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Radial basis functions method for nonlinear time- and space-fractional Fokker-Planck equation

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Abstract In this study, a radial basis functions (RBFs) method for solving nonlinear time- and space-fractional Fokker-Planck equation is presented. The time-fractional derivative is of the Caputo type, and the space-fractional derivatives are considered in the sense of Caputo or Riemann-Liouville. The Caputo and Riemann-Liouville fractional derivatives of RBFs are computed and utilized for approximating the spatial fractional derivatives of the unknown function. Also, in each time step, the time-fractional derivative is approximated by the high order formulas introduced in [6], and then a collocation method is applied. The centers of RBFs are chosen as suitable collocation points. Thus, in each time step, the computations of fractional Fokker-Planck equation are reduced to a nonlinear system of algebraic equations. Several numerical examples are included to demonstrate the applicability, accuracy, and stability of the method. Numerical experiments show that the experimental order of convergence is $4-\alpha$ where α is the order of time derivative.

Keywords. Fokker-Planck equation; Fractional derivative; Newton method; Radial basis functions.
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1. Introduction

In this article, we consider a type of nonlinear time- and space-fractional Fokker-Planck equation (FFPE)

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \left[-\frac{\partial^\beta}{\partial x^\beta} A\big(x,t,u(x,t)\big) + \frac{\partial^{2\beta}}{\partial x^{2\beta}} B\big(x,t,u(x,t)\big) \right] u(x,t) + f(x,t), \quad 0 \le x \le 1,\ t \ge 0, \tag{1.1}$$

with the initial condition

$$u(x,0) = g(x), \quad 0 \le x \le 1, \tag{1.2}$$

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$*$ corresponding.

and boundary conditions

$$u(0,t) = h_1(t), \tag{1.3}$$

$$u(1,t) = h_2(t), \tag{1.4}$$

where $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ denotes the Caputo time fractional derivative of order $\alpha \in (0,1)$. Also, $\frac{\partial^\beta u(x,t)}{\partial x^\beta}$ and $\frac{\partial^{2\beta} u(x,t)}{\partial x^{2\beta}}$ are either Caputo or Riemann-Liouville spatial fractional derivatives of orders β $(0.5 < \beta < 1)$ and 2β, respectively. Moreover, $h_1(t)$, $h_2(t)$ and $g(x)$ are known functions and $f(x,t)$ is the source term.

The FFPE arise in some important fields of Physics such as electromagnetic waves [3], dispersive transport [20], etc. In addition, this equation is applied in modeling of anomalous diffusive and sub-diffusive systems [4, 19, 28].

There is not any method for finding the exact solution of FFPEs. Several approximate and numerical methods for solving the FFPEs have been introduced. Chen et al. introduced a finite difference scheme for FFPE [8]. In [9], a method of lines for solving the time-FFPE was presented. Also, the homotopy perturbation method [30], a variational iterative method [21], and the iterative Laplas transform method [29] were applied for the time- and space-FFPE. For more studies, e. g., see [2, 15, 17, 25].

In most of the methods introduced for solving fractional PDEs, the finite difference and finite elements methods are applied for discretizing the fractional derivatives, while the fractional derivatives are non-local differential operators and so the non-local methods such as the radial basis functions (RBFs) method are more efficient for discretizing them. The RBFs methods are performed without any mesh generation and are efficient especially for solving problems with arbitrary geometry [22]. Furthermore, the RBFs methods are usually more accurate than low order methods, such as finite differences, finite volumes, and finite elements.

In Table 1, some well-known globally supported RBFs are presented. Let $x^* \in \mathbb{R}^d$ be a fixed point and $r = \|x - x^*\|_2$ for any $x \in \mathbb{R}^d$. A radial function $\phi^* = \phi(r)$ depends only on the distance between $x \in \mathbb{R}^d$ and fixed point $x^* \in \mathbb{R}^d$. Hence, the RBF ϕ^* is radially symmetric about x^*. Clearly, the functions in Table 1 are globally supported, infinitely differentiable and depend to a free parameter c which is called shape parameter. There are some strategies to choose the valuable shape parameter, although finding the optimal values of c which produce the most accurate interpolation is still an open problem. Some of these strategies are listed in Table 2.

Let $x_1, x_2, ..., x_M$ be a given set of distinct points in $\mathbb{R}^d$. The idea behind the use of

Table 1. Some well-known functions that generate RBFs.

