found by our method and by DRBIE method in* [3] *at time* $t = 3$*. Figure 2 shows absolute error for different values of time and Figure 3 shows that the solution obtained by our method is close to the exact solution. From Figure 4, we can see that the numerical solution shows the same behavior as the exact solution.*

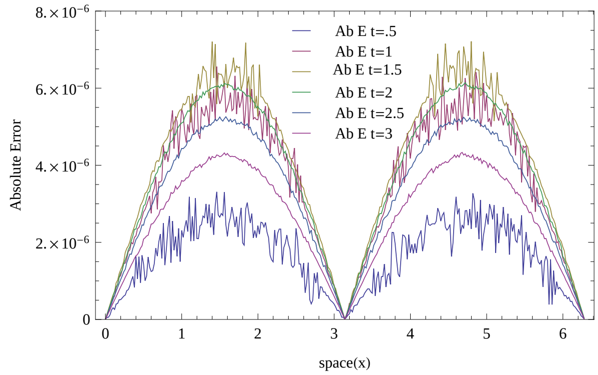

FIGURE 2. Absolute error graph of Example 5.2 for different times with $\Delta t = 0.01$ and $h = 0.005$.

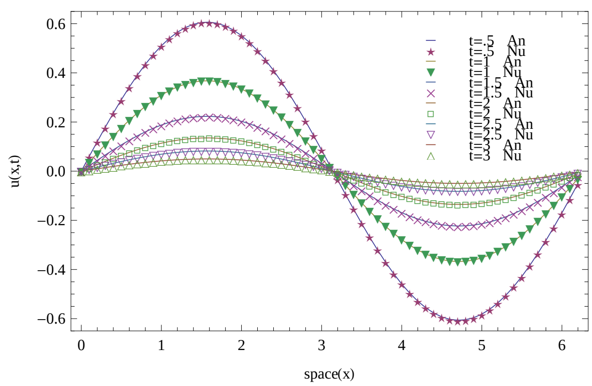

FIGURE 3. Comparisons between numerical and exact solution of Example 5.2 for different values of time with $\Delta t = 0.01$ and $h = 0.005$.

TABLE 5.

h	Our method	Cubic RBF:$1+r^2$	TPS RBF:$r^4\log(r)$	Linear RBF:$1+r$
0.05	5.616E-006	7.125E-005	9.017E-005	3.010E-004
0.02	3.394E-006	1.712E-005	2.943E-005	7.128E-005
0.01	3.077E-006	8.218E-006	8.991E-006	4.320E-005

Example 5.3. *We consider the hyperbolic telegraph (1.1) with $\alpha = \pi$ and $\beta = \pi$, in the interval $0 \le x \le 1$, the analytical solution is given in [4] as $u(x,t) = \sin(\pi x)\sin(\pi t)$. In this case $f(x,t) = \pi^2 \sin(\pi x)\big(\sin(\pi t) + 2\cos(\pi t)\big)$. The initial*

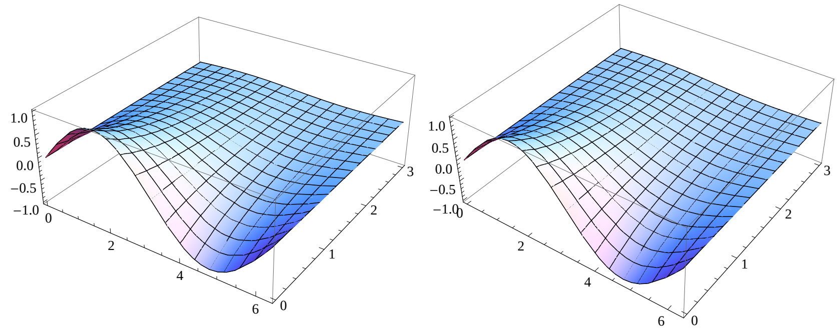

FIGURE 4. Exact solution (left) and numerical solution (right) with $\Delta t = 0.01$ and $h = 200$, of Example 5.2

TABLE 6. $L\infty$, L_2 and RMS error for Example 5.3 at different times with $\Delta t = 0.01, h = 0.005$.

