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Non-uniform L1/DG method for one-dimensional time-fractional convection equation

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Abstract In this paper, we present an efficient numerical method to solve a one-dimensional time-fractional convection equation whose solution has a certain weak regularity at the starting time, where the time-fractional derivative in the Caputo sense with order in $(0, 1)$ is discretized by the L1 finite difference method on non-uniform meshes and the spatial derivative by the discontinuous Galerkin (DG) finite element method. The stability and convergence of the method are analyzed. Numerical experiments are provided to confirm the theoretical results.

Keywords. Time-fractional convection equation, L1 scheme, Discontinuous Galerkin method, Stability and convergence.

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1. Introduction

In recent years, fractional calculus has drawn the interest of many researchers due to its wide applications in science and engineering, such as physics, chemistry, materials, biology and finance (for more details, see [9, 10, 17]). As is known, the analytic solutions of the fractional differential equations (FDEs) are not easy to obatin, so developing efficient numerical methods for FDEs is of necessity and importance.

In this article, we study the non-uniform L1/discontinuous Galerkin (DG) method for solving one-dimensional time-fractional convection equation [11]

$$\begin{cases} {}_C\mathrm{D}_{0,t}^{\alpha} u + \gamma u_x = g(x,t),\ (x,t) \in \Omega \times (0,T], \\ u(x,0) = u_0(x),\ x \in \Omega, \\ u(a,t) = 0,\ t \in (0,T], \end{cases} \tag{1.1}$$

where $\Omega = (a, b)$ is a bounded domain, $\gamma \neq 0$ is a given constant, the source term $g(x,t) \in L^\infty(0,T; L^2(\Omega))$ and initial value $u_0(x) \in L^2(\Omega)$ are given functions, and ${}_C\mathrm{D}_{0,t}^{\alpha}$ is the αth-order Caputo derivative operator defined by

$${}_C\mathrm{D}_{0,t}^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial u}{\partial s} \mathrm{d}s,\ 0 < \alpha < 1, \tag{1.2}$$

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in which $\Gamma(z) := \int_0^\infty s^{z-1}e^{-s}\mathrm{d}s$ denotes the Gamma function.

A vast amount of literature can be found on the numerical approximation of the convection-diffusion(-reaction) equations of fractional order, e.g., [1, 5, 6, 7]. As for the pure convection equation, the investigation on this kind of equations with integer order has a long-term history, while the development of this kind of equations with fractional order is very slow and just the beginning. Very recently, Li et al. [12] derived fractional convection diffusion equations to model anomalous convection process. Shortly after, they constructed various numerical approximations to fractional convection equation in [14]. However, the study of fractional convection equations is far from complete.

The DG method is a class of finite element methods using discontinuous piecewise polynomials as the solution and the test spaces. In the past few years, some interesting papers are concerned with the DG method for FDEs. Mustapha and McLean [18] proposed an effective numerical method to solve an evolution equation with a memory term based on the DG method in time and continuous piecewise-linear finite elements in space. Wei and He [21] presented a fully discrete local DG scheme for the time-fractional fourth-order problems, where the time fractional derivative was approximated by the L1 method on uniform meshes. Du et al. [4] proposed a fully discrete local DG scheme for the nonlinear time-fractional partial differential equation with fourth-order spacial derivative. A DG method for the time discretization of the time-fractional Cable equation was presented in [23]. Huang et al. [8] investigated the time-fractional reaction-diffusion problem by applying the L1 discretisation on non-uniform meshes in time and a direct DG method in space.

The typical solution of time-fractional differential equation often has weak (or low) regularity at the starting (ending) point for the left (right) fractional derivative. In all most all this situations, fractional derivative conventionally means the left one. For example, in Eq. (1.1), $u(\cdot,t)$ can be continuous at $[0,T]$ for a given T, but $\frac{\partial u(\cdot,t)}{\partial t}$ very likely blows up when $t \to 0^+$. To deal with such a problem, many efforts have been made to develop effective numerical methods, such as the corrected method [13, 22], L1 method on non-uniform meshes [8, 16, 20], and so on. The objective in this paper is to present a methodology for the time-fractional convection equation with weak regular solution. The Caputo time-fractional derivative is discretized by the L1 finite difference method on non-uniform meshes and the spatial derivative by the DG method. The stability and convergence of the method are analyzed.

The rest of this paper is outlined as follows. In section 2, we present the non-uniform L1/DG method for the one-dimensional time-fractional convection equation and give a detailed proof of the L^2-stability and optimal error estimate for the fully discrete scheme. In section 3, we provide the illustrative numerical experiments which support the theoretical analyses. Concluding remarks are given in the last section.

2. The non-uniform L1/DG method

In this section, we consider the non-uniform L1/DG method for the one-dimensional time-fractional convection equation (1.1).