Name of function	Definition
Gaussian (GS)	$\phi(r) = \exp(-cr^2)$
Hardy multiquadric (MQ)	$\phi(r) = \sqrt{1 + c^2 r^2}$
Inverse multiquadric (IMQ)	$\phi(r) = (\sqrt{1 + c^2 r^2})^{-1}$
Inverse quadric (IQ)	$\phi(r) = (1 + c^2 r^2)^{-1}$

RBFs is interpolation by translations of a single function i.e.

$$F(x) = \sum_{i=1}^{M} \lambda_i \phi_i(x), \tag{1.5}$$

where $\phi_i(x) = \phi(\|x - x_i\|)$ and λ_i are unknown scalars for $i = 1, ..., M$. The unknown scalars λ_i are found so that $F(x_j) = f_j$ for $j = 1, ..., M$. Thus, the following linear system of equations is obtained

$$Az = f, \tag{1.6}$$

where $A = [a_{i,j}]$ with $a_{i,j} = \phi_i(x_j)$ for $1 \leq i, j \leq M, z = [\lambda_1, ..., \lambda_M]^T$ and $f = [f_1, ..., f_M]^T$. For distinct interpolation points for GS, IMQ and IQ, the matrix A is positive definite, and therefore, nonsingular [27]. Moreover, the matrix A is usually very ill-conditioned i.e. the condition number of A

$$\kappa_s(A) = \|A\|_s \|A^{-1}\|_s, \quad s = 1, 2, \infty, \tag{1.7}$$

is a very large number. Therefore, we have to use more precision arithmetics than the standard floating point arithmetic in our computations.
In [5, 7, 16, 18], the authors showed that the interpolating of smooth data using global, infinitely differentiable RBFs has spectral accuracy. For more information, e. g., see [10, 31].

In this work, the high order difference formulas introduced in [6] are applied for discretizing on time variable. In each time step, the solution of Eqs. (1.1)-(1.4) is approximated by a linear combination of RBFs with unknown coefficients. To find the coefficients, these linear combinations and their fractional derivatives must be substituted in FFPE (1). So, the fractional derivatives of RBFs are computed and applied for approximating spatial fractional derivatives of unknown function. In each time step, using a collocation method the computations of FFPE are reduced to a nonlinear system of algebraic equations. These nonlinear systems can be solved by the Newton iteration method. Our method gives a closed form approximate solution, in each time step. The numerical examples show that the experimental order of convergence is $4 - \alpha$ where α is the order of time derivative.

The organization of the paper is as follows: In section 2, some basic definitions and theorems on the fractional calculus are presented, and the Newton iteration method for solving the systems of nonlinear algebraic equations is described . In section 3, the Caputo and Riemann-Liouville fractional derivatives of RBFs are obtained. In section 4, the solution of Eq. (1.1) by RBFs is considered. Section 5 is devoted to the numerical experiments.

2. Preliminaries

2.1. **Basic definitions and theorems.**

Table 2. Some common shape parameter strategies [11, 12, 13, 14, 26]

Name of strategies	$\mathrm{c_j},\ \ \mathrm{j} = 1, ..., \mathrm{M}$
EPS	$\mathrm{c_j} = \left(\mathrm{c}^2_{\mathrm{min}}\left(\frac{\mathrm{c}^2_{\mathrm{max}}}{\mathrm{c}^2_{\mathrm{min}}}\right)^{\frac{\mathrm{j}-1}{\mathrm{M}-1}}\right)^{\frac{1}{2}}$
ILSP	$\mathrm{c_j} = \mathrm{c_{min}} + (\frac{\mathrm{c_{max}}-\mathrm{c_{min}}}{\mathrm{M}-1})(\mathrm{j}-1)$
DLSP	$\mathrm{c_j} = \mathrm{c_{max}} + (\frac{\mathrm{c_{min}}-\mathrm{c_{max}}}{\mathrm{M}-1})(\mathrm{j}-1)$
SSP	$\mathrm{c_j} = \mathrm{c_{min}} + (\mathrm{c_{max}} - \mathrm{c_{min}})\mathrm{sin}\left(\frac{(\mathrm{j}-1)\pi}{2(\mathrm{M}-1)}\right)$
CR	$\mathrm{c_j} = \left(\mathrm{c}^3_{\mathrm{min}}\mathrm{c}^2_{\mathrm{max}}\frac{\mathrm{j}-1}{\mathrm{M}-1}\right)^{\frac{1}{3}}$
SR	$\mathrm{c_j} = \left(\mathrm{c}^3_{\mathrm{min}}\mathrm{c}^2_{\mathrm{max}}\frac{\mathrm{j}-1}{\mathrm{M}-1}\right)^{\frac{1}{2}}$
HSP	$\begin{cases} SSP_j, & j = 3k+1, \\ DLSP_j, & j = 3k+2,\ \ k = 0, ..., \lfloor \frac{M}{3} \rfloor \\ ESP_j, & j = 3k+3 \end{cases}$