$\Delta t = 0.01, h = 0.01$				
Time	0.4	0.6	0.8	1
L_∞	2.654E-004	3.829E-004	3.935E-004	2.676E-004
L_2	1.877E-003	2.708E-003	2.782E-003	1.893E-003
RMS	1.877E-004	2.708E-004	2.782E-004	1.893E-004
$\Delta t = 0.001, h = 0.01$				
Time	0.4	0.6	0.8	1
L_∞	3.692E-006	9.357E-006	1.334E-005	1.290E-005
L_2	2.611E-005	6.617E-005	9.437E-005	9.125E-005
RMS	2.611E-006	6.617E-006	9.437E-006	9.125E-006
$\Delta t = 0.001, h = 0.005$				
Time	0.4	0.6	0.8	1
L_∞	1.115E-006	6.320E-007	2.549E-007	1.102E-006
L_2	1.116E-005	6.320E-006	2.550E-006	1.102E-005
RMS	7.889E-007	4.469E-007	1.803E-007	7.790E-007

conditions and boundary conditions are taken from the exact solution. The estimated-exact solution graph, for some different times is presented in Figure 5. Absolute error between the analytical and our method is depicted at different times in Figure 5. Also we compute L_∞, L_2 and RMS error for different values of time and different partitions in Table 6.

Example 5.4. *As a last study we consider here a numerical solution of the hyperbolic telegraph (1.1) with different values of α and β, in the interval $0 \leq x \leq 4$ and $f(x,t) = -2\alpha \sin(x)\sin(t) + \beta^2 \cos(t)\sin(x)$, the exact solution $u(x,t) = \cos(t)\sin(x)$ [4]. The initial conditions and boundary conditions are taken from the exact solution.*

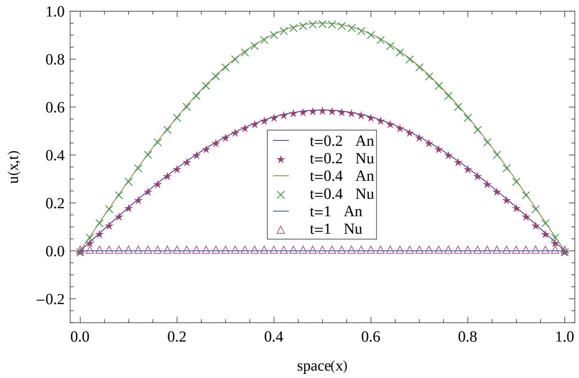

FIGURE 5. Comparisons between numerical and exact solution of Example 5.3 for different values of time with $\Delta t = 0.001$ and $h = 0.01$.

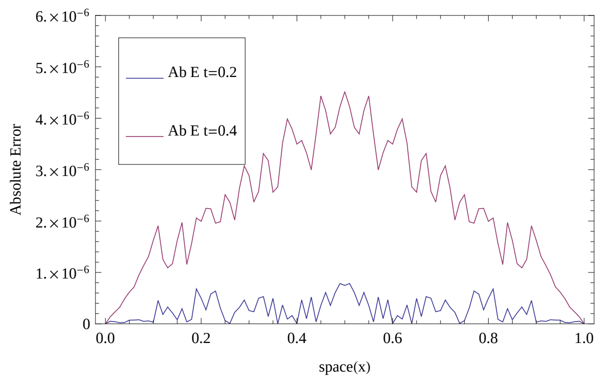

FIGURE 6. Absolute error graph of Example 5.3 for different times with $\Delta t = 0.001$ and $h = 0.01$.

The L_∞, L_2 and (RMS) are obtained in Table 7 for different values of α, β and different partitions.

Example 5.5. *Consider the one-dimensional hyperbolic telegraph equation with the following properties:*

$$\alpha = \beta = b = 1, \; a = 0, \; f(x,t) = f_1(x) = g_1(x) = g_2(x) = 0,$$
$$f_0(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{2}, \\ 1, & \frac{1}{2} \le x \le 1. \end{cases}$$

Table 7. $L\infty$, L_2 and RMS error for Example 5.4 at different times with different values of α, β.

