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An infinite number of nonnegative solutions for iterative system of singular fractional order Boundary value problems

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Abstract In this paper, we consider the iterative system of singular Rimean-Liouville fractional-order boundary value problems with Riemann-Stieltjes integral boundary conditions involving increasing homeomorphism and positive homomorphism operator(IHPHO). By using Krasnoselskiis cone fixed point theorem in a Banach space, we derive sufficient conditions for the existence of an infinite number of nonnegative solutions. The sufficient conditions are also derived for the existence of a unique nonnegative solution to the addressed problem by fixed point theorem in complete metric space. As an application, we present an example to illustrate the main results.

Keywords. Iterative system, Riemann-Stieltjes integral, Homeomorphism, Nonegative solutions.

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1. Introduction

In recent times, the application of fractional calculus become an essential part of the mathematical background needed for researchers and scientists. fractional differential equations have gained its popularity and significance due to its distinguished applications in different areas of applied different areas such as fluid flows, electrical networks, rheology, biology chemical physics, [2, 12, 14, 24, 28, 29]. In 1983, Leibenson [15] introduced the $\mathtt{p}$-Laplacian equation,

$$\frac{d}{dt}\big(\varphi_{\mathtt{p}}(\varpi')\big) = \mathtt{g}(t, \varpi, \varpi'),$$

in order to study the turbulent flow through porous media. For $\mathtt{p} > 1$, the operator $\varphi_{\mathtt{p}}(\varpi) = |\varpi|^{\mathtt{p}-2}\varpi$ is invertible and its inverse is $\phi_{\mathtt{q}}$, where $\mathtt{q} = \mathtt{p}/(\mathtt{p} - 1)$.

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The recent works on the existence of positive solutions for fractional order boundary value problems involving $\mathtt{p}$-Laplacian operator, see [9,16–18,20,30,32]. However, in this paper we use new operator φ called increasing homeomorphism and positive homomorphism operator(IHPHO), which improves and generates classical $\mathtt{p}$-Laplacian operator $\varphi_{\mathtt{p}}$ for some $\mathtt{p} > 1$. For recent works on fractional order boundary value problems with IHPHO, see [8,21,27,33].

On the other hand, in the papers of Heymans and Podlubny [11], Agarwal et al. [1], Baleanu et al. [3], it was shown that RiemannLiouville fractional differential equations are useful in physics to model viscoelasticity and have different properties from the Caputo derivative. As the RiemannLiouville fractional derivative has a singularity at zero, the mathematical analysis to RiemannLiouville fractional differential equations is more complicated [13]. In [23], Padhi, Graef and Pati considered the following fractional order boundary value problem with Riemann-Stieltjes integral boundary conditions,

$$\begin{gathered}
\mathtt{D}_{0+}^{\varsigma}\varpi(t)+q(t)\mathtt{f}(t,\varpi(t))=0,\ 0<t<1,\\
\varpi(0)=\varpi'(0)=\cdots=\varpi^{(n-2)}(0)=0,\\
\mathtt{D}_{0+}^{\sigma}\varpi(1)=\int_0^1 h(\tau,\varpi(\tau))dA(\tau),
\end{gathered}$$

where $n>2$, $n-1<\varsigma\leq n$, $\sigma\in[1,\varsigma-1]$, $\mathtt{D}_{0+}^{\star}$ is a Riemann-Liouville fractional derivative of order $\star\in\{\varsigma,\sigma\}$ and established existence of positive solutions by various fixed point theorems on a Banach space. In [25], Prasad, Krushna and Wesen studied the solvability of iterative system of fractional order boundary value problem

$$\begin{gathered}
\mathtt{D}_{0+}^{\mathtt{q}_1}\varpi_i(t)\big)+\lambda_i\mathtt{p}_i(t)\mathtt{g}_i(\varpi_{i+1}(t))=0,\ 0<t<1,\ i=1,2,\cdots,n,\\
\varpi_{i+1}(t)=\varpi_1(t),\ 0<t<1,\\
\varpi_i(0)=\mathtt{D}_{0+}^{\mathtt{q}_2}x_i(0)=0,\ \varpi_i'(1)-\zeta\varpi_i'(\xi_1)=\vartheta\varpi_i'(\xi_2),\ i=1,2,\cdots,n,
\end{gathered}$$

where $2<\mathtt{q}_1\leq 3$, $0<\mathtt{q}_1\leq 1$, $0<\xi_1<\xi_2<1$, ζ,ϑ are positive constants and $\mathtt{D}_{0+}^{\star}$ is a Riemann-Liouville fractional derivative of order $\star\in\{\mathtt{q}_1,\mathtt{q}_2\}$, by Krasnoselskii's cone fixed point theorem on a Banach space. Recently, Prasad, Khuddush and Rashmita [26] established denumerably many positive solutions for the following problem,

$$\begin{gathered}
\varphi\left[{}^{\mathtt{C}}\mathtt{D}_{0+}^{\sigma}\varpi_{\mathtt{j}}(t)\right]+\psi(t)\mathtt{g}_{\mathtt{j}}(\varpi_{\mathtt{j}+1}(t))=0,\ 0<t<1,\ \mathtt{j}=1,2,\cdots,\ell,\\
\varpi_{\ell+1}(t)=\varpi_1(t),\ 0<t<1,
\end{gathered}$$

satisfying integral boundary conditions

$$\begin{gathered}
\varpi_{\mathtt{j}}(0)-a\varpi_{\mathtt{j}}'(0)=\mathcal{I}_{0+}^{\alpha}\varpi_{\mathtt{j}}(1),\\
\varpi_{\mathtt{j}}(1)+b\varpi_{\mathtt{j}}'(1)=\mathcal{I}_{0+}^{\beta}\varpi_{\mathtt{j}}(1),
\end{gathered}$$

where ${}^{\mathtt{C}}\mathtt{D}_{0+}^{\sigma}$ denote Caputo fractional derivatives with $1<\sigma\leq 2$, $\mathcal{I}_{0+}^{\alpha}$, $\mathcal{I}_{0+}^{\beta}$ denote Riemann-Liouville fractional integrals, $a,b\in\mathbb{R}$, $\alpha,\beta>0$, Motivated by the aforementioned works, in this paper we consider the following singular Rimean-Liouville fractional order boundary value problem with Riemann-Stieltjes integral boundary conditions involving IHPHO,

$$\begin{cases} \varphi\big(\mathtt{D}^{\varsigma}_{0+}\varpi_j(t)\big)+\Upsilon(t)\mathtt{f}_j(\varpi_{j+1}(t)\big)=0, & t\in(0,1),\\ \varpi_{\ell+1}(t)=\varpi_1(t), & j=1,2,\cdots,\ell, \end{cases} \tag{1.1}$$

$$\begin{cases} \varpi^{(\mathtt{r})}_j(0)=0,\ \mathtt{r}=0,1,2,\cdots,\quad m-2, & j=1,2,3,\cdots,\ell,\\ \mathtt{D}^{\sigma}_{0+}\varpi_j(1)=\int_0^1\varpi_j(\tau)d\chi(\tau), & j=1,2,3,\cdots,\ell, \end{cases} \tag{1.2}$$

where $\ell\in\mathbb{N}$, $m\geq 2$, $m-1<\varsigma\leq m,\sigma\in[1,\varsigma)$ and $\mathtt{D}^{\star}_{0+}$ is a Riemann-Liouville fractional derivative of order $\star\in\{\varsigma,\sigma\}$, $\int_0^1\varpi_j(\tau)d\chi(\tau)$ denotes Riemann-Stieltjes integral, $\Upsilon=\prod_{i=1}^n\Upsilon_i(t)$ and each $\Upsilon_i:[0,1]\to[0,\infty)$ has a singularity in $(0,\frac{1}{2})$. $\mathtt{f}_j(t):[0,1]\to[0,+\infty)$ are continuous functions and $\varphi^{-1}(\Upsilon)\in\mathcal{L}_{\mathtt{p}}[0,1]$ for some $\mathtt{p}\geq 1$ and $\varphi:\mathbb{R}\to\mathbb{R}$ is an IHPHO satisfying $\varphi(0)=0$.

