Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 9, No. 4, 2021, pp. 1223-1232
DOI:10.22034/cmde.2020.41145.1785

The symmetry analysis and analytical studies of the rotational Green-Naghdi (R-GN) equation

Zehra Pinar*
Tekirdağ Namık Kemal University, Faculty of Arts and Science,
Department of Mathematics, 59030 Merkez-Tekirdağ, Turkey.
E-mail: zpinar@nku.edu.tr

Abstract The simplified phenomenological model of long-crested shallow-water wave propagations are considered without/with the Coriolis effect. Symmetry analysis is taken into consideration to obtain exact solutions. Both classical wave transformation and transformations are obtained with symmetries and solvable equations are kept thanks to these transformations. Additionally, the exact solutions are obtained via various methods which are ansatz based methods. The obtained results have a major role in the literature so that the considered equation is seen in a large scale of applications in the area of geophysical.

Keywords. Rotational Green-Naghdi equations, Group transformations, Exact solutions.

2010 Mathematics Subject Classification. 35C08, 35QXX, 35A22, 35A25.

1. Introduction

One of the real applications of wave phenomena is the fluid dynamics of geophysical water waves which is modelled by Euler equations [3, 9]. The complexity and difficulties encountered in theoretical and numerical studies have been solved with simpler model equations which are an approach to Euler equations in some specific physical regimes. The most known is the Green-Naghdi equations (GN), which is also called the Serre equations [23], model the propagation of surface waves especially in coastal oceanography. Physically, the characteristics of the flow have a significant role in the model, and also it is explained by Figure 1.

In Figure 1, $h(x,t) = h_0 + \eta(x,t) - b(x)$, $\eta(x,t)$, h_0, $b(x)$, represent total water depth, free surface elevation, steel water depth and bottoms topography variation, respectively. Additionally, nonlinearity $\epsilon = \frac{a}{h_0}$ and shallowness $\mu = \frac{h_0^2}{\lambda^2}$, where is a the wave amplitude, λ is the wavelength and h_0 is mean depth, are important to constitute the model more realistic [2, 9]. For shallow water, generally $\mu = 1$.

Hence, Green et al. [7, 8] proposed GN equations as an alternative model to the Kortewegde Vries (KdV) and Benjamin-Bona-Mahoney (BBM) equations for the unidirectional propagation of long waves.

Received: 05 August 2020 ; Accepted: 06 December 2020.
∗ corresponding author.

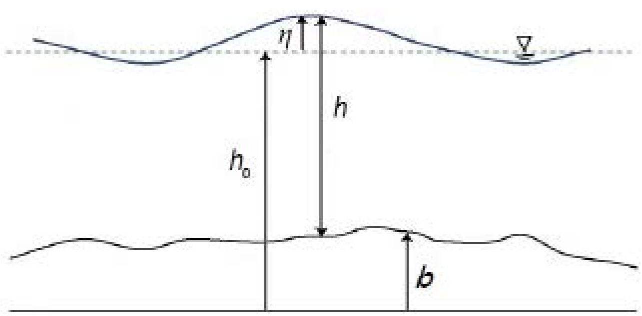


FIGURE 1. The fluid dynamics of geophysical water waves

$$\begin{cases} u_t + \eta_x + \epsilon u u_x = \frac{\mu}{3(1+\epsilon\eta)} \left[\left(1 + \epsilon\eta^3\right) \left(u_{xt} + \epsilon u u_{xx} - \epsilon u_x^2\right) \right]_x, \\ \eta_t + \left(\left(1 + \epsilon\eta\right) u\right)_x = 0, \end{cases} \tag{1.1}$$

where $u(x,t)$ and $\eta(x,t)$ subtend the vertically averaged horizontal and the surface component of the velocity at time t , respectively. Respect to the nonlinearity ϵ with $\mu = 1$, Eq. (1.1) yields KdV equations, BBM equations, the usual Boussinesq models, the Camassa-Holm (CH) and Degasperis-Procesi (DP) equations [2, 4, 9]. The geophysical water waves are impressed by the gravity and rotation of the Earth which is known as the Coriolis effect which has first seen in Ostrovsky equation [15]. Eq. (1.1) ignores the Coriolis effect for free surface water waves. When the Coriolis effect force is taken into consideration, Eq.(1.1) has an extra term which refers the Coriolis effect,