Definition 2.1. The αth order Caputo fractional derivative of function $f(x)$ is defined as follows

$$D^{\alpha}_{C} f(x) = \begin{cases} \frac{1}{\Gamma(k-\alpha)} \int_0^x (x-\xi)^{k-1-\alpha} f^{(k)}(\xi) d\xi, & k-1 < \alpha < k,\ x > 0, \\ f^{(k)}(x), & \alpha = k \in \mathbb{N}. \end{cases} \tag{2.1}$$

Definition 2.2. The αth order Riemann-Liouville fractional derivative of function $f(x)$ is defined as

$$D^{\alpha}_{RL} f(x) = \begin{cases} \frac{1}{\Gamma(k-\alpha)} \frac{d^k}{dx^k} \int_0^x (x-\xi)^{k-1-\alpha} f(\xi) d\xi, & k-1 < \alpha < k,\ x > 0, \\ f^{(k)}(x), & \alpha = k \in \mathbb{N}. \end{cases} \tag{2.2}$$

Theorem 2.3. *The Caputo and Riemann-Liouville fractional derivatives are linear operators, i.e.* [23]

$$D^{\alpha}_{C}\Big(\lambda f(x) + g(x)\Big) = \lambda D^{\alpha}_{C} f(x) + D^{\alpha}_{C} g(x),$$

and

$$D^{\alpha}_{RL}\Big(\lambda f(x) + g(x)\Big) = \lambda D^{\alpha}_{RL} f(x) + D^{\alpha}_{RL} g(x),$$

where $\lambda \in \mathbb{C}$.

Theorem 2.4. *For the Caputo and Riemann-Liouville fractional derivatives, we have* [23]

$$D^{\alpha}_{C} K = 0, \tag{2.3}$$

and

$$D^{\alpha}_{RL} K = \frac{K}{\Gamma(1-\alpha)} x^{-\alpha} \neq 0, \tag{2.4}$$

where K is constant.

Theorem 2.5. *The Caputo and Riemann-Liouville fractional derivatives of the power functions satisfiy* [23]

$$D_C^{\alpha} x^p = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} x^{p-\alpha}, & n-1<\alpha<n,\ p>n-1,\ p\in\mathbb{R}, \\ 0, & n-1<\alpha<n,\ p\leq n-1,\ p\in\mathbb{N}, \end{cases} \tag{2.5}$$

and

$$D_{RL}^{\alpha} x^p = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} x^{p-\alpha},\ \ n-1<\alpha<n,\ p>-1,\ p\in\mathbb{R}. \tag{2.6}$$

Theorem 2.6. (Leibniz Rule) [23]
Let $\alpha\in\mathbb{R}$, $n-1<\alpha<n\in\mathbb{N}$ and $L>0$. If $f(x)$ and $g(x)$ and all its derivatives are continuous in $[0,L]$, then the following hold

$$\begin{aligned} D_C^{\alpha}\Big(f(x)g(x)\Big) &= \sum_{k=0}^{\infty}\binom{\alpha}{k}\Big(D_{RL}^{\alpha-k}f(x)\Big)D^k g(x) \\ &\quad -\sum_{k=0}^{n-1}\frac{x^{k-\alpha}}{\Gamma(k+1-\alpha)}\left(\Big(f(x)g(x)\Big)^{(k)}(0)\right), \end{aligned} \tag{2.7}$$

$$D_{RL}^{\alpha}\Big(f(x)g(x)\Big) = \sum_{k=0}^{\infty}\binom{\alpha}{k}\Big(D_{RL}^{\alpha-k}f(x)\Big)D^k g(x). \tag{2.8}$$

2.2. **Newton Iteration Method.** Consider the nonlinear system of equation [1]

$$F(X)=0, \tag{2.9}$$

where $F(X)=\big(F_1(X),...,F_n(X)\big)^T, F:D\to\mathbb{R}^n$, D convex subset of $\mathbb{R}^n$, $X\in\mathbb{R}^n$, and $F_i:D\to\mathbb{R}$ is continuously differentiable in an open neighborhood $D\subseteq\mathbb{R}^n$. For any initial vector X_0 close to X^*, where X^* is the exact solution of (2.9), Newton-Raphson method generates the sequence of vectors $\{X_k\}_{k=0}^{\infty}$ by using the following iterative scheme:

- Set an initial guess X_0.
- Compute $X_{k+1}=X_k-\Big(J(X_k)\Big)^{-1}F(X_k)$, $k=0,1,2,...$,

where $J(X)$ is the Jacobian matrix of $F(X)$.