2.1. **Notations, definitions and projections.** In order to define the non-uniform L1/DG scheme for Eq. (1.1), we first use $\mathcal{I}_h$ to denote a tessellation of the interval

$\overline{\Omega}$ $=[a,b]$, consisting of cells $I_i=(x_{i-\frac{1}{2}},x_{i+\frac{1}{2}})$, for $1\le i\le N$, where $a=x_{\frac{1}{2}}<x_{\frac{3}{2}}<\cdots<x_{N+\frac{1}{2}}=b$. The cell center and cell length are denoted by $x_i=(x_{i-\frac{1}{2}}+x_{i+\frac{1}{2}})/2$ and $h_i=x_{i+\frac{1}{2}}-x_{i-\frac{1}{2}}$, respectively. We use $h=\max_i h_i$ to represent the length of the largest cell. The associated finite element space is defined as

$$V_h=\left\{v\in L^2(\Omega): v|_{I_i}\in\mathcal{P}^k(I_i), i=1,\ldots,N\right\},$$

where $\mathcal{P}^k(I_i)$ denotes the set of polynomials of degree up to $k\ge 0$ defined on the cell I_i. As usual, we use $u^-|_{x_{i+\frac{1}{2}}}$ and $u^+|_{x_{i+\frac{1}{2}}}$ to represent the values of u at the discontinuity point $x_{i+\frac{1}{2}}$, from the left cell I_i, and from the right cell I_{i+1}, respectively. The jump of u at each element boundary point is denoted by $[\![u]\!]_{x_{i+\frac{1}{2}}}=u^+|_{x_{i+\frac{1}{2}}}-u^-|_{x_{i+\frac{1}{2}}}$.

Define the L^2-inner product over the interval $I_i\,(i=1,\ldots,N)\subset\Omega$ and the associated norm by

$$\|v\|_{I_i}^2=(v,v)_{I_i},\ (u,v)_{I_i}=\int_{I_i}uv\mathrm{d}x.$$

Summing over all the elements, we denote

$$\|v\|_{\Omega}^2=\sum_{i=1}^{N}\|v\|_{I_i}^2,\ (u,v)_{\Omega}=\sum_{i=1}^{N}(u,v)_{I_i}.$$

For any nonnegative integer m, denote by $\|\cdot\|_{H^m(I_i)}$ the standard Sobolev norm on the cell I_i. Then the broken Sobolev space $H^m(\mathcal{I}_h)$ is given by [19]

$$H^m(\mathcal{I}_h):=\left\{v\in L^2(\Omega):\ v|_{I_i}\in H^m(I_i),\ \forall i=1,\ldots,N\right\},$$

and is endowed with the following norm

$$\|v\|_{H^m(\mathcal{I}_h)}=\left(\sum_{i=1}^{N}\|v\|_{H^m(I_i)}^2\right)^{\frac{1}{2}}.$$

Now we define two kinds of Gauss-Radau projections $\mathbb{P}_h^{\pm}$ of a given function $q\in H^1(\Omega)$ into the finite element space V_h, which were introduced by Castillo et al. [2], i.e., for each i,

$$\int_{I_i}\left(\mathbb{P}_h^+q(x)-q(x)\right)v_h\mathrm{d}x=0,\ \forall v_h\in\mathcal{P}^{k-1}(I_i),\ (\mathbb{P}_h^+q)^+|_{x_{i-\frac{1}{2}}}=q^+|_{x_{i-\frac{1}{2}}},\tag{2.1}$$

and

$$\int_{I_i}\left(\mathbb{P}_h^-q(x)-q(x)\right)v_h\mathrm{d}x=0,\ \forall v_h\in\mathcal{P}^{k-1}(I_i),\ (\mathbb{P}_h^-q)^-|_{x_{i+\frac{1}{2}}}=q^-|_{x_{i+\frac{1}{2}}}.\tag{2.2}$$

Let $\Pi=q(x)-\mathbb{Q}_hq(x)$ with $\mathbb{Q}_h=\mathbb{P}_h^{\pm}$ defined in (2.1) and (2.2). Then a standard scaling argument as that in [3] yields

$$\|\Pi\|_{\Omega}+h\|\Pi_x\|_{\Omega}+h^{\frac{1}{2}}\|\Pi\|_{\Gamma_h}\le Ch^{k+1}\|\Pi\|_{H^{k+1}(\mathcal{I}_h)},\tag{2.3}$$

where C is a positive constant independent of h, and

$$\|\Pi\|_{\Gamma_h}^2=\sum_{i=1}^{N}\left((\Pi^+|_{x_{i-\frac{1}{2}}})^2+(\Pi^-|_{x_{i+\frac{1}{2}}})^2\right).$$

2.2. **Fully discrete non-uniform L1/DG scheme.** In [11], the (possible) weak regularity of the solution at the starting time is not considered. As compensation, here we study the case of the weak regularity of the solution at the initial time. Let $t_n = T(n/M)^r$, $n = 0, 1, \ldots, M$ be the mesh points, $r \geq 1$. Denote $\tau_n = t_n - t_{n-1}$, $n = 1, \ldots, M$ be the time mesh sizes. If $r = 1$, then the mesh is just uniform.