$$\begin{cases} u_t + \eta_x + \epsilon u u_x + 2\Omega\eta_t = \frac{\mu}{3(1+\epsilon\eta)} \left[\left(1 + \epsilon\eta^3\right) \left(u_{xt} + \epsilon u u_{xx} - \epsilon u_x^2\right) \right]_x, \\ \eta_t + \left(\left(1 + \epsilon\eta\right) u\right)_x = 0, \end{cases} \tag{1.2}$$

where Ω is a constant rotational frequency as a result of the Coriolis effect. Eq. (1.2) is known as the rotational-Green-Naghdi equations [2, 9]. When $\Omega \to 0$, Eq. (1.2) is reduced into Eq. (1.1). The given theorem by Chen et al. [2] is that the rotation-Camasa-Holm (R-CH) equation is obtained by considering the vertical average velocity $u(x,t)$ as the horizontal velocity $u^\theta(x,t)$ calculated at a given depth in Eq. (1.2) where $\theta \in [0,1]$ is the level of fluid domain i.e. $\theta = 0$ correspond to the bottom while $\theta = 1$ is the surface. Therefore, the R-CH equation with the given parameters

$$u_t + cu_x + \frac{2c^2}{c^2+1}\epsilon u u_x + \omega_1 \epsilon^2 u^2 u_x + \omega_2 \epsilon^3 u^3 u_x + \mu\left(\alpha u_{xxx} + \beta u_{xxt}\right) - \epsilon\mu\left(\sigma u u_{xxx} + \delta u_x u_{xx}\right). \tag{1.3}$$

$p \in i$ i.e. complex numbers, $\lambda = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right), c = \sqrt{1+\Omega^2} - \Omega,$

$$\alpha = c\left(p+\lambda\right), \beta = -\frac{c^2}{3\left(c^2+1\right)} + p + \lambda, \sigma = -\frac{c^2\left(5c^2-1\right)}{3\left(c^2+1\right)^3} - \frac{3c^2 p}{\left(c^2+1\right)},$$

$$\delta = -\frac{c^2\left(3c^4+16c^2+4\right)}{3\left(c^2+1\right)^3} - \frac{3c^2\left(3p+\lambda\right)}{\left(c^2+1\right)},$$

$$\omega_1 = -\frac{3c\left(c^2-1\right)\left(c^2-2\right)}{2\left(c^2+1\right)^3}, \omega_2 = \frac{\left(c^2-2\right)\left(c^2-1\right)^2\left(8c^2-1\right)}{2\left(c^2+1\right)^5},$$

is consistent with the R-GN equations i.e. Eq. (1.2) [2]. In this paper, the R-GN equations in the CamassaHolm regime (Eq. (1.3)) is considered instead of Eq. (1.2). For the considered model, not only the solitary wave solutions but also group symmetries are investigated. The symmetries of GN equations are obtained [1, 24] but for Eq. (1.3) is the first attempt in the literature to our knowledge. Existence and numerical solutions of R-GN equations are seen in the literature. The existence of the solutions of R-GN equations is also proved [2, 9]. Now, in our study, we focus on obtaining exact solutions in open form, as there is no study in the literature on the exact solutions of the model. Eq. (1.3) is reduced into solvable equations via two transformations: wave transformation and group transformation obtained by symmetries. To obtain exact solutions, the approximate methods, which are modifications of the auxiliary equation method, are considered. The obtained results have a major role in the literature so that the considered equation is seen in a large scale of applications in the area of geophysical.

2. Group Analysis Method

Group Analysis method, which is also known as Lie group analysis, is important for not only numerical methods but also analytical methods. By means of this method, the equations are reduced to solvable equations by the obtained transformations via symmetries. Conservation laws have an importance in physics and engineering, one way to obtain them is Group analysis method [11, 13, 16, 17, 25, 26]. For many nonlinear models, these are the only explicit, exact solutions which are available, have a significant role in both the mathematical analysis and applications of the system in the science.