The following assumptions hold throughout the paper:

(H_1) $\mathtt{f}_j:[0,+\infty)\to[0,+\infty)$ are continuous,

(H_2) there is a sequence $\{t_k\}_{k=1}^{\infty}$ such that

$$0<t_{k+1}<t_k<\frac{1}{2},\ \lim_{k\to\infty}t_k=t^*<\frac{1}{2},\lim_{t\to t_k}\Upsilon_i(t)=+\infty,\ k\in\mathbb{N},$$

$i=1,2,\cdots,n$ and $\Upsilon_i(t)$ does not vanish identically on any subinterval of $[0,1]$. Moreover, thereexists $\lambda_i>0$ such that

$$\lambda_i<\varphi^{-1}(\Upsilon_i(t))<\infty\ \text{ for }\ 0\leq t\leq 1,\ i=1,2,\cdots,n.$$

(H_3) χ be nondecreasing and of bounded variation function such that $0<\eta<1$ where

$$\eta=\frac{\Gamma(\varsigma-\sigma)}{\Gamma(\varsigma)}\int_0^1\tau^{\varsigma-1}d\chi(\tau).$$

The rest of the paper is organized in the following fashion. In section 2, we construct the kernel for the homogeneous problem corresponding to (1.1)-(1.2), estimate bounds for the kernel, and some lemmas which are needed in establishing our main results are provided. In section 3, we establish a criteria for the existence of infinite number of nonnegative solutions for the boundary value problem (1.1)-(1.2) by applying Hölder's inequality and Krasnoselskiis cone fixed point theorem in a Banch space. Also we derive sufficient conditions for the existence of unique nonnegative solution to the problem by an application of fixed point theorem in a complete metric space. Finally, we provide an example to illustrate the main results of the paper.

2. Preliminaries, Kernel and Its Bounds

In this section, we list some definitions and lemmas which are useful for our later discussions. Next, we constructed kernel to the homogeneous BVP corresponding to (1.1)-(1.2), and established certain lemmas for the bounds of the kernel.

Definition 2.1. *[14] The Riemann-Liouville fractional integral of order $\gamma > 0$ for a function $f : (0, \infty) \to \mathbb{R}$ is defined as*

$$\mathtt{I}_{0+}^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s) ds,$$

provided that the right side is pointwise defined on $(0, \infty)$.

Definition 2.2. *[14] The Riemann-Liouville fractional derivative of order $\gamma > 0$ for a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined as*

$$\mathtt{D}_{0+}^{\gamma} f(t) = \frac{1}{\Gamma(m-\gamma)} \left(\frac{d}{dt}\right)^m \int_0^t \frac{f(s)}{(t-s)^{\gamma-m-1}} ds,$$

where $m = [\gamma] + 1$, provided that the right side is pointwise defined on $(0, \infty)$.

Lemma 2.1. *[14] The general solution to $\mathtt{D}_{0+}^{\gamma} y(t) = 0$ with $\gamma \in (m-1, m]$ and $m > 1$ is the function*

$$y(t) = c_1 t^{\gamma-1} + c_2 t^{\gamma-2} + \cdots + c_m t^{\gamma-m},\ c_i \in \mathbb{R},\ i = 1, 2, \cdots, m.$$

Lemma 2.2. *[14] Let $\gamma > 0$. Then the following equality holds for $y(t)$:*

$$\mathtt{D}_{0+}^{-\gamma} \mathtt{D}_{0+}^{\gamma} y(t) = y(t) + c_1 t^{\gamma-1} + c_2 t^{\gamma-2} + \cdots + c_m t^{\gamma-m},\ c_i \in \mathbb{R},\ i = 1, 2, \cdots, m$$

and m is the smallest integer greater than or equal to γ.

Lemma 2.3. *Let $\mathtt{Q} \in \mathcal{C}[0,1]$. Then the unique solution of FBVP*

$$\varphi\big(\mathtt{D}_{0+}^{\varsigma} \varpi_1(t)\big) + \mathtt{Q}(t) = 0,\ \ 0 < t < 1, \tag{2.1}$$

$$\begin{cases} \varpi_1^{(\mathtt{r})}(0) = 0, & \mathtt{r} = 0, 1, 2, \cdots, m-2, \\ \mathtt{D}_{0+}^{\sigma} \varpi_1(1) = \int_0^1 \varpi_1(\tau) d\chi(\tau), & \end{cases} \tag{2.2}$$

is given by

$$\varpi_1(t) = \int_0^1 \aleph(t, \tau) \varphi^{-1}\big(\mathtt{Q}(\tau)\big) d\tau, \tag{2.3}$$

where

$$\aleph(t, \tau) = \aleph_0(t, \tau) + \frac{\Gamma(\varsigma - \sigma)}{\Gamma(\varsigma)(1-\eta)} t^{\varsigma-1} \mathtt{g}_{\chi}(\tau),$$

$$\aleph_0(t, \tau) = \frac{1}{\Gamma(\varsigma)} \begin{cases} t^{\varsigma-1}(1-\tau)^{\varsigma-\sigma-1} - (t-\tau)^{\varsigma-1}, & \tau \le t, \\ t^{\varsigma-1}(1-\tau)^{\varsigma-\sigma-1}, & t \le \tau, \end{cases}$$

and

$$\mathtt{g}_{\chi}(\tau) = \int_0^1 \aleph_0(\tau_1, \tau) d\chi(\tau_1).$$

Proof. From Lemmas 2.1 and 2.2, the equation (2.1) reduces to the fractional integral equation

$$\varpi_1(t) = c_1 t^{\varsigma-1} + c_2 t^{\varsigma-2} + \cdots + c_n t^{\varsigma-n} - \int_0^t \frac{(t-\tau)^{\varsigma-1}}{\Gamma(\varsigma)} \varphi^{-1}\big(\mathtt{Q}(\tau)\big) d\tau.$$

By using boundary conditions (2.2), we determined $c_2 = c_3 = \cdots = c_n = 0$ and

$$c_1 = \frac{1}{\Gamma(\varsigma)} \int_0^1 (1-\tau)^{\varsigma-\sigma-1} \varphi^{-1}\big(\mathtt{Q}(\tau)\big) d\tau + \frac{\Gamma(\varsigma-\sigma)}{\Gamma(\varsigma)} \int_0^1 \varpi_1(\tau) d\chi(\tau).$$

So, we get

$$\varpi_1(t) = \int_0^1 \aleph_0(t,\tau) \varphi^{-1}\big(\mathtt{Q}(\tau)\big) d\tau + \frac{\Gamma(\varsigma-\sigma)}{\Gamma(\varsigma)} t^{\varsigma-1} \int_0^1 \varpi_1(\tau) d\chi(\tau). \tag{2.4}$$

After certain computations, we receive

$$\int_0^1 \varpi_1(\tau) d\chi(\tau) = \frac{1}{1-\eta} \int_0^1 \mathtt{g}_\chi(\tau) \varphi^{-1}\big(\mathtt{Q}(\tau)\big) d\tau.$$

Substituting into (2.4), we get (2.3). This completes the proof. □

Lemma 2.4. *The function $\aleph_0(t,\tau)$ satisfies below properties:*

(i) $\aleph_0(t,\tau)$ *is nonnegative and continuous on* $[0,1] \times [0,1]$,
(ii) $\aleph_0(t,\tau) \le \aleph_0(1,\tau)$ *for* $t, \tau \in [0,1]$,
(iii) *there is some* $\delta \in (0, \frac{1}{2})$ *such that* $\delta^{\varsigma-1}\aleph_0(1,\tau) \le \aleph_0(t,\tau)$ *for* $t \in [\delta, 1-\delta], \tau \in [0,1]$.

Proof. It is easy to establish the results (i) and (ii). We prove (iii).
Let $\mathtt{V}(t,\tau) = t^{\varsigma-1}(1-\tau)^{\varsigma-\sigma-1} - (t-\tau)^{\varsigma-1}$ for $0 \le \tau \le t \le 1$. Then for $\delta \in (0, \frac{1}{2})$, we have

$$\begin{aligned} \mathtt{V}(t,\tau) &= t^{\varsigma-1}(1-\tau)^{\varsigma-\sigma-1} - (t-\tau)^{\varsigma-1} \\ &= t^{\varsigma-1}\left[(1-\tau)^{\varsigma-\sigma-1} - \left(1 - \frac{s}{t}\right)^{\varsigma-1}\right] \\ &\ge t^{\varsigma-1}[(1-\tau)^{\varsigma-\sigma-1} - (1-\tau)^{\varsigma-1}] \\ &\ge \delta^{\varsigma-1}\aleph_0(1,\tau). \end{aligned}$$

Other case is trivial and hence proof of the theorem completes. □

Lemma 2.5. *Let* $\mathtt{g}^*_\chi(\tau) = \aleph_0(1,\tau) + \frac{\Gamma(\varsigma-\sigma)}{\Gamma(\varsigma)(1-\eta)} \mathtt{g}_\chi(\tau)$. *The kernel* $\aleph(t,\tau)$ *has the following properties:*

(i) $\aleph(t,\tau)$ *is nonnegative and continuous on* $[0,1] \times [0,1]$,
(ii) $\aleph(t,\tau) \le \mathtt{g}^*_\chi(\tau)$ *for* $t, \tau \in [0,1]$,
(iii) *there exists* $\delta \in (0, \frac{1}{2})$ *such that* $\delta^{\varsigma-1}\mathtt{g}^*_\chi(\tau) \le \aleph(t,\tau)$ *for* $t \in [\delta, 1-\delta], \tau \in [0,1]$.