3. The Caputo and Riemann-Liouville fractional derivatives of RBFs

In this section, we compute the Caputo and Riemann-Liouville fractional derivatives of the Gaussian (GS) and quadric RBFs.

3.1. **Fractional derivatives of GS-RBFs.** The Taylor series expansion of function GS about the point $x = x_j$ is as

$$e^{-c(x-x_j)^2} = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!}(x-x_j)^{2n} = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!} \sum_{k=0}^{2n} \binom{2n}{k}(-1)^k x_j^{2n-k} x^k. \tag{3.1}$$

So,

$$D_C^{\beta} e^{-c(x-x_j)^2} = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!} \sum_{k=0}^{2n} \binom{2n}{k}(-1)^k x_j^{2n-k} D_C^{\beta} x^k.$$

By the above equation and Eq. (2.5), the Caputo fractional derivative of function GS is obtained as

$$D_C^{\beta} e^{-c(x-x_j)^2} = \sum_{n=1}^{\infty} \frac{(-c)^n}{n!} \sum_{k=n'}^{2n} \binom{2n}{k}(-1)^k \; x_j^{2n-k} \frac{\Gamma(k+1)}{\Gamma(k+1-\beta)} x^{k-\beta}, \tag{3.2}$$

where $n' = [Re(\beta)] + 1$.
Similarly, by Eqs. (2.6) and (3.1), we can write

$$D_{RL}^{\beta} e^{-c(x-x_j)^2} = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!} \sum_{k=0}^{2n} \binom{2n}{k}(-1)^k x_j^{2n-k} \frac{\Gamma(k+1)}{\Gamma(k+1-\beta)} x^{k-\beta}. \tag{3.3}$$

3.2. **Fractional derivatives of the quadric functions.** Generally, we consider the quadric functions as

$$\phi(r) = (1 + c^2 r^2)^{\mu},$$

where $\mu = -1, -\frac{1}{2}$ and $\frac{1}{2}$, give IQ, IMQ and MQ functions, respectively.
The binomial series expansion of quadric functions is as

$$\begin{aligned}\Big(1 + c^2(x-x_j)^2\Big)^{\mu} &= 1 + \sum_{n=1}^{\infty} \binom{\mu}{n} \big(c(x-x_j)\big)^{2n} \\ &= 1 + \sum_{n=1}^{\infty} \binom{\mu}{n} \; c^{2n} \sum_{k=0}^{2n} \binom{2n}{k} \; (-1)^k \; x_j^{2n-k} \; x^k, \end{aligned} \tag{3.4}$$

for $-1 < c(x-x_j) < 1$. So,

$$\begin{aligned}&D_C^{\beta}\Big(1 + c^2(x-x_j)^2\Big)^{\mu} = \\ &D_C^{\beta} 1 + \sum_{n=1}^{\infty} \binom{\mu}{n} c^{2n} \sum_{k=0}^{2n} \binom{2n}{k}(-1)^k x_j^{2n-k} D_C^{\beta} x^k. \end{aligned} \tag{3.5}$$

By Eqs. (2.3), (2.5) and (3.5), the Caputo fractional derivative of the quadric functions is obtained as

$$D_C^{\beta}\Big(1+c^2(x-x_j)^2\Big)^{\mu} = \sum_{n=1}^{\infty}\binom{\mu}{n}c^{2n}\sum_{k=n'}^{2n}\binom{2n}{k}(-1)^k x_j^{2n-k}\frac{\Gamma(k+1)}{\Gamma(k+1-\beta)}x^{k-\beta}, \tag{3.6}$$

where $n' = [Re(\beta)] + 1$.
Similarly by Eqs. (2.4), (2.6) and (3.4), the Riemann-Liouville fractional derivative of the quadric functions is given as

$$D_{RL}^{\beta}\Big(1+c^2(x-x_j)^2\Big)^{\mu} = \frac{1}{\Gamma(1-\beta)}x^{-\beta} + \sum_{n=1}^{\infty}\binom{\mu}{n}c^{2n}\sum_{k=0}^{2n}\binom{2n}{k}(-1)^k x_j^{2n-k}\frac{\Gamma(k+1)}{\Gamma(k+1-\beta)}x^{k-\beta}. \tag{3.7}$$