For the implementation of Group analysis method, the reader can find many documents and papers in the literature [12, 13, 14, 18, 21].

3. The results

3.1. Symmetry analysis of R-GN equation. The classical Lie group analysis is considered to obtain the symmetries of R-GN equation (Eq. (1.3)). The innitesimal generator of one-parameter group with ξ_i is the tangent vector field is $X = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x^i}$. The determining equations to determine ξ_i are given

$$\frac{\partial \xi_x(x,t,u)}{\partial t} = -\frac{\left(3\theta^2 c^2 - c^2 + 6pc^2 + 3\theta^2 - 1 + 6p\right) c\xi_u(x,t,u)}{\left(3\theta^2 c^2 - c^2 + 6pc^2 + 3\theta^2 - 1 + 6p\right) u},$$

$$\frac{\partial \xi_x(x,t,u)}{\partial u} = 0, \frac{\partial \xi_t(x,t,u)}{\partial u} = 0, \frac{\partial \xi_t(x,t,u)}{\partial x} = 0,$$

$$\frac{\partial \xi_x(x,t,u)}{\partial x} = \frac{u\left(\frac{\partial \xi_t(x,t,u)}{\partial t}\right) + \xi_u(x,t,u)}{u}.$$

As a result of the determining system, tangent vector field is obtained,

$$\begin{aligned}\xi_x(x,t,u) &= \frac{c(c^2+1)(3\theta^2+6p-1)F_1(t)}{(3\theta^2+6p-3)c^2+3\theta^2+6p-1} \\ &+ F_2\left(\frac{(3\theta^2+6p-1)(c^3t+ct-x)-xc^2(3\theta^2+6p-3)}{c(c^2+1)(3\theta^2+6p-1)}\right),\end{aligned}$$

$$\xi_t(x,t,u) = F_1(t),$$

$$\begin{aligned}\xi_u(x,t,u) &= -u\frac{dF_1(t)}{dt} \\ &- \frac{u\left((3\theta^2+6p-3)c^2+3\theta^2+6p-1\right) D\left(F_2\right)\left(\frac{(3\theta^2+6p-1)(c^3t+ct-x)-xc^2(3\theta^2+6p-3)}{c(c^2+1)(3\theta^2+6p-1)}\right)}{c(c^2+1)(3\theta^2+6p-1)},\end{aligned}$$

where F_1 and F_2 are arbitrary functions.
Now, assuming $F_2\left(\frac{(3\theta^2+6p-1)(c^3t+ct-x)-xc^2(3\theta^2+6p-3)}{c(c^2+1)(3\theta^2+6p-1)}\right) = 0$ and $F_1(t) = A$ the transformation is obtained as $u(x,t) = H(\zeta), \zeta = \frac{-x+tc\left(\left(c^2+1\right)\left(3\theta^2+6p\right)-c^2\right)}{c(c^2+1)(3\theta^2+6p-1)}$.Using the transformation, Eq. (1.3) is reduced into Eq. (3.1) which is a third-order nonlinear ordinary differential equation.

$$
\begin{aligned}
&H^{'}\left(1-\frac{1}{\left(c^{2}+1\right)\left(3\theta^{2}+6p-1\right)}\right) \qquad (3.1)\\
&-\frac{3c\epsilon HH^{'}}{\left(c^{2}+1\right)^{2}\left(3\theta^{2}+6p-1\right)}+\frac{3\left(c^{2}-1\right)\left(c^{2}-2\right)\epsilon^{2}H^{2}H^{'}}{2\left(c^{2}+1\right)^{4}\left(3\theta^{2}+6p-1\right)}\\
&\quad-\frac{\left(c^{2}-1\right)^{2}\left(c^{2}-2\right)\epsilon^{3}H^{3}H^{'}}{c\left(c^{2}+1\right)^{6}\left(3\theta^{2}+6p-1\right)}\\
&+\mu H^{'''}\left(\frac{\left(3\theta^{2}+6p\right)\left(c^{2}+1\right)+3c^{2}}{6c^{2}\left(c^{2}+1\right)^{3}\left(3\theta^{2}+6p-1\right)^{2}}\right)\\
&+\epsilon\mu\left(\frac{5c^{2}-1+9p\left(c^{2}+1\right)^{2}}{3c\left(c^{2}+1\right)^{6}\left(3\theta^{2}+6p-1\right)^{3}}HH^{'''}+\frac{\left(c^{2}+1\right)^{2}\left(54c^{2}p-9\theta^{2}\right)}{6\left(c^{2}+1\right)^{3}}H^{'}H^{''}\right)\\
&=0
\end{aligned}
$$