The L1 approximation on the non-uniform meshes to the Caputo derivative is given as follows,

$$\begin{aligned} {}_C\mathrm{D}^{\alpha}_{0,t}u|_{t=t_n} &= \frac{1}{\Gamma(1-\alpha)}\int_0^{t_n}(t_n-s)^{-\alpha}\frac{\partial u}{\partial s}\mathrm{d}s \\ &= \frac{1}{\Gamma(2-\alpha)}\Bigg[d_{n,1}u(x,t_n)+\sum_{i=1}^{n-1}(d_{n,i+1}-d_{n,i})u(x,t_{n-i}) \\ &\quad -d_{n,n}u(x,t_0)\Bigg]+Q^n. \end{aligned} \tag{2.4}$$

Here Q^n is the truncation error and $d_{n,i} = \frac{(t_n-t_{n-i})^{1-\alpha}-(t_n-t_{n-i+1})^{1-\alpha}}{\tau_{n-i+1}}$. In particular, $d_{n,1} = \tau_n^{-\alpha}$. Besides, the coefficients $d_{n,i}$ have the following properties,

$$d_{n,i+1} \leq d_{n,i}, \text{ for } 1 \leq i \leq n-1, \tag{2.5}$$

and

$$(1-\alpha)(t_n-t_{n-i})^{-\alpha} < d_{n,i} < (1-\alpha)(t_n-t_{n-i+1})^{-\alpha}, \text{ for } 1 \leq i \leq n. \tag{2.6}$$

Although the solution $u(x,t)$ to (1.1) can be continuous at the interval $[0,T]$, $\dfrac{\partial u(x,t)}{\partial t}$ and/or $\dfrac{\partial^2 u(x,t)}{\partial t^2}$ most likely blow(s) up (that is, approach(es) to ∞) as $t \to 0^+$ for $\alpha \in (0,1)$. That is to say, $u(x,t)$ has weak regularity at the starting time $t = 0$. Throughout this paper, we assume that there exists a unique solution $u(x,t)$ of Eq. (1.1) such that

$$\left|\frac{\partial^l u(x,t)}{\partial t^l}\right| \leq C(1+t^{\alpha-l}),\ l = 0,1,2. \tag{2.7}$$

Such assumptions are reasonable, for example, see [20] for details. And they have often been used, e.g., [15]. Then with the above assumptions, we have the following lemma.

Lemma 2.1 ([20])**.** *Assume that (2.7) holds for any fixed x. Then there exists a constant C such that for all t_n*

$$|Q^n| \leq Cn^{-\min\{2-\alpha, r\alpha\}},\ n = 1, \ldots, M.$$

Now we return to Eq. (1.1). Denote $u^n = u(x,t_n)$ and

$$\Upsilon^{\alpha}_t u^n = \frac{1}{\Gamma(2-\alpha)}\left[d_{n,1}u^n + \sum_{i=1}^{n-1}(d_{n,i+1}-d_{n,i})u^{n-i} - d_{n,n}u^0\right], n = 1, \ldots, M. \tag{2.8}$$

The weak form of (1.1) at t_n is formulated as

$$({}_C\mathrm{D}^{\alpha}_{0,t}u^n, v)_\Omega - (\gamma u^n, v_x)_\Omega + \sum_{i=1}^{N}\Big((\gamma u^n v)^-|_{x_{i+\frac{1}{2}}} - (\gamma u^n v)^+|_{x_{i-\frac{1}{2}}}\Big)$$
$$= (g^n, v)_\Omega. \tag{2.9}$$

Here $v \in V_h$ is an arbitrary test function.

Let $u_h^n \in V_h$ be the approximation of u^n. Then we can define the fully discrete non-uniform L1/DG scheme as follows: find $u_h^n \in V_h$ such that for all test function $v_h \in V_h$, it holds that

$$(\Upsilon_t^\alpha u_h^n, v_h)_\Omega - \big(\gamma u_h^n, (v_h)_x\big)_\Omega + \sum_{i=1}^{N}\Big((\gamma \widehat{u}_h^n v_h)^-|_{x_{i+\frac{1}{2}}} - (\gamma \widehat{u}_h^n v_h)^+|_{x_{i-\frac{1}{2}}}\Big)$$
$$= (g^n, v_h)_\Omega. \tag{2.10}$$

The hat terms in (2.10) obtained from integration by parts are the so-called "numerical fluxes", which are single-valued functions defined on the edges. The freedom in choosing numerical fluxes can be utilized for designing a scheme that enjoys certain stability properties. It turns out that if $\gamma > 0$, we can take the numerical flux as

$$\widehat{u}_h^n = (u_h^n)^- \tag{2.11}$$

on each cell interface. If $\gamma < 0$, we can take the numerical flux as

$$\widehat{u}_h^n = (u_h^n)^+ \tag{2.12}$$

on each cell interface.

Without loss of generality, we consider the case $\gamma > 0$ with flux choice (2.11) in its numerical analysis. The cases of $\gamma < 0$ can be considered in a same manner so is left out here.

2.3. **Stability analysis.** The non-uniform L1/DG scheme using the numerical flux (2.11) for the one-dimensional time-fractional convection equation satisfies the following L^2-stability.