4. Method of solution

First, we discretize equation (1.1) in the time direction, as

$$\frac{\partial^{\alpha}u^{n+1}}{\partial t^{\alpha}} = \Big[-\frac{\partial^{\beta}}{\partial x^{\beta}}A\big(x,t^{n+1},u^{n+1}\big) + \frac{\partial^{2\beta}}{\partial x^{2\beta}}B\big(x,t^{n+1},u^{n+1}\big)\Big]u^{n+1} + f^{n+1}, \tag{4.1}$$

where $u^{n+1} = u(x,t^{n+1})$, $f^{n+1} = f(x,t^{n+1})$, $t^n = n\tau$, $n = 0,1,...,N$, the time step τ, and the time length $N\tau$. The values of $\frac{\partial^{\alpha}u^{n+1}}{\partial t^{\alpha}}$ for $n=0$, $n=1$ and $n \geq 2$ are obtained as follows: [6]

$$\frac{\partial^{\alpha}u^{1}}{\partial t^{\alpha}} = \mu a_0(u^1-u^0) + O(\tau^{2-\alpha}), \tag{4.2}$$

$$\frac{\partial^{\alpha}u^{2}}{\partial t^{\alpha}} = \mu\Big[(b_0-a_1)u^0 + (a_1-a_0-2b_0)u^1 + (a_0+b_0)u^2\Big] + O(\tau^{3-\alpha}), \tag{4.3}$$

$$\frac{\partial^{\alpha}u^{n+1}}{\partial t^{\alpha}} = \mu\Big[(b_{n-1}-a_n)u^0 + (a_n-a_{n-1}-2b_{n-1})u^1 + (a_{n-1}+b_{n-1}) u^2 + \sum_{k=3}^{n}\Big(w_{1,n-k+1}u^k + w_{2,n-k+1}u^{k-1} + w_{3,n-k+1}u^{k-2} + w_{4,n-k+1} u^{k-3}\Big) + w_{1,0}u^{n+1} + w_{2,0}u^n + w_{3,0}u^{n-1} + w_{4,0}u^{n-2}\Big] + O(\tau^{4-\alpha}), \tag{4.4}$$

in which

$$\mu = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)},$$
$$a_i = (i+1)^{1-\alpha} - i^{1-\alpha},$$
$$b_i = \frac{(i+1)^{2-\alpha} - i^{2-\alpha}}{2-\alpha} - \frac{(i+1)^{1-\alpha} + i^{1-\alpha}}{2},$$
$$\begin{aligned} w_{1,n-k+1} =& \frac{1}{6}\Big[2(n-k+2)^{1-\alpha} - 11(n-k+1)^{1-\alpha}\Big] \\ &- \frac{1}{2-\alpha}\Big[2(n-k+1)^{2-\alpha} - (n-k+2)^{2-\alpha}\Big] \\ &- \frac{1}{(2-\alpha)(3-\alpha)}\Big[(n-k+1)^{3-\alpha} - (n-k+2)^{3-\alpha}\Big], \end{aligned}$$

$$\begin{aligned} w_{2,n-k+1} =& \frac{1}{2}\Big[6(n-k+1)^{1-\alpha} + (n-k+2)^{1-\alpha}\Big] \\ &+ \frac{1}{2-\alpha}\Big[5(n-k+1)^{2-\alpha} - 2(n-k+2)^{2-\alpha}\Big] \\ &+ \frac{3}{(2-\alpha)(3-\alpha)}\Big[(n-k+1)^{3-\alpha} - (n-k+2)^{3-\alpha}\Big], \\ w_{3,n-k+1} =& -\frac{1}{2}\Big[3(n-k+1)^{1-\alpha} + 2(n-k+2)^{1-\alpha}\Big] \\ &- \frac{1}{2-\alpha}\Big[4(n-k+1)^{2-\alpha} - (n-k+2)^{2-\alpha}\Big] \\ &- \frac{3}{(2-\alpha)(3-\alpha)}\Big[(n-k+1)^{3-\alpha} - (n-k+2)^{3-\alpha}\Big], \end{aligned}$$

and

$$\begin{aligned} w_{4,n-k+1} =& \frac{1}{6}\Big[2(n-k+1)^{1-\alpha} + (n-k+2)^{1-\alpha}\Big] \\ &+ \frac{1}{2-\alpha}(n-k+1)^{2-\alpha} \\ &+ \frac{1}{(2-\alpha)(3-\alpha)}\Big[(n-k+1)^{3-\alpha} - (n-k+2)^{3-\alpha}\Big]. \end{aligned}$$