Numerical or analytical [5, 6, 10, 17, 18, 19, 20, 21, 25, 26, 27, 28] methods can be considered to obtain the solutions of Eq.(3.1). Now, the Bernoulli approximation method [17, 20, 21] is considered that its auxiliary equation is Bernoulli type differential equation

$$z^{'}+P(\zeta)z+Q(\zeta)z^{k}=0, k\neq 0,1. \qquad (3.2)$$

Due to the balancing principle, the ansatz is determined as $H(\zeta)=g_0+g_1 z(\zeta)$, where $z(\zeta)$ is the solution of Eq. (3.2) and g_0 , g_1 are the coefficients determined as a result of the system.

Case 3.1. *When* $P(\zeta)=\frac{1}{\zeta}, Q(\zeta)=\zeta, k=3$ *the parameters* $c=\pm\frac{\sqrt{-14\pm 2i\sqrt{31}}}{4}, p=-\frac{25}{18}\pm i\frac{\sqrt{31}}{18}, \theta=\pm\frac{\sqrt{133\pm i\sqrt{31}}}{6}, g_0=0$ *are obtained.*

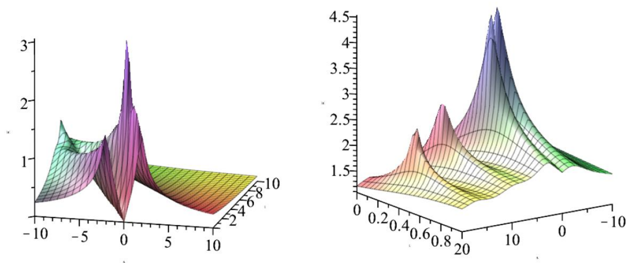
Case 3.2. *When* $P(\zeta)=P=constant, Q(\zeta)=\zeta, k=3$ *the parameters* $c=\pm\sqrt{-1mi}, p=-\frac{2}{3}\pm\frac{5}{9}i$ *are obtained.*

Finally, all obtained parameters and the solution of Eq. (3.2) are substituted in the ansatz the exact solution is hold, they are given in Figure 2, respectively.

As seen Figure 2, the solutions of Case 3.1 have soliton behavior, whereas for Case 3.2 the solutions are traveling wave solutions. Additionally, plot of Case 3.2 is given for surface so $\theta=1$.

3.2. **The exact solutions of R-GN equation.** In this section, instead of Lie symmetry transformations, the classical wave transformation $\zeta=x-mt$ is considered to reduce Eq. (1.3). For the exact solutions of the reduced equation, two analytical methods are used:

(1) Bernoulli approximation method [17, 20, 21] which is mentioned above. The solution sets are given below for $k=2$. Their plots are given in Figure 3.



(a) (b)

FIGURE 2. The solutions obtained via Lie symmetries for Case 3.1 and Case 3.2, respectively.