Proof. (i) and (ii) are obvious. To prove (iii), let $\delta \in (0, \frac{1}{2})$ and $\tau \in [0,1]$. Then from Lemma 2.4, we have

$$\begin{aligned}\aleph(t,\tau) &= \aleph_0(t,\tau) + \frac{\Gamma(\varsigma-\sigma)}{\Gamma(\varsigma)(1-\eta)} t^{\varsigma-1} g_\chi(\tau)\\ &\geq \delta^{\varsigma-1}\aleph_0(1,\tau) + \frac{\Gamma(\varsigma-\sigma)}{\Gamma(\varsigma)(1-\eta)}\delta^{\varsigma-1} g_\chi(\tau) = \delta^{\varsigma-1} g_\chi^*(\tau).\end{aligned}$$

□

We note that an ℓ−tuple $(\varpi_1(t), \varpi_2(t), \varpi_3(t), \cdots, \varpi_\ell(t))$ is a solution of the iterative system (1.1)–(1.2) if

$$\begin{aligned}&\varpi_j(t) = \int_0^1 \aleph(t,\tau)\varphi^{-1}\big[\Upsilon(\tau)\mathtt{f}_j(\varpi_{j+1}(\tau))\big]d\tau,\ 1 \leq j \leq \ell,\\ &\varpi_{\ell+1}(t) = \varpi_1(t),\ 0 < t < 1,\end{aligned}$$

i.e.,

$$\begin{aligned}\varpi_1(t) = &\int_0^1 \aleph(t,\tau_1)\varphi^{-1}\Bigg[\Upsilon(\tau_1)\mathtt{f}_1\Bigg(\int_0^1 \aleph(\tau_1,\tau_2)\varphi^{-1}\Bigg[\Upsilon(\tau_2)\mathtt{f}_2\Bigg(\int_0^1 \aleph(\tau_2,\tau_3)\\ &\times \varphi^{-1}\Bigg[\Upsilon(\tau_3)\mathtt{f}_3\Bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\\ &\times \mathtt{f}_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\big[\Upsilon(\tau_\ell)\mathtt{f}_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\Bigg)\cdots d\tau_3\Bigg]d\tau_2\Bigg]d\tau_1.\end{aligned}$$

Let $\mathscr{B}$ denotes the Banach space $\mathcal{C}([0,1],\mathbb{R})$ with the norm $\|\varpi\| = \max_{t\in[0,1]}|\varpi(t)|$. For $\delta \in (0, \frac{1}{2})$, define the cone $\mathcal{P}_\delta \subset \mathscr{B}$ by

$$\mathcal{P}_\delta = \Big\{\varpi \in \mathscr{B} : \varpi(t) \geq 0 \text{ and } \min_{t\in[\delta,\,1-\delta]} \varpi(t) \geq \delta^{\varsigma-1}\|\varpi(t)\|\Big\}.$$

For $\varpi_1 \in \mathcal{P}_\delta$, define an operator $\Omega : \mathcal{P}_\delta \to \mathscr{B}$ by

$$\begin{aligned}(\Omega\varpi_1)(t) = &\int_0^1 \aleph(t,\tau_1)\varphi^{-1}\Bigg[\Upsilon(\tau_1)\mathtt{f}_1\Bigg(\int_0^1 \aleph(\tau_1,\tau_2)\varphi^{-1}\Bigg[\Upsilon(\tau_2)\mathtt{f}_2\Bigg(\int_0^1 \aleph(\tau_2,\tau_3)\\ &\times \varphi^{-1}\Bigg[\Upsilon(\tau_3)\mathtt{f}_3\Bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\\ &\times \mathtt{f}_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\big[\Upsilon(\tau_\ell)\mathtt{f}_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\Bigg)\cdots d\tau_3\Bigg]d\tau_2\Bigg]d\tau_1.\end{aligned}$$

Lemma 2.6. *Suppose $(H_1), (H_2)$ and (H_3) hold. Then $\Omega(\mathcal{P}_\delta) \subset \mathcal{P}_\delta$ and $\Omega : \mathcal{P}_\delta \to \mathcal{P}_\delta$ is completely continuous for each $\delta \in (0, \frac{1}{2})$.*

Proof. Fix $\delta \in (0, \frac{1}{2})$. It is clear that $\mathtt{f}_j(\varpi_1(t)) \geq 0$ for all $t \in [0,1]$, and $\varpi_1 \in \mathcal{P}_\delta$.

Now, by Lemma 2.5, we have

$$\begin{aligned}
(\Omega\varpi_1)(t) = &\int_0^1 \aleph(t,\tau_1)\varphi^{-1}\Bigg[\Upsilon(\tau_1)\mathtt{f}_1\Bigg(\int_0^1 \aleph(\tau_1,\tau_2)\varphi^{-1}\Bigg[\Upsilon(\tau_2)\mathtt{f}_2\Bigg(\int_0^1 \aleph(\tau_2,\tau_3)\\
&\times \varphi^{-1}\Bigg[\Upsilon(\tau_3)\mathtt{f}_3\Bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\\
&\times \mathtt{f}_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\big[\Upsilon(\tau_\ell)\mathtt{f}_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\Bigg)\cdots d\tau_3\Bigg]d\tau_2\Bigg]d\tau_1\\
\le &\int_0^1 \mathtt{g}^*_\chi(\tau_1)\varphi^{-1}\Bigg[\Upsilon(\tau_1)\mathtt{f}_1\Bigg(\int_0^1 \aleph(\tau_1,\tau_2)\varphi^{-1}\Bigg[\Upsilon(\tau_2)\mathtt{f}_2\Bigg(\int_0^1 \aleph(\tau_2,\tau_3)\\
&\times \varphi^{-1}\Bigg[\Upsilon(\tau_3)\mathtt{f}_3\Bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\\
&\times \mathtt{f}_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\big[\Upsilon(\tau_\ell)\mathtt{f}_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\Bigg)\cdots d\tau_3\Bigg]d\tau_2\Bigg]d\tau_1
\end{aligned}$$

and

$$\begin{aligned}
&\min_{t\in[\delta,1-\delta]}(\Omega\varpi_1)(t) = \min_{t\in[\delta,1-\delta]}\Bigg\{\int_0^1 \aleph(t,\tau_1)\varphi^{-1}\Bigg[\Upsilon(\tau_1)\mathtt{f}_1\Bigg(\int_0^1 \aleph(\tau_1,\tau_2)\\
&\times \varphi^{-1}\Bigg[\Upsilon(\tau_2)\mathtt{f}_2\Bigg(\int_0^1 \aleph(\tau_2,\tau_3)\cdots\\
&\times \mathtt{f}_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\big[\Upsilon(\tau_\ell)\mathtt{f}_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\Bigg)\cdots d\tau_3\Bigg]d\tau_2\Bigg]d\tau_1\Bigg\}\\
&\ge \delta^{\varsigma-1}\Bigg\{\int_0^1 \mathtt{g}^*_\chi(\tau_1)\varphi^{-1}\Bigg[\Upsilon(\tau_1)\mathtt{f}_1\Bigg(\int_0^1 \aleph(\tau_1,\tau_2)\varphi^{-1}\Bigg[\Upsilon(\tau_2)\mathtt{f}_2\Bigg(\int_0^1 \aleph(\tau_2,\tau_3)\\
&\times \varphi^{-1}\Bigg[\Upsilon(\tau_3)\mathtt{f}_3\Bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\\
&\times \mathtt{f}_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\big[\Upsilon(\tau_\ell)\mathtt{f}_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\Bigg)\cdots d\tau_3\Bigg]d\tau_2\Bigg]d\tau_1\Bigg\}\\
&\ge \delta^{\varsigma-1}(\Omega\varpi_1)(t),
\end{aligned}$$

for all $t \in [0,1]$. Thus $\Omega(\mathcal{P}_\delta) \subset \mathcal{P}_\delta$. Furthermore, by an application of the ArzelaAscoli theorem, the operator Ω is completely continuous. □

3. Main results

By utilizing following theorems, in this section we establish the existence of infinite number of nonnegative solutions for the iterative system (1.1)–(1.2).