By Eqs. (4.1)-(4.4), the following finite differences equations are obtained:

$$\mu u^1 + D_\gamma^\beta (Au)^1 - D_\gamma^{2\beta}(Bu)^1 - \mu u^0 - f^1 = 0, \tag{4.5}$$
$$\begin{aligned} &\mu(a_0+b_0)u^2 + D_\gamma^\beta (Au)^2 - D_\gamma^{2\beta}(Bu)^2 + \\ &\mu\Big[(b_0-a_1)u^0 + (a_1-a_0-2b_0)u^1\Big] - f^2 = 0, \end{aligned} \tag{4.6}$$

and

$$\mu w_{1,0}u^{n+1} + D_{\gamma}^{\beta}(Au)^{n+1} - D_{\gamma}^{2\beta}(Bu)^{n+1} + \mu\Bigg[(b_{n-1}-a_n)u^0 +$$
$$(a_n - a_{n-1} - 2b_{n-1})u^1 + (a_{n-1}+b_{n-1})u^2 + \sum_{k=3}^{n}\Big(w_{1,n-k+1}u^k +$$
$$w_{2,n-k+1}u^{k-1} + w_{3,n-k+1}u^{k-2} + w_{4,n-k+1}u^{k-3}\Big) + w_{2,0}u^n + w_{3,0}u^{n-1}$$
$$+ w_{4,0}u^{n-2}\Bigg] - f^{n+1} = 0, \quad n = 2,3,..., \tag{4.7}$$

where $\gamma = C$ or RL.
Now, using the radial basis functions, we consider the solution $u(x, t^{n+1})$ as follows

$$u^{n+1}(x) = \sum_{j=1}^{M}\lambda_j^{n+1}\phi(\|x - x_j\|), \ n = 0,1,2,..., \tag{4.8}$$

where $\lambda_j^{n+1}, j = 1, ..., M$ is unknown.

To construct the approximations for $u^1(x)$, first we substitute (4.8) in (1.3), (1.4) and (4.5). Then, we collocate the resulted equations. For suitable collocation points, we choose the centers, $x_i, i = 1, ..., M$ $(x_i = (i-1)\Delta x,\ \Delta x = \frac{1}{M-1})$, as collocation points. Thus, a nonlinear system of M equations in M unknowns is obtained as follows:

$$\begin{aligned}
&\tilde{F}_1^1 = \sum_{j=1}^{M}\lambda_j^1\phi(\|x_1 - x_j\|) - h_1(t^1) = 0,\\
&\tilde{F}_i^1 = \mu\sum_{j=1}^{M}\lambda_j^1\phi(\|x_i - x_j\|) +\\
&D_{\gamma}^{\beta}\left[A\Big(x_i, t^1, \sum_{j=1}^{M}\lambda_j^1\phi(\|x_i - x_j\|)\Big)\Big(\sum_{j=1}^{M}\lambda_j^1\phi(\|x_i - x_j\|)\Big)\right]\\
&-D_{\gamma}^{2\beta}\left[B\Big(x_i, t^1, \sum_{j=1}^{M}\lambda_j^1\phi(\|x_i - x_j\|)\Big)\Big(\sum_{j=1}^{M}\lambda_j^1\phi(\|x_i - x_j\|)\Big)\right]\\
&-\mu u_i^0 - f_i^1 = 0, \qquad i = 2, ..., M-1,\\
&\tilde{F}_M^1 = \sum_{j=1}^{M}\lambda_j^1\phi(\|x_M - x_j\|) - h_2(t^1) = 0,
\end{aligned} \tag{4.9}$$

where h_1 and h_2 are respectively the boundary conditions (1.3) and (1.4). Also, the fractional derivatives for Caputo case are obtained by Eq. (2.7) together with Eq. (3.2) (for GS-RBF) or Eq. (3.6) (for quadric-RBFs) and similarly, for Riemann-Liouville case by Eqs. (2.8), (3.3), and (3.7). By solving the system (4.9), $\lambda_j^1,\ j = 1, \ldots, M$ is computed. Similarly, $u^2(x)$ is obtained by substituting (4.8) in (1.3), (1.4) and (4.6), and using the collocation method with the same collocation points and solving the nonlinear system