- Set 1

$$P(\zeta)=\frac{1}{\zeta},Q(\zeta)=\zeta,m=\frac{c\left(9\theta^2\left(c^2+1\right)^2-c^2\left(3c^2+16\right)\right)}{9\theta^2\left(c^2+1\right)^2-c^2\left(9c^2+22\right)-1},p=-\frac{5c^2-1}{9\left(c^2+1\right)^2},$$

$$\mu_1=9\zeta^2\left(c^2+1\right)^2\left(\left(6c-9\epsilon g_0\theta^2\right)\left(c^2+1\right)^2+\epsilon g_0\left(9c^4+22c^2+1\right)\right)^2$$

$$\begin{aligned}\mu_2=4\Big[&g_0\epsilon\left(9\theta^2\left(c^2+1\right)^2-c^2\left(9c^2-22\right)-1\right)\\&\times\left(9\theta^2\left(c^2+1\right)^2+c^2\left(3c^2-4\right)+11\right)\\&\times s\left(\left(4c-9\epsilon g_0\theta^2\right)\left(c^2+1\right)^2+\epsilon g_0\left(9c^4+22c^2+1\right)\right)\Big]\end{aligned}$$

$$\mu=\frac{\mu_1}{\mu_2}$$

$$g_1=\frac{g_0\zeta^2\left(\left(6c-\epsilon g_0\theta^2\right)\left(c^2+1\right)^2+\epsilon g_0\left(9c^4+22c^2+1\right)\right)}{2\left(\left(4c-9\epsilon g_0\theta^2\right)\left(c^2+1\right)^2+\epsilon g_0\left(9c^4+22c^2+1\right)\right)},$$

- Set 2

$$P(\zeta)=\frac{1}{\zeta},Q(\zeta)=\zeta,m=\frac{c\left(9\theta^2\left(c^2+1\right)^2-c^2\left(3c^2+16\right)\right)}{9\theta^2\left(c^2+1\right)^2-c^2\left(9c^2+22\right)-1},p=-\frac{5c^2-1}{9\left(c^2+1\right)^2},$$

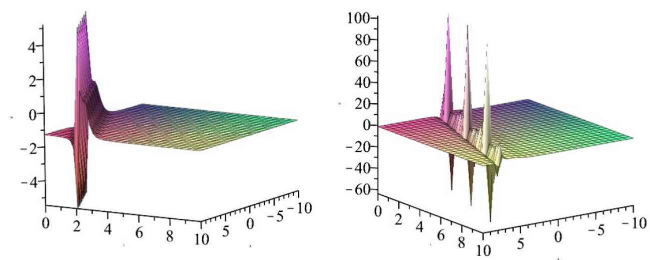
$$\mu = \frac{27\zeta^2\left(c^2+1\right)^2\left(9\theta^2\left(c^2+1\right)^2-c^2\left(3c^2-13\right)+5+9\theta^2\right)^2}{4\left(9\theta^2\left(c^2+1\right)^2-c^2\left(9c^2+22\right)-1\right)\left(9\theta^2\left(c^2+1\right)^2+c^2\left(3c^2-4\right)+11\right)^2},$$

$$g_0 = \frac{6c\left(c^2+1\right)^2}{\epsilon\left(18\theta^2\left(c^2+1\right)^2-c^2\left(12c^2+35\right)+4\right)},$$

$$g_1 = \frac{9c\zeta^2\left(\left(9c^4\left(6+c^4\right)+9+36c^2\left(1+c^4\right)\right)\theta^2+5-c^2\left(3+24c^2+19c^4+3c^6\right)\right)}{\epsilon\left(18\theta^2\left(c^2+1\right)^2+4-c^2\left(12c^2+35\right)\right)\left(9\theta^2\left(c^2+1\right)^2+c^2\left(3c^2-4\right)+11\right)}$$

.

In addition, the given solution sets, the trivial solutions are obtained from the algebraic system.



(a) (b)

FIGURE 3. The solutions obtained via Bernoulli approximation method at the surface and bottom of the fluid, respectively.

(2) Chebyshev approximation method [22] is considered that its auxiliary equation is Chebyshev differential equation

$$\left(1-\zeta^2\right)z''(\zeta)-\zeta z'(\zeta)+n^2z(\zeta)=0$$

,

with the transformation $\omega = cos\zeta$ reducing Eq.(6) to $z''(\omega)+n^2z(\omega)=0$ and it has a solution $T_n(\omega)$ known as Chebyshev function. The solution sets are given below.