Theorem 3.1 (Krasnosellski)**.** *Let $\mathcal{P}$ be a cone in $\mathscr{B}$ and let Λ_1, Λ_2 be open open subsets of $\mathscr{B}$ with $0 \in \Lambda_1, \overline{\Lambda}_1 \subset \Lambda_2$. Then the operator Ω has a fixed point in $\mathcal{P} \cap (\overline{\Lambda}_2 \backslash \Lambda_1)$. If $\Omega : \mathcal{P} \cap (\overline{\Lambda}_2 \backslash \Lambda_1) \to \mathcal{P}$ be a completely continuous operator such that either*

(1) $\|\Omega\varpi\| \le \|\varpi\|$, $\varpi \in \mathcal{P} \cap \partial\Lambda_1$, *and* $\|\Omega\varpi\| \ge \|\varpi\|$, $\varpi \in \mathcal{P} \cap \partial\Lambda_2$, *or*
(2) $\|\Omega\varpi\| \ge \|\varpi\|$, $\varpi \in \mathcal{P} \cap \partial\Lambda_1$, *and* $\|\Omega\varpi\| \le \|\varpi\|$, $\varpi \in \mathcal{P} \cap \partial\Lambda_2$.

Theorem 3.2 (Hölder's Inequality)**.** *Let $\mathtt{f} \in \mathcal{L}^{\mathtt{p}_i}[0,1]$ with $\mathtt{p}_i > 1$, for $i = 1, 2, \cdots, n$ and $\sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} = 1$. Then $\prod_{i=1}^{n} \mathtt{f}_i \in \mathcal{L}^1[0,1]$ and $\|\prod_{i=1}^{n} \mathtt{f}_i\|_1 \le \prod_{i=1}^{n} \|\mathtt{f}_i\|_{\mathtt{p}_i}$. Further, if $\mathtt{f} \in \mathcal{L}^1[0,1]$ and $\mathtt{g} \in \mathcal{L}^\infty[0,1]$. Then $\mathtt{fg} \in \mathcal{L}^1[0,1]$ and*

$$\|\mathtt{fg}\|_1 \le \|\mathtt{f}\|_1 \|\mathtt{g}\|_\infty.$$

Consider the following three possible cases for $\omega \in \mathcal{L}^{\mathtt{p}_i}[0,1]$:

$$\sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} < 1, \ \sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} = 1, \ \sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} > 1.$$

Firstly, we seek denumerably many positive solutions for the case $\sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} < 1$.

Theorem 3.3. *Suppose $(H_1), (H_2)$ and (H_3) hold and let $\{\delta_k\}_{k=1}^\infty$ be such that $t_{k+1} < \delta_k < t_k$, $k = 1, 2, 3, \cdots$. Let $\{R_k\}_{k=1}^\infty$ and $\{S_k\}_{k=1}^\infty$ be two sequences satisfies the relation*

$$R_{k+1} < \delta_k^{\varsigma-1} S_k < \theta S_k < R_k, \ k \in \mathbb{N},$$

where

$$\theta = \max\left\{ \left[\delta_1^{\varsigma-1} \prod_{i=1}^{n} \lambda_i \int_{\delta_1}^{1-\delta_1} \mathtt{g}_\chi^*(\tau_\ell) d\tau_\ell \right]^{-1}, \ 1 \right\}.$$

Further, assume that $\mathtt{f}_j$ satisfies

(A_1) $\mathtt{f}_j(\varpi(t)) \le \varphi(\mathfrak{N}_1 R_k)$ *for all* $t \in [0,1]$, $0 \le \varpi \le R_k$,
where $\mathfrak{N}_1 < \left[\left\|\mathtt{g}_\chi^*\right\|_{\mathtt{q}} \prod_{i=1}^{n} \left\|\varphi^{-1}(\Upsilon_i)\right\|_{\mathtt{p}_i} \right]^{-1}$.
(A_2) $\mathtt{f}_j(\varpi(t)) \ge \varphi(\theta S_k)$ *for all* $t \in [\delta_k, 1-\delta_k]$, $\delta_k^{\varsigma-1} S_k \le \varpi \le S_k$.

Then the iterative system (1.1)–(1.2) has infinite number of solutions $\{(\varpi_1^{[k]}, \varpi_2^{[k]}, \cdots, \varpi_\ell^{[k]})\}_{k=1}^\infty$ such that $\varpi_j^{[k]}(t) \ge 0$ on $(0,1)$, $j = 1, 2, \cdots, \ell$ and $k \in \mathbb{N}$.

Proof. Let $\{\Lambda_{1,k}\}_{k=1}^\infty$ and $\{\Lambda_{2,k}\}_{k=1}^\infty$ be any two sequences in $\mathcal{B}$ such that

$$\Lambda_{1,k} = \{\varpi \in \mathscr{B} : \|\varpi\| < R_k\}, \ \Lambda_{2,k} = \{\varpi \in \mathscr{B} : \|\varpi\| < S_k\}.$$

Then $\{\Lambda_{1,k}\}_{k=1}^\infty$ and $\{\Lambda_{2,k}\}_{k=1}^\infty$ are open. Let $\{\delta_k\}_{k=1}^\infty$ be a sequence as mentioned in the hypothesis. Then $t^* < t_{k+1} < \delta_k < t_k < \frac{1}{2}$, for all $k \in \mathbb{N}$. Define a cone $\mathcal{P}_{\delta_k}$ by

$$\mathcal{P}_{\delta_k} = \left\{ \varpi \in \mathscr{B} : \varpi(t) \ge 0 \text{ and } \min_{t \in [\delta_k, 1-\delta_k]} \varpi(t) \ge \delta_k^{\varsigma-1} \|\varpi(t)\| \right\}$$

, for each $k \in \mathbb{N}$. Let $\varpi_1 \in \mathcal{P}_{\delta_k} \cap \partial\Lambda_{1,k}$. Then, $\varpi_1(\tau) \leq R_k = \|\varpi_1\|$ for all $\tau \in [0,1]$. By (A_1) and $0 < \tau_{\ell-1} < 1$, we have

$$\begin{aligned}
\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\big[\Upsilon(\tau_\ell)\mathtt{f}_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell &\leq \int_0^1 \mathtt{g}^*_\chi(\tau_\ell)\varphi^{-1}\big[\Upsilon(\tau_\ell)\mathtt{f}_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell \\
&\leq \mathfrak{N}_1 R_k \int_0^1 \mathtt{g}^*_\chi(\tau_\ell)\varphi^{-1}\Big[\prod_{i=1}^{n}\Upsilon_i(\tau_\ell)\Big]d\tau_\ell \\
&\leq \mathfrak{N}_1 R_k \int_0^1 \mathtt{g}^*_\chi(\tau_\ell)\prod_{i=1}^{n}\varphi^{-1}(\Upsilon_i(\tau_\ell))d\tau_\ell.
\end{aligned}$$

There exists a $\mathtt{q} > 1$ such that $\dfrac{1}{\mathtt{q}} + \displaystyle\sum_{i=1}^{n}\dfrac{1}{\mathtt{p}_i} = 1$. So,

$$\begin{aligned}
\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\big[\Upsilon(\tau_\ell)\mathtt{f}_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell &\leq \mathfrak{N}_1 R_k \Big\|\mathtt{g}^*_\chi\Big\|_{\mathtt{q}} \Big\|\prod_{i=1}^{n}\varphi^{-1}(\Upsilon_i)\Big\|_{\mathtt{p}_i} \\
&\leq \mathfrak{N}_1 R_k \Big\|\mathtt{g}^*_\chi\Big\|_{\mathtt{q}} \prod_{i=1}^{n}\big\|\varphi^{-1}(\Upsilon_i)\big\|_{\mathtt{p}_i} \leq R_k.
\end{aligned}$$