Lemma 2.2. *The solution u_h^n of the non-uniform L1/DG method defined by (2.10) satisfies*

$$\|u_h^n\|_\Omega \le \|u_h^0\|_\Omega + \tau_n^\alpha \Gamma(2-\alpha)\sum_{j=1}^{n}\theta_{n,j}\,\|g^j\|_\Omega,\ n = 1, \dots, M, \tag{2.13}$$

where $\theta_{n,n} = 1,\ \theta_{n,j} = \displaystyle\sum_{i=1}^{n-j}\tau_{n-i}^\alpha(d_{n,i} - d_{n,i+1})\theta_{n-i,j},\ j = 1, \dots, n-1.$

Proof. Taking the test functions $v_h = u_h^n$ in the scheme (2.10) and integrating by parts, we obtain

$$(\Upsilon_t^\alpha u_h^n, u_h^n)_\Omega + \frac{1}{2}\gamma\sum_{i=1}^{N}[\![u_h^n]\!]^2_{x_{i-\frac{1}{2}}} = (g^n, u_h^n)_\Omega,$$

which implies

$$(\Upsilon_t^\alpha u_h^n, u_h^n)_\Omega \leq (g^n, u_h^n)_\Omega, \tag{2.14}$$

By definition (2.8) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\|u_h^n\|_\Omega \quad &\leq \quad \tau_n^\alpha \Bigg[d_{n,n}\, \|u_h^0\|_\Omega + \sum_{i=1}^{n-1} (d_{n,i} - d_{n,i+1}) \|u_h^{n-i}\|_\Omega \\ &\qquad + \Gamma(2-\alpha) \|g^n\|_\Omega \Bigg]. \end{aligned} \tag{2.15}$$

Now, we prove this lemma by mathematical induction. When $n = 1$, (2.15) becomes

$$\|u_h^1\|_\Omega \leq \tau_1^\alpha \left(d_{1,1}\, \|u_h^0\|_\Omega + \Gamma(2-\alpha) \|g^1\|_\Omega \right), \tag{2.16}$$

which is true.

Supposing the following estimates hold

$$\|u_h^m\|_\Omega \leq \|u_h^0\|_\Omega + \tau_m^\alpha \Gamma(2-\alpha) \sum_{j=1}^{m} \theta_{m,j}\, \|g^j\|_\Omega,\ m = 2, \ldots, s, \tag{2.17}$$

we only need to prove

$$\|u_h^{s+1}\|_\Omega \leq \|u_h^0\|_\Omega + \tau_{s+1}^\alpha \Gamma(2-\alpha) \sum_{j=1}^{s+1} \theta_{s+1,j}\, \|g^j\|_\Omega.$$

Letting $n = s+1$ in (2.15) and using (2.17), we have

$$\begin{aligned}\|u_h^{s+1}\|_\Omega \quad &\leq \quad \tau_{s+1}^\alpha \Bigg[d_{s+1,s+1}\, \|u_h^0\|_\Omega + \sum_{i=1}^{s} (d_{s+1,i} - d_{s+1,i+1}) \|u_h^{s+1-i}\|_\Omega \\ &\qquad + \Gamma(2-\alpha) \|g^{s+1}\|_\Omega \Bigg] \\ &\leq \quad \tau_{s+1}^\alpha \Bigg[d_{s+1,s+1}\, \|u_h^0\|_\Omega + \Gamma(2-\alpha) \|g^{s+1}\|_\Omega + \sum_{i=1}^{s} (d_{s+1,i} - d_{s+1,i+1}) \\ &\qquad \times \Big(\|u_h^0\|_\Omega + \tau_{s+1-i}^\alpha\, \Gamma(2-\alpha) \sum_{j=1}^{s+1-i} \theta_{s+1-i,j} \|g^j\|_\Omega \Big) \Bigg] \\ &= \quad \tau_{s+1}^\alpha \Bigg[d_{s+1,1}\, \|u_h^0\|_\Omega + \Gamma(2-\alpha) \|g^{s+1}\|_\Omega \\ &\qquad + \Gamma(2-\alpha) \sum_{i=1}^{s} \Big(\tau_{s+1-i}^\alpha (d_{s+1,i} - d_{s+1,i+1}) \sum_{j=1}^{s+1-i} \theta_{s+1-i,j}\, \|g^j\|_\Omega \Big) \Bigg] \\ &= \quad \|u_h^0\|_\Omega + \tau_{s+1}^\alpha \Gamma(2-\alpha) \sum_{j=1}^{s+1} \theta_{s+1,j}\, \|g^j\|_\Omega. \end{aligned}$$

The proof is thus completed. $\square$

Lemma 2.3 ([20]). *Let the parameter η satisfies $\eta \leq r\alpha$. Then for $n = 1, 2, \dots, M$, one has*

$$\tau_n^\alpha \sum_{j=1}^{n} j^{-\eta}\theta_{n,j} \leq \frac{T^\alpha M^{-\eta}}{1-\alpha}.$$

Theorem 2.4. *The solution u_h^n to the scheme (2.10) satisfies the L^2-stability*

$$\|u_h^n\|_\Omega \leq \|u_h^0\|_\Omega + T^\alpha \Gamma(1-\alpha) \max_{1\leq j\leq n} \left\|g^j\right\|_\Omega, n = 1, \dots, M. \tag{2.18}$$

Proof. It follows from Lemma 2.3 that

$$\tau_n^\alpha \sum_{j=1}^{n} \theta_{n,j} \leq \frac{T^\alpha}{1-\alpha}.$$

Combining the above inequality with Lemma 2.2, we immediately get the desired inequality (2.18). □

2.4. **Error estimate.** Now we show the optimal error estimate of the fully discrete non-uniform L1/DG scheme (2.10) with flux (2.11) for the time-fractional convection equation (1.1).