- Set 1

$$m = \frac{c\left(\left(9\mu n^2\theta^2-18\right)\left(c^2+1\right)^2\right)-5\mu n^2\left(c^4+4c^2+1\right)}{\left(9\mu n^2\theta^2-18\right)\left(c^2+1\right)^2-\mu n^2\left(13c^4+28c^2+5\right)},$$

$$p = -\frac{\left(3\mu n^2\theta^2-6\right)\left(c^2+1\right)^2+\mu n^2\left(c^4+12c^2+1\right)}{24\mu n^2\left(c^2+1\right)^2},$$

$$g_0 = 0,$$

$$g_1 = \frac{\omega_1}{\epsilon\omega_2\left(C_1 sin(n\zeta)+C_2 cos(n\zeta)\right)}$$

- Set 2

$$m=\frac{c\left(9\theta^2\left(c^2+1\right)^2-9c^4-28c^2-13\right)}{9\theta^2\left(c^2+1\right)^2-21c^4-40c^2-13},$$

$$g_0=-\frac{4c\left(c^2+1\right)^2}{\epsilon\left(9\theta^2\left(c^2+1\right)^2-21c^4-40c^2-13\right)},$$

$$\begin{aligned}g_{11}=4\Big[&4\omega_1\left(c^2+1\right)^5+\left(567-243\theta^2\right)c^9+\left(-972\theta^2+2214\right)c^7\\&+\left(-1458\theta^2+3078\right)c^5+\left(-972\theta^2+1782\right)c^3+\left(-243\theta^2+351\right)c\Big]\end{aligned}$$

$$\begin{aligned}g_{12}=9\epsilon\Big[&\left(C_1 sin(n\zeta)+C_2 cos(n\zeta)\right)\Big(81\theta^4\left(c^2+1\right)^4\\&-\theta^2\left(234+1188c^2+2052c^4+1476c^6+378c^8\right)+169+2146c^4\\&+1040c^2+441c^8+1680c^6\Big)\Big]\end{aligned}$$

$$g_1=-\frac{g_{11}}{g_{12}}$$

- Set 3

$$\begin{aligned}m_1=\Big[&\mu n^2\left(6c\left(1+3c^2\left(c^2+1\right)c^6\right)-9\epsilon g_0c^2\left(c^2+1\right)^2\right]c^7\left(12pn^2\mu-12-2n^2\mu\right)\\&\times c^6\epsilon g_0\left(-36pn^2\mu+6\omega_2\epsilon^2g_0^2-3\mu n^2-18\right)\\&+c^5\left(36pn^2\mu-36-6n^2\mu\right)+c^4\epsilon g_0\left(-72pn^2\mu+18\omega_2\epsilon^2g_0^2-16\mu n^2-36\right)\\&+c^3\left(36pn^2\mu-36-6n^2\mu\right)+c^2\epsilon g_0\left(-36pn^2\mu+18\omega_2\epsilon^2g_0^2-7\mu n^2-18\right)\\&+c\left(12pn^2\mu-12-2n^2\mu\right)+6\omega_2\epsilon^3g_0^3,\end{aligned}$$

$$\begin{aligned}m_2=2\Big(&c^6\left(3\mu n^2\theta^2-6+6pn^2\mu-3n^2\mu\right)+c^4\left(9\mu n^2\theta^2-18+18pn^2\mu-7n^2\mu\right)\\&+c^2\left(9\mu n^2\theta^2-18+18pn^2\mu-5n^2\mu\right)+3\mu n^2\theta^2-6+6pn^2\mu-n^2\mu\Big),\end{aligned}$$

$$m=\frac{m_1}{m_2},$$

$$g_1=-\frac{\omega_1+3\epsilon g_0\omega_2}{\epsilon\omega_2\left(C_1 sin(n\zeta)+C_2 cos(n\zeta)\right)}$$