It follows in similar manner (for $0 < \tau_{\ell-2} < 1$) that

$$\begin{aligned}
&\int_0^1 \aleph(\tau_{\ell-2},\tau_{\ell-1})\varphi^{-1}\Big[\Upsilon(\tau_{\ell-1})\mathtt{f}_{\ell-1}\Big(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\big[\Upsilon(\tau_\ell)\mathtt{f}_\ell(\vartheta_1(\tau_\ell))\big]d\tau_\ell\Big)\Big]d\tau_{\ell-1} \\
&\qquad\leq \int_0^1 \aleph(\tau_{\ell-2},\tau_{\ell-1})\varphi^{-1}\big[\Upsilon(\tau_{\ell-1})\mathtt{f}_{\ell-1}(R_k)\big]d\tau_{\ell-1} \\
&\qquad\leq \int_0^1 \mathtt{g}^*_\chi(\tau_{\ell-1})\varphi^{-1}\big[\Upsilon(\tau_{\ell-1})\mathtt{f}_{\ell-1}(R_k)\big]d\tau_{\ell-1} \\
&\qquad\leq \mathfrak{N}_1 R_k \int_0^1 \mathtt{g}^*_\chi(\tau_{\ell-1})\varphi^{-1}\big[\Upsilon(\tau_{\ell-1})\big]d\tau_{\ell-1} \\
&\qquad\leq \mathfrak{N}_1 R_k \int_0^1 \mathtt{g}^*_\chi(\tau_{\ell-1})\varphi^{-1}\Big[\prod_{i=1}^{n}\Upsilon_i(\tau_{\ell-1})\Big]d\tau_{\ell-1} \\
&\qquad\leq \mathfrak{N}_1 R_k \int_0^1 \mathtt{g}^*_\chi(\tau_{\ell-1})\prod_{i=1}^{n}\varphi^{-1}(\Upsilon_i(\tau_{\ell-1}))d\tau_{\ell-1} \\
&\qquad\leq \mathfrak{N}_1 R_k \|\mathtt{g}^*_\chi\|_{\mathtt{q}} \prod_{i=1}^{n}\big\|\varphi^{-1}(\Upsilon_i)\big\|_{\mathtt{p}_i} \\
&\qquad\leq R_k.
\end{aligned}$$

Proceeding with this bootstrapping argument, we get

$$\begin{aligned}(\Omega\varpi_1)(t) &= \int_0^1 \aleph(t,\tau_1)\varphi^{-1}\Bigg[\Upsilon(\tau_1)\mathtt{f}_1\Bigg(\int_0^1 \aleph(\tau_1,\tau_2)\varphi^{-1}\Bigg[\Upsilon(\tau_2)\mathtt{f}_2\Bigg(\int_0^1 \aleph(\tau_2,\tau_3)\\ &\quad\times \varphi^{-1}\Bigg[\Upsilon(\tau_3)\mathtt{f}_3\Bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\\ &\quad\times \mathtt{f}_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\big[\Upsilon(\tau_\ell)\mathtt{f}_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\Bigg)\cdots d\tau_3\Bigg]d\tau_2\Bigg]d\tau_1\\ &\leq R_k.\end{aligned}$$

Since $R_k = \|\varpi_1\|$ for $\varpi_1 \in \mathcal{P}_{\delta_k} \cap \partial\Lambda_{1,k}$, we get

$$\|\Omega\varpi_1\| \leq \|\varpi_1\|. \tag{3.1}$$

Let $t \in [\delta_k, 1-\delta_k]$. Then,

$$r_k = \|\varpi_1\| \geq \varpi_1(t) \geq \min_{t\in[\delta_k,1-\delta_k]} \varpi_1(t) \geq \delta_k^{\varsigma-1}\|\varpi_1\| \geq \delta_k^{\varsigma-1} r_k.$$

By (A_2) and for $\tau_{\ell-1} \in [\delta_k, 1-\delta_k]$, we have

$$\begin{aligned}&\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\big[\Upsilon(\tau_\ell)\mathtt{f}_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\\ &\qquad\geq \delta_k^{\varsigma-1}\int_{\delta_k}^{1-\delta_k} \mathtt{g}_\chi^*(\tau_\ell)\varphi^{-1}\big[\Upsilon(\tau_\ell)\mathtt{f}_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\\ &\qquad\geq \delta_k^{\varsigma-1}\theta S_k\int_{\delta_k}^{1-\delta_k} \mathtt{g}_\chi^*(\tau_\ell)\varphi^{-1}(\Upsilon(\tau_\ell))d\tau_\ell\\ &\qquad\geq \delta_k^{\varsigma-1}\theta S_k\int_{\delta_k}^{1-\delta_k} \mathtt{g}_\chi^*(\tau_\ell)\prod_{i=1}^n\varphi^{-1}(\Upsilon_i(\tau_\ell))d\tau_\ell\\ &\qquad\geq \delta_1^{\varsigma-1}\theta S_k\prod_{i=1}^n\lambda_i\int_{\delta_1}^{1-\delta_1} \mathtt{g}_\chi^*(\tau_\ell)d\tau_\ell\\ &\qquad\geq S_k.\end{aligned}$$

Proceeding with bootstrapping argument, we get

$$\begin{aligned}(\Omega\varpi_1)(t) &= \int_0^1 \aleph(t,\tau_1)\varphi^{-1}\Bigg[\Upsilon(\tau_1)\mathtt{f}_1\Bigg(\int_0^1 \aleph(\tau_1,\tau_2)\varphi^{-1}\Bigg[\Upsilon(\tau_2)\mathtt{f}_2\Bigg(\int_0^1 \aleph(\tau_2,\tau_3)\\ &\quad\times \varphi^{-1}\Bigg[\Upsilon(\tau_3)\mathtt{f}_3\Bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\\ &\quad\times \mathtt{f}_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\big[\Upsilon(\tau_\ell)\mathtt{f}_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\Bigg)\cdots d\tau_3\Bigg]d\tau_2\Bigg]d\tau_1\\ &\geq S_k.\end{aligned}$$

Thus, if $\varpi_1 \in \mathcal{P}_{\delta_k} \cap \partial\Lambda_{2,k}$, then

$$\|\Omega\varpi_1\| \geq \|\varpi_1\|. \tag{3.2}$$

It is evident that $0 \in \Lambda_{2,k} \subset \overline{\Lambda}_{2,k} \subset \Lambda_{1,k}$. It follows from (3.1), (3.2) and Theorem 3.1 that the operator Ω has a fixed point $\varpi_1^{[k]} \in \mathcal{P}_{\delta_k} \cap \left(\overline{\Lambda}_{1,k} \backslash \Lambda_{2,k}\right)$ such that $\varpi_1^{[k]}(t) \geq 0$ on $(0,1)$, and $k \in \mathbb{N}$. Next setting $\varpi_{\ell+1} = \varpi_1$, we obtain an infinite number of nonnegative solutions $\{(\varpi_1^{[k]}, \varpi_2^{[k]}, \cdots, \varpi_\ell^{[k]})\}_{k=1}^{\infty}$ of (1.1)-(1.2) given iteratively by

$$\varpi_j(t) = \int_0^1 \aleph(t,\tau)\varphi^{-1}\big[\Upsilon(\tau) f_j(\varpi_{j+1}(\tau))\big]d\tau,\ t \in (0,1),\ j = \ell, \ell-1, \cdots, 1.$$

The proof is completed. □

For $\displaystyle\sum_{i=1}^{n} \frac{1}{p_i} = 1$, we have the following theorem.

Theorem 3.4. *Suppose* $(H_1), (H_2)$ *and* (H_3) *hold and let* $\{\delta_k\}_{k=1}^{\infty}$ *be such that* $t_{k+1} < \delta_k < t_k$, $k = 1, 2, 3, \cdots$. *Let* $\{R_k\}_{k=1}^{\infty}$ *and* $\{S_k\}_{k=1}^{\infty}$ *be two sequences satisfies the relation*

$$R_{k+1} < \delta_k^{\varsigma-1} S_k < \theta S_k < R_k,\ k \in \mathbb{N}.$$

where θ *is defined in Theorem 3.3. Further, assume that* $\mathtt{f}_j$ *satisfies* (A_2) *and*

(A_3) $\mathtt{f}_j(\varpi(t)) \leq \varphi(\mathfrak{N}_2 R_k)$ *for all* $t \in [0,1]$, $0 \leq \varpi \leq R_k$,

$$\text{where } \mathfrak{N}_2 < \left\{\left[\left\|\mathsf{g}_\chi^*\right\|_\infty \prod_{i=1}^{n} \left\|\varphi^{-1}(\Upsilon_i)\right\|_{\mathtt{p}_i}\right]^{-1}, \theta\right\}.$$

Then the iterative system (1.1)-(1.2) *has an infinite number of solutions* $\{(\varpi_1^{[k]}, \varpi_2^{[k]}, \cdots, \varpi_\ell^{[k]})\}_{k=1}^{\infty}$ *such that* $\varpi_j^{[k]}(t) \geq 0$ *on* $(0,1)$, $j = 1, 2, \cdots, \ell$ *and* $k \in \mathbb{N}$.