Theorem 2.5. *Let $u(x, t_n)$ be the exact solution of (1.1) and u_h^n be the numerical solution of the fully discrete non-uniform L1/DG scheme (2.10), respectively. Suppose that $u(x, t)$ satisfies the temporal regularity assumption (2.7) and $u(\cdot, t) \in H^{k+1}(\mathcal{I}_h)$. Then, it holds that*

$$\|u(x, t_n) - u_h^n\|_\Omega \leq C\left(M^{-\min\{2-\alpha, r\alpha\}} + h^{k+1}\right), \tag{2.19}$$

where C is a positive constant independent of M and h.

Proof. Denote

$$e_u^n = u(x, t_n) - u_h^n = \mathbb{P}_h^- e_u^n + \left(u(x, t_n) - \mathbb{P}_h^- u^n\right) := \xi_u^n + \eta_u^n. \tag{2.20}$$

Subtracting (2.10) from (2.9), we obtain the error equation

$$\begin{aligned} &\left({}_C\mathrm{D}_{0,t}^\alpha u(x, t_n) - \Upsilon_t^\alpha u_h^n, v_h\right)_\Omega - \left(\gamma e_u^n, (v_h)_x\right)_\Omega + \sum_{i=1}^{N}\Big(\gamma(e_u^n)^-(v_h)^-|_{x_{i+\frac{1}{2}}} \\ &-\gamma(e_u^n)^-(v_h)^+|_{x_{i-\frac{1}{2}}}\Big) = 0. \end{aligned} \tag{2.21}$$

Substituting (2.20) into (2.21), one has

$$\begin{aligned} &(\Upsilon_t^\alpha \xi_u^n, v_h)_\Omega - \gamma\left(\xi_u^n, (v_h)_x\right)_\Omega + \sum_{i=1}^{N} \gamma\left((\xi_u^n)^- v_h^-|_{x_{i+\frac{1}{2}}} - (\xi_u^n)^- v_h^+|_{x_{i-\frac{1}{2}}}\right) \\ &= -(Q^n, v_h)_\Omega - (\Upsilon_t^\alpha \eta_u^n, v_h)_\Omega + \gamma\left(\eta_u^n, (v_h)_x\right)_\Omega - \gamma\sum_{i=1}^{N}\Big((\eta_u^n)^- v_h^-|_{x_{i+\frac{1}{2}}} \\ &\quad -(\eta_u^n)^- v_h^+|_{x_{i-\frac{1}{2}}}\Big). \end{aligned} \tag{2.22}$$

Taking the test function $v_h = \xi_u^n$ in (2.22) and using (2.2) and the Cauchy-Schwarz inequality, we arrive at

$$\begin{aligned}&(\Upsilon_t^\alpha \xi_u^n, \xi_u^n)_\Omega + \frac{1}{2}\gamma \sum_{i=1}^{N} [\![\xi_u^n]\!]^2_{x_{i-\frac{1}{2}}} \\ &\quad = -(Q^n, \xi_u^n)_\Omega - (\Upsilon_t^\alpha \eta_u^n, \xi_u^n)_\Omega \le \|Q^n\|_\Omega \|\xi_u^n\|_\Omega + \|\Upsilon_t^\alpha \eta_u^n\|_\Omega \|\xi_u^n\|_\Omega.\end{aligned} \tag{2.23}$$

Then in view of the definition of operator Υ_t^α and Lemma 2.1, we obtain

$$\|\xi_u^n\|_\Omega \le \tau_n^\alpha \left[\sum_{i=1}^{n-1} (d_{n,i} - d_{n,i+1}) \|\xi_u^{n-i}\|_\Omega + C\left(n^{-\min\{2-\alpha, r\alpha\}} + h^{k+1}\right) \right]. \tag{2.24}$$

Now we prove that the interpolation error ξ_u^n satisfies

$$\|\xi_u^n\|_\Omega \le C\tau_n^\alpha \sum_{j=1}^{n} \theta_{n,j} \left(h^{k+1} + j^{-\min\{2-\alpha, r\alpha\}}\right),\ n = 1, \dots, M, \tag{2.25}$$

where C is the constant in (2.24). We prove (2.25) by mathematical induction.

When $n = 1$, (2.24) is

$$\|\xi_u^1\|_\Omega \quad \le \quad C\tau_1^\alpha (1 + h^{k+1}).$$

It is true.