- Set4

$$
\begin{aligned}
\mu_1 = 2 \big(&\omega_2 c^{12} (c - 3m) + c^{11} \left(\omega_2 \left(5 + 3m^2\right) - 81\right) \\
&+ c^{10} m \left(\omega_2 \left(-15 - m^2\right) - 81\right) + c^9 \left(\omega_2 \left(10 + 15m^2\right) - 162\right) \\
&+ c^8 m \left(\omega_2 \left(-30 - 5m^2\right) + 162\right) + c^7 \left(\omega_2 \left(10 + 30m^2\right) - 81\right) \\
&+ c^6 m \left(\omega_2 \left(-30 - 10m^2\right) + 81\right) + c^5 \omega_2 \left(5 + 30m^2\right) \\
&- c^4 \omega_2 m \left(15 + 10m^2\right) + c^3 \omega_2 \left(1 + 15m^2\right) \\
&- c^2 \omega_2 m \left(3 + 5m^2\right) + m^2 \omega_2 (3c - m)
\end{aligned}
$$

$$
\begin{aligned}
\mu_2 = 3c^6 n^2 \big(&9c^5 \left(\theta^2 - 1\right) + c^4 m \left(-9\theta^2 + 21\right) + c^3 \left(18\theta^2 - 28\right) \\
&+ c^2 m \left(-18\theta^2 + 40 + c \left(9\theta^2 - 13\right) + m \left(-9\theta^2 + 13\right)\right)
\end{aligned}
$$

$$
\mu = -\frac{\mu_1}{\mu_2}
$$

$$
g_0 = \frac{(c - m) \left(c^2 + 1\right)}{3\epsilon c^2},
$$

$$
g_1 = -\frac{\omega_1 c^2 + \omega_2 (c - m)(c^2 + 1)}{\epsilon \omega_2 c^2 \left(C_1 sin(n\zeta) + C_2 cos(n\zeta)\right)}
$$

As it is seen that the new solutions of Eq. (1.3) are obtained via various considered methods. In case of the problem is the initial/boundary value problem, the parameters can be defined. The physical part of the obtained solutions is open to research.

4. Conclusion

In this article, group transformations and exact solutions of the rotation-Green-Naghdi equations with the Coriolis effect have been achieved. As can be seen from the solution sets, new exact solutions are obtained with two of the various modifications of the sub-equation method in the literature, and the physical interpretation of the exact solutions depends on the initial / boundary conditions. Due to the expected behavior of the solutions, these two modifications of the sub-equation method are considered.

The exact solutions of the model have been examined for the first time in the literature. The results that can be easily verified in this article relate to general geophysics and coastal engineering. Therefore, the discovery of new mathematical results can have a significant impact on future research in the relevant field.

References

[1] Y. Y. Bagderina and A. P. Chupakhin, *Invariant and Partially Invariant Solutions of The GreenNaghdi Equations*, Journal of Applied Mechanics and Technical Physics, *46*(6) (2005), 791799.

[2] R. M. Chen, G. Gui, and Y. Liu, *On a shallow-water approximation to the GreenNaghdi equations with the Coriolis effect*, Advances in Mathematics, *340* (2018), 106137.

[3] A. Constantin, *On the modelling of equatorial waves*, Geophys. Res. Lett., *39* (2012), L05602.

[4] R. C. Ertekin, M. Hayatdavoodi, and J.W. Kim, *On some solitary and cnoidal wave diffraction solutions of the GreenNaghdi equations* , Applied Ocean Research, *47* (2014), 125137.

[5] W. Gao, G. Yel, H. M. Baskonus, and C. Cattani, *Complex solitons in the conformable (2+1)-dimensional Ablowitz-Kaup-Newell-Segur equation*, AIMS Mathematics, *5*(1) (2020), 507521.

[6] W. Gao, H. F. Ismael,H. Bulut, and H. M. Baskonus, *Instability modulation for the (2+1)-dimension paraxial wave equation and its new optical soliton solutions in Kerr media*, Physica Scripta, *95* (2019), 035207 . .

[7] A.E. Green and P.M. Naghdi, *A derivation of equations for wave propagation in water of variable depth*, J. Fluid Mech, *78* (1976), 237246.

[8] A.E. Green and P.M. Naghdi, *On the theory of water waves*, Proc R Soc Lond Ser AMath Phys Sci, *338* (1974), 4355.