$$\begin{aligned}
&\tilde{F}_1^2 = \sum_{j=1}^{M} \lambda_j^2 \phi(\|x_1 - x_j\|) - h_1(t^2) = 0,\\
&\tilde{F}_i^2 = \mu(a_0 + b_0) \sum_{j=1}^{M} \lambda_j^2 \phi(\|x_i - x_j\|) +\\
&D_\gamma^\beta \Bigg[A\Big(x_i, t^2, \sum_{j=1}^{M} \lambda_j^2 \phi(\|x_i - x_j\|)\Big)\Big(\sum_{j=1}^{M} \lambda_j^2 \phi(\|x_i - x_j\|)\Big)\Bigg] -\\
&D_\gamma^{2\beta} \Bigg[B\Big(x_i, t^2, \sum_{j=1}^{M} \lambda_j^2 \phi(\|x_i - x_j\|)\Big)\Big(\sum_{j=1}^{M} \lambda_j^2 \phi(\|x_i - x_j\|)\Big)\Bigg] +\\
&\mu\Big[(b_0 - a_1)u_i^0 + (a_1 - a_0 - 2b_0)u_i^1\Big] - f_i^2 = 0, \quad i = 2, \ldots, M-1,\\
&\tilde{F}_M^2 = \sum_{j=1}^{M} \lambda_j^2 \phi(\|x_M - x_j\|) - h_2(t^2) = 0.
\end{aligned} \tag{4.10}$$

Inductively, to obtain $u^{n+1}(x)$, $n = 2, 3, 4, \ldots$, first Eq. (4.7) is substituted in Eqs. (1.3), (1.4) and (4.8), and then the same technique is applied. These lead to the following nonlinear system

$$\begin{aligned}
&\tilde{F}_1^{n+1} = \sum_{j=1}^{M} \lambda_j^{n+1} \phi(\|x_1 - x_j\|) - h_1(t^{n+1}) = 0,\\
&\tilde{F}_i^{n+1} = \mu w_{1,0} \sum_{j=1}^{M} \lambda_j^{n+1} \phi(\|x_i - x_j\|) +\\
&D_\gamma^\beta \Bigg[A\Big(x_i, t^{n+1}, \sum_{j=1}^{M} \lambda_j^{n+1} \phi(\|x_i - x_j\|)\Big)\Big(\sum_{j=1}^{M} \lambda_j^{n+1} \phi(\|x_i - x_j\|)\Big)\Bigg] -\\
&D_\gamma^{2\beta} \Bigg[B\Big(x_i, t^{n+1}, \sum_{j=1}^{M} \lambda_j^{n+1} \phi(\|x_i - x_j\|)\Big)\Big(\sum_{j=1}^{M} \lambda_j^{n+1} \phi(\|x_i - x_j\|)\Big)\Bigg] +\\
&\mu\Bigg[(b_{n-1} - a_n)u_i^0 + (a_n - a_{n-1} - 2b_{n-1})u_i^1 + (a_{n-1} + b_{n-1})u_i^2 +\\
&\sum_{k=3}^{n}\Big(w_{1,n-k+1}u_i^k + w_{2,n-k+1}u_i^{k-1} + w_{3,n-k+1}u_i^{k-2} + w_{4,n-k+1}u_i^{k-3}\Big)\\
&+ w_{2,0}u_i^n + w_{3,0}u_i^{n-1} + w_{4,0}u_i^{n-2}\Bigg] - f_i^{n+1} = 0, \quad i = 2, ..., M-1,\\
&\tilde{F}_M^{n+1} = \sum_{j=1}^{M} \lambda_j^{n+1} \phi(\|x_M - x_j\|) - h_2(t^{n+1}) = 0.
\end{aligned} \tag{4.11}$$

We solve the resulted nonlinear systems by utilizing the Newton iteration method presented in Section 2.2, as follows: In $(n+1)$th, $n = 0, 1, 2, ...$ time step, the unknown vector, $\Lambda^{n+1} = \big[\lambda_1^{n+1}, \lambda_2^{n+1}, ..., \lambda_M^{n+1}\big]^T$, is given by

$$\Lambda_{k+1}^{n+1} = \Lambda_k^{n+1} - \Big(J^{n+1}(\Lambda_k^{n+1})\Big)^{-1} \tilde{F}^{n+1}(\Lambda_k^{n+1}), \qquad k = 0, 1, 2, ..., \tag{4.12}$$

with a suitable choice of Λ_0^{n+1}. In (4.12), $\tilde{F}^{n+1} = \left[\tilde{F}_1^{n+1}, \tilde{F}_2^{n+1}, ..., \tilde{F}_M^{n+1}\right]^T$ is a function of Λ^{n+1} which is given by (4.9), (4.11) and (**??**) for $n = 0$, $n = 1$ and $n \geq 2$, respectively.
By substituting the values of $\lambda_j^{n+1}, j = 1, 2, ..., M$ obtained by (4.12) in Eq. (4.8), the values of unknown function $u^{n+1}(x)$, $n = 0, 1, 2, ...$ are computed.