Suppose that the following inequality holds,

$$\|\xi_u^m\|_\Omega \le C\tau_m^\alpha \sum_{j=1}^{m} \theta_{m,j} \left(h^{k+1} + j^{-\min\{2-\alpha, r\alpha\}}\right),\ m = 2, \dots, s. \tag{2.26}$$

We need prove that

$$\|\xi_u^{s+1}\|_\Omega \le C\tau_{s+1}^\alpha \sum_{j=1}^{s+1} \theta_{s+1,j} \left(h^{k+1} + j^{-\min\{2-\alpha, r\alpha\}}\right).$$

Let $n = s+1$ in (2.24) and use the induction hypothesis (2.26). One has

$$
\begin{aligned}
\|\xi_u^{s+1}\|_\Omega \quad &\leq \quad \tau_{s+1}^\alpha \left[\sum_{i=1}^{s} (d_{s+1,i} - d_{s+1,i+1}) \|\xi_u^{s+1-i}\|_\Omega + C\big(h^{k+1} + (s+1)^{-\min\{2-\alpha, r\alpha\}}\big) \right] \\
&\leq \quad \tau_{s+1}^\alpha \left[\sum_{i=1}^{s} (d_{s+1,i} - d_{s+1,i+1}) \Big(C\tau_{s+1-i}^\alpha \sum_{j=1}^{s+1-i} \theta_{s+1-i,j} \big(h^{k+1} \right. \\
&\qquad \left. + j^{-\min\{2-\alpha, r\alpha\}}\big)\Big) + C\big(h^{k+1} + (s+1)^{-\min\{2-\alpha, r\alpha\}}\big) \right] \\
&= \quad \tau_{s+1}^\alpha \left[\sum_{j=1}^{s} C\big(h^{k+1} + j^{-\min\{2-\alpha, r\alpha\}}\big) \right. \\
&\qquad \times \left(\sum_{i=1}^{s+1-j} \tau_{s+1-i}^\alpha (d_{s+1,i} - d_{s+1,i+1}) \theta_{s+1-i,j} \right) \\
&\qquad \left. + C\big(h^{k+1} + (s+1)^{-\min\{2-\alpha, r\alpha\}}\big) \right] \\
&= \quad C\tau_{s+1}^\alpha \sum_{j=1}^{s+1} \theta_{s+1,j} \big(h^{k+1} + j^{-\min\{2-\alpha, r\alpha\}}\big).
\end{aligned}
$$

This ends the proof of (2.25).

Hence, applying Lemma 2.3 and (2.25) gives

$$
\begin{aligned}
\|\xi_u^n\|_\Omega &\leq C\tau_n^\alpha \sum_{j=1}^{n} \theta_{n,j}\, h^{k+1} + C\tau_n^\alpha \sum_{j=1}^{n} \theta_{n,j}\, j^{-\min\{2-\alpha, r\alpha\}} \\
&\leq \frac{CT^\alpha}{1-\alpha} \left(h^{k+1} + M^{-\min\{2-\alpha, r\alpha\}} \right).
\end{aligned}
\tag{2.27}
$$

Finally, Theorem 2.5 follows from the triangle inequality and the interpolating property (2.3). □

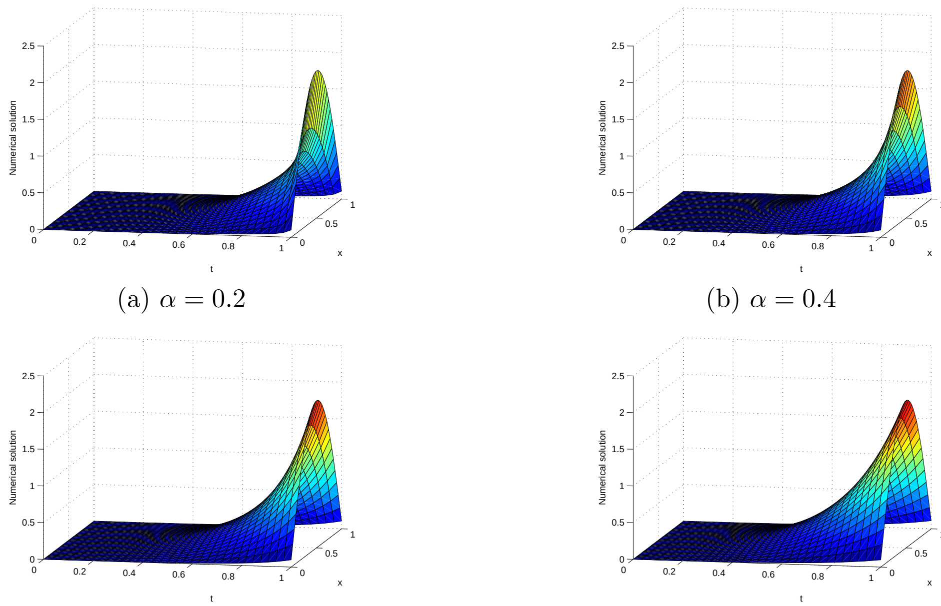
Remark 2.6. Even though the proof in this paper is presented only for time-fractional convection equation (1.1) with the Dirichlet boundary conditions on the inflow boundary (i.e., $x = a$). The same optimal convergence results can be obtained for (1.1) with the periodic boundary conditions in a similar way.