[9] G. Gui, Y. Liu, and T. Luo, *Model Equations and Traveling Wave Solutions for Shallow-Water Waves with the Coriolis Effect*, Journal of Nonlinear Science, *29* (2019), 9931039.

[10] G. Guirao, H. M. Baskonus A. Kumar, M. S. Rawat, and G. Yel, *Complex Patterns to the (3+1)-Dimensional B-type Kadomtsev-Petviashvili-Boussinesq Equation*, Symmetry, *12* (2020). DOI:10.3390/sym12010017.

[11] N. H. Ibragimov, *Elementary Lie Group Analysis and Ordinary Differential Equations*, Chichester: John Wiley and Sons, (1999).

[12] S. Lie, *Geometrie der Berhrungstransformationen*, Leipzig: B. G. Teubner, (1896), Reprinted by Chelsea Publishing Company, New York, 1977.

[13] S. Lie, *Theorie der Transformationsgruppen I, II and III* , Leipzig: B. G. Teubner (1888), Reprinted by Chelsea Publishing Company, New York, 1970.

[14] J.P. Olver,*Applications of Lie Groups to Differential Equations (2nd ed.)*. New York: Springer, (1993).

[15] L.A. Ostrovsky, *Nonlinear internal waves in a rotating ocean*, Okeanologia,*18* (1978), 181191.

[16] L.V. Ovsiannikov, *Group Analysis of Differential Equations*, Moscow: Nauka (1978). English translation, Ames, W. F., Ed., published by Academic Press, New York, 1982.

[17] Z. Pinar and H. Kocak,*Exact solutions for the third-order dispersive-Fisher equations*, Nonlinear Dyn, *91* (2018), 421426.

[18] Z. Pnar and T. zis,*An observation on the periodic solutions to nonlinear physical models by means of the auxiliary equation with a sixth degree nonlinear term* , Communications in Nonlinear Science and Numerical Simulation, *18(8)* (2013), 2177-2187.

[19] Z. Pnar and T. zis, *Classical symmetry analysis and exact solutions for generalized Kortewegde Vries models with variable coefficients*, International Journal of Non-Linear Mechanics, *105* (2018), 99104.

[20] Z. Pinar,*Simulations of Surface Corrugations of Graphene Sheets via the Generalized Graphene Thermophoretic Motion Equation*, International Journal of Computational Materials Science and Engineering,*9(1)* (2020), 20050005. DOI:10.1142/S2047684120500050.

[21] Z. Pinar, *The Combination of Conservation Laws and Auxiliary Equation Method* , International Journal of Applied and Computational Mathematics ,*6*(1) (2020). DOI:10.1007/s40819-019- 0764-2.

[22] Z. Pinar, *The symmetry analysis of electrostatic micro-electromechanical system (MEMS)*, Modern Physics Letters B, *34(18)* (2020), 2050199. DOI:10.1142/S0217984920501997.

[23] F. Serre, *Contribution ltude des coulements permeanents et variables ands les canaux*, Houille Blanche, *3* (1953), 374388.

[24] P. Siriwata and S. V. Meleshko, *Group properties of the extended GreenNaghdi equations*, Applied Mathematics Letters, *81* (2018), 16.

[25] Y. Wang and Y. Yang, *Solitary vortex dynamics of two-dimensional harmonically trapped Bose-Einstein condensates with higher-order nonlinear interactions*, AIP Advances, *8* (2018), 095317.

[26] Y. Wang, Q. Chen, J. Guo, and W. Wang, *Sonic horizon dynamics for quantum systems with cubic-quintic-septic nonlinearity* , AIP Advances, *9* (2019) , 075206.

[27] Y. Wang, S. Li, J. Guo, Y. Zhou, Q. Zhou, and W. Wen, *Analytical solution and Soliton-like behavior for the (1+1)-dimensional quantum system with generalized cubic-quintic nonlinearity*, International Journal of Bifurcation and Chaos, *26* (2016), 1650195.

[28] Y. Wang, Y. Yang, S. He, and W. Wang, *Dynamic evolution of vortex solitons for coupled Bose-Einstein condensates in harmonic potential trap* , AIP Advances,7 (2017), 105209.