Proof. For a fixed k, let $\Lambda_{1,k}$ be as in the proof of Theorem 3.3 and let $\varpi_1 \in \mathcal{P}_{\delta_k} \cap \partial\Lambda_{2,k}$. Then, we have $\varpi_1(\tau) \leq R_k = \|\varpi_1\|$, for all $\tau \in (0,1)$. By (A_3) and for $\tau_{\ell-1} \in (0,1)$, we have

$$\begin{aligned}
\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\big[\Upsilon(\tau_\ell)\mathtt{f}_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell &\leq \int_0^1 \mathsf{g}_\chi^*(\tau_\ell)\varphi^{-1}\big[\Upsilon(\tau_\ell)\mathtt{f}_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell \\
&\leq \mathfrak{N}_2 R_k \int_0^1 \mathsf{g}_\chi^*(\tau_\ell)\varphi^{-1}\big[\Upsilon(\tau_\ell)\big]d\tau_\ell \\
&\leq \mathfrak{N}_2 R_k \int_0^1 \mathsf{g}_\chi^*(\tau_\ell)\varphi^{-1}\left[\prod_{i=1}^{n}\Upsilon_i(\tau_\ell)\right]d\tau_\ell \\
&\leq \mathfrak{N}_2 R_k \int_0^1 \mathsf{g}_\chi^*(\tau_\ell)\prod_{i=1}^{n}\varphi^{-1}(\Upsilon_i(\tau_\ell))d\tau_\ell \\
&\leq \mathfrak{N}_2 R_k \left\|\mathsf{g}_\chi^*\right\|_\infty \left\|\prod_{i=1}^{n}\varphi^{-1}(\Upsilon_i)\right\|_{\mathtt{p}_i} \\
&\leq \mathfrak{N}_2 R_k \left\|\mathsf{g}_\chi^*\right\|_\infty \prod_{i=1}^{n}\left\|\varphi^{-1}(\Upsilon_i)\right\|_{\mathtt{p}_i} \\
&\leq R_k.
\end{aligned}$$

It follows in similar manner (for $0 < \tau_{\ell-2} < 1$) that

$$\begin{aligned}
&\int_0^1 \aleph(\tau_{\ell-2},\tau_{\ell-1})\varphi^{-1}\Big[\Upsilon(\tau_{\ell-1})\mathtt{f}_{\ell-1}\Big(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\big[\Upsilon(\tau_\ell)\mathtt{f}_\ell(\vartheta_1(\tau_\ell))\big]d\tau_\ell\Big)\Big]d\tau_{\ell-1}\\
&\qquad\le \int_0^1 \aleph(\tau_{\ell-2},\tau_{\ell-1})\varphi^{-1}\big[\Upsilon(\tau_{\ell-1})\mathtt{f}_{\ell-1}(R_k)\big]d\tau_{\ell-1}\\
&\qquad\le \int_0^1 \mathtt{g}_\chi^*(\tau_{\ell-1})\varphi^{-1}\big[\Upsilon(\tau_{\ell-1})\mathtt{f}_{\ell-1}(R_k)\big]d\tau_{\ell-1}\\
&\qquad\le \mathfrak{N}_2 R_k \int_0^1 \mathtt{g}_\chi^*(\tau_{\ell-1})\varphi^{-1}\big[\Upsilon(\tau_{\ell-1})\big]d\tau_{\ell-1}\\
&\qquad\le \mathfrak{N}_2 R_k \int_0^1 \mathtt{g}_\chi^*(\tau_{\ell-1})\varphi^{-1}\Big[\prod_{i=1}^n \Upsilon_i(\tau_{\ell-1})\Big]d\tau_{\ell-1}\\
&\qquad\le \mathfrak{N}_2 R_k \int_0^1 \mathtt{g}_\chi^*(\tau_{\ell-1})\prod_{i=1}^n \varphi^{-1}(\Upsilon_i(\tau_{\ell-1}))d\tau_{\ell-1}\\
&\qquad\le \mathfrak{N}_2 R_k \|\mathtt{g}_\chi^*\|_{\mathtt{q}}\prod_{i=1}^n \big\|\varphi^{-1}(\Upsilon_i)\big\|_{\mathtt{p}_i}\\
&\qquad\le R_k.
\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}
(\Omega\varpi_1)(t) &= \int_0^1 \aleph(t,\tau_1)\varphi^{-1}\Bigg[\Upsilon(\tau_1)\mathtt{f}_1\Big(\int_0^1 \aleph(\tau_1,\tau_2)\varphi^{-1}\Bigg[\Upsilon(\tau_2)\mathtt{f}_2\Big(\int_0^1 \aleph(\tau_2,\tau_3)\\
&\quad\times\varphi^{-1}\Bigg[\Upsilon(\tau_3)\mathtt{f}_3\Big(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\\
&\quad\times \mathtt{f}_{\ell-1}\Big(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\big[\Upsilon(\tau_\ell)\mathtt{f}_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\Big)\cdots d\tau_3\Bigg]d\tau_2\Bigg]d\tau_1\\
&\le R_k.
\end{aligned}$$

Since $R_k = \|\varpi_1\|$ for $\varpi_1 \in \mathcal{P}_{\delta_k}\cap\partial\Lambda_{1,k}$, we get

$$\|\Omega\varpi_1\| \le \|\varpi_1\|. \tag{3.3}$$

Rest of the proof is similar to the proof of Theorem 3.3. Hence, the theorem. □

Lastly, the case $\displaystyle\sum_{i=1}^n \frac{1}{p_i} > 1$.

Theorem 3.5. *Assume that $(H_1)-(H_3)$ hold and let $\{\delta_k\}_{k=1}^\infty$ be such that $t_{k+1} < \delta_k < t_k$, $k = 1,2,3,\cdots$. Let $\{R_k\}_{k=1}^\infty$ and $\{S_k\}_{k=1}^\infty$ be two sequences satisfies the relation*

$$R_{k+1} < \delta_k^{\varsigma-1} S_k < \theta S_k < R_k,\ \ k\in\mathbb{N}.$$

where θ is defined in Theorem 3.3. Further, assume that $\mathtt{f}_j$ satisfies (A_2) and

(A_4) $\mathtt{f}_j(\varpi(t)) \leq \varphi(\mathfrak{N}_3 R_k)$ *for all* $t \in [0,1]$, $0 \leq \varpi \leq R_k$,

where $\mathfrak{N}_3 < \left\{ \left[\left\| \mathtt{g}_\chi^* \right\|_\infty \prod_{i=1}^{n} \left\| \varphi^{-1}(\Upsilon_i) \right\|_1 \right]^{-1}, \theta \right\}$.

Then the iterative system (1.1)–(1.2) *has an infinite number of solutions* $\{(\varpi_1^{[k]}, \varpi_2^{[k]}, \cdots, \varpi_\ell^{[k]})\}_{k=1}^{\infty}$ *such that* $\varpi_j^{[k]}(t) \geq 0$ *on* $(0,1)$, $j = 1, 2, \cdots, \ell$ *and* $k \in \mathbb{N}$.

Proof. The proof of the present theorem is similar to the proofs of Theorem 3.3 and Theorem 3.4. So, we omit details here. □

4. Uniqueness of Positive Solution

In this section, we establish the existence of a unique positive solution of the problem (1.1)–(1.2) by using fixed point theorem. In this regard, by $\mathcal{Q}$ we denote the class of functions $\Theta : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(i) Θ is strictly increasing;

(ii) For each sequence $\{t_n\} \subset (0, +\infty)$,

$$\lim_{n\to\infty} t_n = 0 \iff \lim_{n\to\infty} \Theta(t_n) = -\infty;$$

(iii) There exists $\alpha \in (0,1)$ such that

$$\lim_{t\to 0^+} t^\alpha \Theta(t) = 0.$$

Examples of such functions are $\Theta(t) = -\frac{1}{\sqrt{t}}$, $\Theta(t) = \ln t$, $\Theta(t) = \ln t + t$, etc.