3. Numerical examples

The purpose of this section is to numerically validate the error estimate of the non-uniform L1/DG method for one-dimensional time-fractional convection equation.

Example 3.1. Consider the following one-dimensional time-fractional convection equation,

$$
\begin{cases}
{}_C\mathrm{D}_{0,t}^\alpha u + u_x = g(x,t),\ (x,t) \in (0,1) \times (0,1],\ \alpha \in (0,1), \\
u(x,0) = 0,\ x \in (0,1), \\
u(0,t) = 0,\ t \in (0,1],
\end{cases}
\tag{3.1}
$$



(a) $\alpha = 0.2$ (b) $\alpha = 0.4$

(c) $\alpha = 0.6$ (d) $\alpha = 0.8$

FIGURE 1. The numerical solutions for Example 3.1 with different α. $\mathcal{P}^1$ elements and non-uniform meshes with $M = N = 32$ cells.

where

$$g(x,t) = \left(\Gamma(\alpha+1) + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}\right)\sin(\pi x) + \pi(t^2 + t^\alpha)\cos(\pi x).$$

The exact solution is $u(x,t) = (t^2 + t^\alpha)\sin(\pi x)$.

Obviously, the exact solution of Eq. (3.1) has weak regularity at the initial time due to the factor t^α. ($\dfrac{\partial u(x,t)}{\partial t}$ and $\dfrac{\partial^2 u(x,t)}{\partial t^2}$ blow up as $t \to 0^+$ albeit $u(x,t)$ is continuous in $[0,T]$ for a given $T > 0$.) We test this example by scheme (2.10) with numerical flux (2.11). In Figure 1, the numerical solutions of $\alpha = 0.2$, 0.4, 0.6, 0.8 are displayed. We clearly see that the initial layer becomes sharper as α decreases. The L^2 errors and convergence rates in temporal direction with different parameters α and r are presented in Tables 1-3. These results indicate that the temporal accuracy is $\mathcal{O}(M^{-\min\{2-\alpha, r\alpha\}})$. In Tables 4 and 5, we take $t = 1$, $r = \frac{2-\alpha}{\alpha}$ and the errors in L^2 form are closed to $(k+1)$-th order of convergence in space, which agrees with the theoretical analysis.

For the discontinuous initial value condition, scheme (2.10) still works well. See the next example.

TABLE 1. Time convergence results for Example 3.1 using scheme (2.10), $M = N$, $k = 1$, $t = 1$, $r = 1$.

	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
M	L^2-error	Order	L^2-error	Order	L^2-error	Order
64	2.49e-02	-	1.17e-02	-	2.26e-03	-
128	1.97e-02	0.34	7.87e-03	0.58	2.26e-03	1.61
256	1.53e-02	0.36	5.23e-03	0.59	1.33e-03	0.77
512	1.18e-02	0.38	3.46e-03	0.60	7.70e-04	0.79
1024	9.03e-03	0.39	2.29e-03	0.60	4.45e-04	0.79

TABLE 2. Time convergence results for Example 3.1 using scheme (2.10), $M = N$, $k = 1$, $t = 1$, $r = \frac{2-\alpha}{\alpha}$.

	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
M	L^2-error	Order	L^2-error	Order	L^2-error	Order
64	7.32e-04	-	1.04e-03	-	2.08e-03	-
128	2.68e-04	1.45	4.18e-04	1.31	9.29e-04	1.16
256	9.50e-05	1.50	1.65e-04	1.34	4.11e-04	1.18
512	3.30e-05	1.53	6.40e-05	1.36	1.80e-04	1.19
1024	1.13e-05	1.55	2.46e-05	1.38	7.88e-05	1.19

TABLE 3. Time convergence results for Example 3.1 using scheme (2.10), $M = N$, $k = 1$, $t = 1$, $r = \frac{2(2-\alpha)}{\alpha}$.

	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
M	L^2-error	Order	L^2-error	Order	L^2-error	Order
64	2.17e-03	-	2.71e-03	-	4.46e-03	-
128	8.00e-04	1.44	1.09e-03	1.31	2.00e-03	1.16
256	2.84e-04	1.49	4.28e-04	1.35	8.82e-04	1.18
512	9.89e-05	1.52	1.66e-04	1.37	3.87e-04	1.19
1024	3.39e-05	1.55	6.37e-05	1.38	1.69e-04	1.19

TABLE 4. Spatial convergence results for Example 3.1 using scheme (2.10), $M = 500$, $k = 1$, $t = 1$, $r = \frac{2-\alpha}{\alpha}$.