$\alpha = 3, \beta = 2,$	$\Delta t = 0.01, h = 0.01$			
Time	0.5	1	1.5	2
L_∞	1.088E-005	1.715E-005	1.230E-005	3.885E-006
L_2	1.398E-004	2.166E-004	1.544E-004	2.914E-005
RMS	6.990E-006	1.083E-005	7.722E-006	1.457E-006
Time	2.5	3	3.5	4
L_∞	1.557E-005	2.883E-005	3.607E-005	3.513E-005
L_2	1.995E-004	3.638E-004	4.539E-004	4.423E-004
RMS	9.975E-006	1.819E-005	2.270E-005	2.212E-005
$\alpha = 10, \beta = 3,$	$\Delta t = 0.001, h = 0.01$			
Time	0.5	1	1.5	2
L_∞	1.777E-009	3.650E-008	9.205E-008	1.474E-007
L_2	1.328E-008	4.733E-007	1.184E-006	1.887E-006
RMS	6.642E-0010	2.366E-008	5.922E-008	9.432E-008
Time	2.5	3	3.5	4
L_∞	1.839E-007	1.887E-007	1.577E-007	9.606E-008
L_2	2.343E-006	2.394E-006	1.991E-006	1.206E-006
RMS	1.172E-007	1.197E-007	9.954E-008	6.033E-008
$\alpha = 5, \beta = 4,$	$\Delta t = 0.001, h = 0.005$			
Time	0.5	1	1.5	2
L_∞	2.139E-007	2.534E-007	1.376E-007	4.427E-008
L_2	3.897E-006	4.555E-006	2.450E-006	8.814E-007
RMS	1.378E-007	1.610E-007	8.663E-008	3.116E-008
Time	2.5	3	3.5	4
L_∞	2.263E-007	3.566E-007	4.009E-007	3.474E-007
L_2	4.112E-006	6.433E-006	7.206E-006	6.226E-006
RMS	1.454E-007	2.275E-007	2.548E-007	2.201E-007
$\alpha = 6, \beta = 1,$	$\Delta t = 0.001, h = 0.01$			
Time	0.5	1	1.5	2
L_∞	1.207E-007	2.623E-007	3.654E-007	4.025E-007
L_2	1.558E-006	3.346E-006	4.628E-006	5.074E-006
RMS	7.790E-008	1.673E-007	2.314E-007	2.537E-007
Time	2.5	3	3.5	4
L_∞	3.628E-007	2.541E-007	1.015E-007	1.155E-007
L_2	4.561E-006	3.214E-006	1.442E-006	1.120E-006
RMS	2.280E-007	1.607E-007	7.211E-008	5.600E-008

The Fourier solution of this problem based on the method of separation of variables can be obtained as follows

$$u(x,t) = \sum_{i=0}^{\infty} e^{-t} \sin(i\pi x)\big(d_i \cos(i\pi t) + \frac{d_i}{i\pi} \sin(i\pi t)\big),$$

TABLE 8. $L\infty$, L_2 and RMS error for Example 5.5 at different times.

$Time$	16	17	18	19	20
L_∞	5.390E-008	1.957E-008	7.018E-009	2.489E-009	8.764E-010
L_2	1.568E-007	5.481E-008	1.916E-008	6.616E-009	2.324E-009
RMS	2.217E-008	7.752E-009	2.710E-009	9.356E-010	3.287E-010

TABLE 9. Numerical results for Example 5.5.

Partitions	$\Delta t = 0.01, h = 0.005$		$\Delta t = 0.001, h = 0.01$		$\Delta t = 0.01, h = 0.1$	
Time	RMS	$L\infty$	RMS	$L\infty$	RMS	$L\infty$
15	1.215E-007	2.744E-007	9.247E-009	1.670E-008	1.057E-007	2.581E-007
16	4.651E-008	1.091E-007	3.868E-009	7.037E-009	3.238E-008	6.460E-008
17	1.617E-008	3.995E-008	1.602E-009	2.934E-009	1.511E-008	3.114E-008
18	5.584E-009	1.381E-008	6.590E-010	1.213E-009	6.007E-009	1.073E-008
19	2.141E-009	4.719E-009	2.688E-010	4.976E-010	3.405E-009	1.856E-009
20	8.695E-010	1.679E-009	1.090E-010	2.027E-010	1.178E-009	2.600E-009

where $d_i = 2\big(\cos(i\pi/2) - \cos(i\pi)\big)/i\pi$. *The numerical results for* $\Delta t = 0.001, h = 0.02$ *have been given in Table 8. Also by using different partition on domains, obtained numerical results have been tabulated in Table 9.*

CONCLUSION

In conclusion, this paper is dedicated to use a collocation method based on quadratic B-spline functions for one-dimensional hyperbolic telegraph equation. At first step, hyperbolic telegraph is discretized to the time variable by finite difference scheme and in the next step collocation method is used to space variable. In the last step, a linear system of equations is created. This system is represented as a matrix equation so that the coefficient matrix is a strictly diagonally dominant matrix. The convergence analysis is studied and it is shown that the order of convergence is $O(h^2 + \Delta t)$. Also the proposed method is studied on some problems and compared with exact solutions and other paper. The obtained numerical experiments demonstrate the good accuracy of the proposed scheme in this research.

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