Theorem 4.1. *[31] Let* $(\mathtt{X}, \mathtt{d})$ *be a complete metric space and* $\mathtt{T} : \mathtt{X} \to \mathtt{X}$ *a mapping such that there exist* $\lambda > 0$ *and* $\Theta \in \mathcal{Q}$ *satisfying for any* $\varpi, \vartheta \in \mathtt{X}$ *with* $\mathtt{d}(\mathtt{T}\varpi, \mathtt{T}\vartheta) > 0$,

$$\lambda + \Theta\big(\mathtt{d}(\mathtt{T}\varpi, \mathtt{T}\vartheta)\big) \leq \Theta\big(\mathtt{d}(\varpi, \vartheta)\big).$$

Then $\mathtt{T}$ *has a unique fixed point.*

In the next theorem, we establish the existence of unique positive solution of the problem (1.1)–(1.2). For this purpose, let us take $\mathtt{g}_j(t, \varpi) = \Upsilon(t)\mathtt{f}_j(\varpi)$ such that

$$\lim_{t\to 0^+} \mathtt{g}_j(t, \cdot) = \infty.$$

Theorem 4.2. *Let* $0 < \alpha < 1$, $\mathtt{g}_j : (0,1] \times [0,\infty)$ *be continuous and* $t^\alpha \mathtt{g}_j(t, \varpi)$ *be continuous function on* $[0,1] \times [0,\infty)$. *Assume that there exists a constant* $\lambda > 0$ *such that for* $\varpi, \vartheta \in [0, +\infty)$ *and* $t \in [0,1]$,

(H_4) $|\varphi^{-1}(\vartheta) - \varphi^{-1}(\varpi)| \leq |\vartheta - \varpi|$.

(H_5) $t^\alpha |\mathtt{g}_j(t, \vartheta) - \mathtt{g}_j(t, \varpi)| \leq \dfrac{|\vartheta - \varpi|}{C\left[1 + \lambda\sqrt{|\vartheta - \varpi|}\right]^2}$, *where* $C = \int_0^1 \mathtt{g}_\chi^*(\tau)\tau^{-\alpha} d\tau$.

Then Problem (1.1)–(1.2) *has a unique positive solution.*

Proof. Consider the cone $\mathtt{X} = \{\vartheta \in C[0,1] : \vartheta \geq 0\}$. Notice that $\mathtt{X}$ is a closed subset of $C[0,1]$ and therefore, $(\mathtt{X}, \mathtt{d})$ is a complete metric space where

$$\mathtt{d}(\varpi, \vartheta) = \sup\left\{|\varpi(t) - \vartheta(t)| : t \in [0,1]\right\} \text{ for } \varpi, \vartheta \in \mathtt{X}.$$

Now, for $\varpi_1 \in \mathtt{X}$ we define the operator $\mathtt{T}$ by

$$\begin{aligned}(\mathtt{T}\varpi_1)(t) = &\int_0^1 \aleph(t, \tau_1)\varphi^{-1}\Bigg[\mathtt{g}_1\Bigg(\tau_1, \int_0^1 \aleph(\tau_1, \tau_2)\varphi^{-1}\Bigg[\mathtt{g}_2\Bigg(\tau_2, \int_0^1 \aleph(\tau_2, \tau_3)\\ &\times \varphi^{-1}\Bigg[\mathtt{g}_3\Bigg(\tau_3, \int_0^1 \aleph(\tau_3, \tau_4) \cdots\\ &\times \mathtt{g}_{\ell-1}\Bigg(\tau_{\ell-1}, \int_0^1 \aleph(\tau_{\ell-1}, \tau_\ell)\varphi^{-1}\big[\mathtt{f}_\ell(\tau_\ell, \varpi_1(\tau_\ell))\big]d\tau_\ell\Bigg) \cdots d\tau_3\Bigg]d\tau_2\Bigg]d\tau_1.\end{aligned}$$

Since $\Upsilon(t)\mathtt{f}_j(\varpi_1(\tau)) \geq 0$ for all $\tau \in [0,1]$, $\vartheta_1 \in \mathtt{X}$ and $\aleph(t,\tau) \geq 0$ for all $t, \tau \in [0,1]$, it follows that $(\mathtt{T}\varpi_1)(t) \geq 0$ for all $t \in [0,1]$, $\varpi_1 \in \mathtt{X}$. Therefore, $\mathtt{T}$ applies $\mathtt{X}$ into itself. For $\tau_\ell \in [0,1]$, we have

$$\begin{aligned}&\left|\int_0^1 \aleph(\tau_{\ell-1}, \tau_\ell)\varphi^{-1}\big[\mathtt{g}_\ell(\tau_\ell, \varpi_1(\tau_\ell))\big]d\tau_\ell - \int_0^1 \aleph(\tau_{\ell-1}, \tau_\ell)\varphi^{-1}\big[\mathtt{g}_\ell(\tau_\ell, \vartheta_1(\tau_\ell))\big]d\tau_\ell\right|\\ &\qquad\leq \int_0^1 \aleph(\tau_{\ell-1}, \tau_\ell)\tau_\ell^{-\alpha}\tau_\ell^{\alpha}\left|\mathtt{g}_\ell(\tau_\ell, \varpi_1(\tau_\ell)) - \mathtt{g}_\ell(\tau_\ell, \vartheta_1(\tau_\ell))\right| d\tau_\ell\\ &\qquad\leq C\frac{|\varpi_1 - \vartheta_1|}{C[1 + \lambda\sqrt{|\varpi_1 - \vartheta_1|}]^2}\\ &\qquad= \frac{\mathtt{d}(\varpi_1, \vartheta_1)}{[1 + \lambda\sqrt{\mathtt{d}(\varpi_1, \vartheta_1)}]^2}\end{aligned}$$

and note that

$$\left[1 + \lambda\sqrt{\mathtt{d}(\varpi_1, \vartheta_1)}\right]^2 \geq [1 + \lambda \cdot 0]^2 = 1.$$

Similarly, for $\tau_{\ell-1} \in [0,1]$, we have

$$\begin{aligned}&\Bigg|\int_0^1 \aleph(\tau_{\ell-2}, \tau_{\ell-1})\varphi^{-1}\big[\mathtt{g}_{\ell-1}(\tau_{\ell-1}, \int_0^1 \aleph(\tau_{\ell-1}, \tau_\ell)\varphi^{-1}\big[\mathtt{g}_\ell(\tau_\ell, \varpi_1(\tau_\ell))\big]d\tau_\ell)]d\tau_{\ell-1}\\ &- \int_0^1 \aleph(\tau_{\ell-2}, \tau_{\ell-1})\varphi^{-1}\big[\mathtt{g}_{\ell-1}(\tau_{\ell-1}, \int_0^1 \aleph(\tau_{\ell-1}, \tau_\ell)\varphi^{-1}\big[\mathtt{g}_\ell(\tau_\ell, \vartheta_1(\tau_\ell))\big]d\tau_\ell)]d\tau_{\ell-1}\Bigg|\\ &\leq \int_0^1 \aleph(\tau_{\ell-2}, \tau_{\ell-1})\tau_{\ell-1}^{-\alpha}\left[\frac{\mathtt{d}(\varpi_1, \vartheta_1)}{[1 + \lambda\sqrt{\mathtt{d}(\varpi_1, \vartheta_1)}]^2}\right]\Big/\left[C(1 + \lambda \cdot 0)^2\right]d\tau_{\ell-1}\\ &\leq \int_0^1 \mathtt{g}_{\mathtt{X}}^*(\tau_{\ell-1})\tau_{\ell-1}^{-\alpha}d\tau_{\ell-1}\frac{\mathtt{d}(\varpi_1, \vartheta_1)}{C[1 + \lambda\sqrt{\mathtt{d}(\varpi_1, \vartheta_1)}]^2}\\ &= \frac{\mathtt{d}(\varpi_1, \vartheta_1)}{[1 + \lambda\sqrt{\mathtt{d}(\varpi_1, \vartheta_1)}]^2}.\end{aligned}$$

Continuing in this way, finally we get

$$\begin{aligned}\mathtt{d}(\mathtt{T}\varpi_1,\mathtt{T}\vartheta_1) &= \max_{t\in[0,1]}|(\mathtt{T}\varpi_1)(t)-(\mathtt{T}\vartheta_1)(t)|\\ &\le \frac{\mathtt{d}(\varpi_1,\vartheta_1)}{[1+\lambda\sqrt{\mathtt{d}(\varpi_1,\vartheta_1)}]^2}.\end{aligned}$$

That is

$$\lambda-\frac{1}{\sqrt{\mathtt{d}(\mathtt{T}\varpi_1,\mathtt{T}\vartheta_1)}}\le-\frac{1}{\sqrt{\mathtt{d}(\varpi_1,\vartheta_1)}}$$

and the contractivity condition of the Theorem 4.1 is satisfied with the function $\Theta(t)=-\frac{1}{\sqrt{t}}$ which belongs to the class $\mathcal{Q}$. Consequently, by Theorem 4.1, the operator $\mathtt{T}$ has a unique fixed point in $\mathtt{X}$. This means that Problem (1.1)–(1.2) has a unique positive solution in $C[0,1]$. □