	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
N	L^2-error	Order	L^2-error	Order	L^2-error	Order
4	3.81e-02	-	3.83e-02	-	3.85e-02	-
8	9.32e-03	2.03	9.32e-03	2.04	9.26e-03	2.05
16	2.28e-03	2.03	2.25e-03	2.05	2.17e-03	2.09
32	5.44e-04	2.07	5.19e-04	2.12	4.38e-04	2.31

Example 3.2. Consider the following one-dimensional time-fractional convection equation with discontinuous initial value condition,

$$\begin{cases} {}_C\mathrm{D}_{0,t}^{\alpha} u + u_x = f(x,t),\ (x,t) \in (0,1) \times (0,1],\ \alpha \in (0,1), \\ u(x,0) = \begin{cases} x^2(1-x)^2, & x \in (0, \frac{1}{2}), \\ -x^2(1-x)^2, & x \in [\frac{1}{2}, 1), \end{cases} \\ u(0,t) = 0,\ t \in (0,1], \end{cases}$$

Table 5. Spatial convergence results for Example 3.1 using scheme (2.10), $M = 10000$, $k = 2$, $t = 1$, $r = \frac{2-\alpha}{\alpha}$.

	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.8$	
N	L^2-error	Order	L^2-error	Order	L^2-error	Order
4	2.75e-03	-	2.74e-03	-	2.73e-03	-
8	3.51e-04	2.97	3.50e-04	2.97	3.47e-04	2.97
16	4.41e-05	2.99	4.37e-05	3.00	4.16e-05	3.06
32	5.40e-06	3.03	5.09e-06	3.10	5.17e-06	3.01

where the source term is given by

$$f(x,t) = \begin{cases} \left(\Gamma(\alpha+1) + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}\right) x^2(1-x)^2 \\ \quad +(t^2+t^\alpha+1)2x(1-x)(1-2x), \; x \in (0, \frac{1}{2}), \\ -\left(\Gamma(\alpha+1) + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}\right) x^2(1-x)^2 \\ \quad -(t^2+t^\alpha+1)2x(1-x)(1-2x), \; x \in [\frac{1}{2}, 1). \end{cases}$$

The exact solution of the above equation is

$$u = \begin{cases} (t^2+t^\alpha+1)x^2(1-x)^2, \;\; x \in (0, \frac{1}{2}), \\ -(t^2+t^\alpha+1)x^2(1-x)^2, \; x \in [\frac{1}{2}, 1). \end{cases}$$

We calculate this example by using the proposed numerical algorithm (2.10). Figure 2 shows the numerical solution agrees the exact solution with different values of α.

4. Concluding remarks

In this paper, we have studied the one-dimensional time-fractional convection equations. Considering the time-fractional partial differential equation often has weak regularity at the starting time, we use the non-uniform L1 scheme to discretize the time fractional derivative, and the DG method to discretize the space derivative. Then we prove that the established scheme is stable and convergent. Finally, several numerical examples are provided which are in line with the theoretical analysis. In a future work, we would extend the present method to deal with high-dimensional problems.

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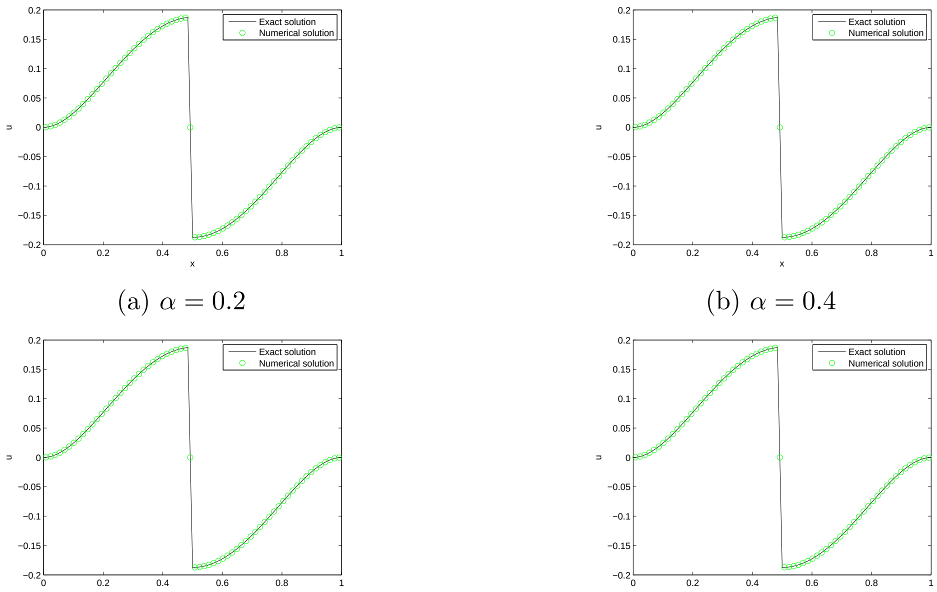

(a) $\alpha = 0.2$ (b) $\alpha = 0.4$

(c) $\alpha = 0.6$ (d) $\alpha = 0.8$

Figure 2. The comparison of exact and numerical solutions for Example 3.2 with different α at $t = 1$. $\mathcal{P}^1$ elements and non-uniform meshes with $M = N = 32$ cells.

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