5. Examples

Example 5.1. *Consider the following fractional order boundary value problem,*

$$\left.\begin{aligned}\varphi(\mathtt{D}_{0+}^{5/2}\varpi_j(t))+\Upsilon(t)\mathtt{f}_j(\varpi_{j+1}(t))=0,\ 0<t<1,\ j=1,2,\\ \varpi_{j+1}(t)=\varpi_1(t),\ 0<t<1,\\ \varpi_j(0)=\varpi_j'(0)=0,\ \mathtt{D}_{0+}^{3/2}\varpi_j(1)=\int_0^1\varpi_j(\tau)\,d\chi(\tau),\end{aligned}\right\}\tag{5.1}$$

where

$$\varphi(\varpi)=\begin{cases}\dfrac{\varpi^3}{1+\varpi^2}, & \varpi\le 0,\\ \varpi^2, & \varpi>0,\end{cases}$$

and

$$\Upsilon(t)=\Upsilon_1(t)\cdot\Upsilon_2(t),$$

in which

$$\Upsilon_1(t)=\frac{1}{|t-\frac14|^{\frac12}}\ \text{and}\ \Upsilon_2(t)=\frac{1}{|t-\frac13|^{\frac12}},$$

$$\mathtt{f}_j(\varpi)=\begin{cases}0.09\times10^{-16}, & \varpi\in(10^{-16},+\infty),\\ \frac{15400\times10^{-(16k+8)}-0.09\times10^{-16k}}{10^{-(16k+8)}-10^{-16k}}(\varpi-10^{-16k})+0.09\times10^{-16k}, & \\ & \varpi\in\left[10^{-(16k+8)},10^{-16k}\right],\\ 15400\times10^{-(16k+8)}, & \varpi\in\left(\frac{1}{5^{3/2}}\times10^{-(16k+8)},10^{-(16k+8)}\right),\\ \frac{15400\times10^{-(16k+8)}-0.09\times10^{-(16k+16)}}{\frac{1}{5^{3/2}}\times10^{-(16k+8)}-10^{-(16k+16)}}(\varpi-10^{-(16k+16)})+0.09\times10^{-(16k+16)}, & \\ & \varpi\in\left(10^{-(16k+16)},\frac{1}{5^{3/2}}\times10^{-(16k+8)}\right],\end{cases}$$

(5.2)

for j=1,2, and

$$\chi(t)=\begin{cases} t, & t\in[0,1/2)\cup[2/3,5/6),\\ \frac{1}{2}, & t\in[1/2,2/3),\\ \frac{5}{6}, & t\in[5/6,1].\end{cases}$$

Let

$$t_j=\frac{31}{64}-\sum_{r=1}^{j}\frac{1}{4(r+1)^4},\ \delta_j=\frac{1}{2}(t_j+t_{j+1}),\ j=1,2,3,\cdots,$$

then

$$\delta_1=\frac{15}{32}-\frac{1}{648}<\frac{15}{32}$$

and

$$t_{j+1}<\delta_j<t_j,\ \delta_j>\frac{1}{5}.$$

Therefore,

$$\delta_j^{\varsigma-1}>\frac{1}{5^{3/2}},\ j=1,2,3,\cdots.$$

It is easy to see

$$t_1=\frac{15}{32}<\frac{1}{2},\ t_j-t_{j+1}=\frac{1}{4(j+2)^4},\ j=1,2,3,\cdots.$$

Since $\sum_{j=1}^{\infty}\frac{1}{j^4}=\frac{\pi^4}{90}$ and $\sum_{j=1}^{\infty}\frac{1}{j^2}=\frac{\pi^2}{6}$, it follows that

$$t^*=\lim_{j\to\infty}t_j=\frac{31}{64}-\sum_{i=1}^{\infty}\frac{1}{4(i+1)^4}=\frac{47}{64}-\frac{\pi^4}{360}>\frac{1}{5},$$

$$\Upsilon_1,\Upsilon_2\in\mathcal{L}^{\mathrm{p}}[0,1]\text{ for all }0<p<2,\ \text{so }\lambda_1=\lambda_2=\frac{1}{\sqrt{3}}.$$

$$\begin{aligned}\eta=&\frac{\Gamma(\varsigma-\sigma)}{\Gamma(\varsigma)}\int_0^1\tau^{\varsigma-1}d\chi(\tau)\\ =&\frac{4}{3\sqrt{\pi}}\times 0.2698528306\approx 0.203\end{aligned}$$

$$\begin{aligned}\delta_1^{\varsigma-1}\prod_{i=1}^{n}\lambda_i\int_{\delta_1}^{1-\delta_1}\mathrm{g}_\chi^*(\tau_\ell)d\tau_\ell=&\left(\frac{15}{32}-\frac{1}{648}\right)^{3/2}\times\frac{1}{3}\int_{\frac{15}{32}-\frac{1}{648}}^{1-\frac{15}{32}+\frac{1}{648}}1.786\times\aleph_0(1,\tau)d\tau\\ \approx&0.0080582873.\end{aligned}$$

So, $\theta=\max\left\{\frac{1}{0.0080582873},1\right\}=124.0958$

$$\|\mathrm{g}_\chi^*\|_q=\left[\int_0^1|\mathrm{g}_\chi^*(\tau)|^q d\tau\right]^{\frac{1}{q}}<1.198085222\text{ for }q=2.$$

Next, let $0 < \varepsilon < 1$ be fixed. Then $\Upsilon_1, \Upsilon_2 \in \mathcal{L}^{1+\varepsilon}[0,1]$. It follows that

$$\|\varphi^{-1}(\Upsilon_1)\|_{1+\varepsilon} = \left[\frac{1}{3-\varepsilon}\left(3^{\frac{3-\varepsilon}{4}}+1\right)2^{\frac{1+\varepsilon}{2}}\right]^{\frac{1}{1+\varepsilon}}$$

$$\|\varphi^{-1}(\Upsilon_2)\|_{1+\varepsilon} = \left[\frac{4}{3-\varepsilon}\left(2^{\frac{3-\varepsilon}{4}}+1\right)(1/3)^{\frac{3-\varepsilon}{4}}\right]^{\frac{1}{1+\varepsilon}}.$$

So, for $0 < \varepsilon < 1$, we have

$$0.3024442806 \leq \left[\left\|\mathtt{g}_\chi^*\right\|_{\mathtt{q}}\prod_{i=1}^{n}\left\|\varphi^{-1}(\Upsilon_i)\right\|_{\mathtt{p}_i}\right]^{-1} \leq 0.3441810494.$$

Taking $M_1 = 0.302$. In addition if we take

$$R_k = 10^{-8k},\ r_k = 10^{-(8k+4)},$$

then

$$\begin{aligned}R_{k+1} = 10^{-(8k+8)} &< \frac{1}{5^{3/2}} \times 10^{-(8k+4)} < \delta_k^{\varsigma-1} r_k\\ &< r_k = 10^{-(8k+4)} < R_k = 10^{-8k},\end{aligned}$$

$\theta r_k = 124.0958 \times 10^{-(8k+4)} < 0.302 \times 10^{-8k} = M_1 R_k$, $k = 1, 2, 3, \cdots$, and $\mathtt{f}_j (j = 1, 2)$ satisfies the following growth conditions:

$$\begin{aligned}\mathtt{f}_j(\varpi) &\leq \varphi(K_1 R_k) = M_1^2 R_k^2 = 0.091204 \times 10^{-16k},\ \varpi \in \left[0, 10^{-16k}\right]\\ \mathtt{f}_j(\varpi) &\geq \varphi(\theta r_k) = \theta^2 r_k^2\\ &= 15399.56902 \times 10^{-(16k+8)},\ \varpi \in \left[\frac{1}{5^{3/2}} \times 10^{-(16k+8)}, 10^{-(16k+8)}\right].\end{aligned}$$

Thus, all conditions of Theorem 3.3 are satisfied. Therefore, by Theorem 3.3, the boundary value problem (5.1) has denumerably many positive solutions $\{\varpi_j^{[k]}\}_{k=1}^{\infty}$ such that $10^{-(8k+4)} \leq \|\varpi_j^{[k]}\| \leq 10^{-8k}$ for $j = 1, 2$, and $k = 1, 2, 3, \cdots$.

6. Conclusion

The research of fractional order differential equations with integral boundary conditions has become a new area of investigation. Moreover, our problem may have singularities. By the use of cone fixed point theorem in a Banach space, we deived the sufficient conditions for the existence of infinite number of nonnegative solutions and by fixed point theorem in a complete metric spaces, uniqueness of nonnegative solutions for the problem are acquired. An example is presented to illustrate the main results. The conclusion obtained in this paper will be useful in the application point of view. Also, we expect to find some applications in more nonlinear problems.

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