5. Numerical Examples

In this section, we solve three examples by the proposed method. The space-fractional derivative, in the first and second examples, is of Caputo type and in the third, is of Riemann-Liouville type. We use, $\delta = 100$ floating point arithmetics in our computations. Also, we solve the resulted nonlinear systems via corresponding Newton iteration method with the stop condition

$$\frac{\|\Lambda_{k+1} - \Lambda_k\|_\infty}{\|\Lambda_{k+1}\|_\infty} < 10^{-3}.$$

We calculate the errors and the experimental convergence order ($C - order$) by the following formulas

$$\begin{aligned} E_\infty &= \left\| u_{exact}(x, t^N) - u_{approx}(x, t^N) \right\|_\infty \\ &= \max_{1 \leq i \leq M} \left| u_{exact}(x_i, t^N) - u_{approx}(x_i, t^N) \right|, \\ E_2 &= \sqrt{\sum_{i=1}^{M} \Big(u_{exact}(x_i, t^N) - u_{approx}(x_i, t^N) \Big)^2}, \\ RMSE &= \sqrt{\frac{1}{M} \sum_{i=1}^{M} \Big(u_{exact}(x_i, t^N) - u_{approx}(x_i, t^N) \Big)^2}, \\ C - order &= log_2 \left(\frac{E_\infty(\Delta x, 2\tau)}{E_\infty(\Delta x, \tau)} \right). \end{aligned}$$

In practice, in the Eqs. (3.2), (3.3), (3.6) and (3.7) we have to put a positive integer "q" instead of "∞". We perform our computations with q=30 and by using **Maple 16** software. Also, in Eqs. (2.7) and (2.8), we put q'=3 instead of "∞".

5.1. **Caputo case.**

In this section, we will solve two examples in which the fractional space derivatives are of the Caputo type.

5.1.1. *Example 1.* Consider the nonlinear time- and space-FFPE (1.1) with $A\big(x, t, u(x, t)\big) = \dfrac{7}{2} u(x, t)$ and $B\big(x, t, u(x, t)\big) = u(x, t)$ as

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \left[-\frac{\partial^\beta}{\partial x^\beta} \Big(\frac{7}{2} u(x, t) \Big) + \frac{\partial^{2\beta}}{\partial x^{2\beta}} \Big(u(x, t) \Big) \right] u(x, t) + f(x, t),$$

with the initial and boundary conditions,

$$u(x,0) = 0,\ 0 \le x \le 1,$$
$$u(0,t) = t^5,\ t \ge 0,$$
$$u(1,t) = 2t^5,\ t \ge 0,$$

and

$$f(x,t) = \frac{120}{\Gamma(6-\alpha)}(x^2+1)t^{5-\alpha} + t^{10}\Big[\frac{84}{\Gamma(5-\beta)}x^{4-\beta} + \frac{14}{\Gamma(3-\beta)}x^{2-\beta} - \frac{24}{\Gamma(5-2\beta)}x^{4-2\beta} - \frac{4}{\Gamma(3-2\beta)}x^{2-2\beta}\Big].$$

The exact solution of this problem is $u(x,t) = t^5(x^2+1)$.

We have solved the problem by the present method using GS-RBFs, IQ-RBFs, MQ-RBFs and IMQ-RBFs. Table 3 depicts the E_∞ errors and the experimental convergence orders ($C-orders$) obtained by our method with $M = 16$ and DLSP strategy for the case $\alpha = 0.3$ and $\beta = 0.7$. Table 3 shows that the $C-order$ is approximately $4-\alpha$ and also as τ becomes smaller, the smaller errors are obtained. Moreover, the numerical approximations obtained by large number of iterations are very accurate and thus the method has a good stability.

In Table 4, we report the E_∞, E_2 and RMSE errors obtained by IQ-RBFs and HSP strategy with $c_{min} = 0.4$ and $c_{max} = 0.5$ for various values of α. Table 4 shows that as α becomes larger, the larger errors are obtained.

Table 5 presents the errors resulted by GS-RBFs and the SSP strategy for different values of β.

Table 6 depicts the errors and κ_∞(A) resulted by GS-RBFs and EPS strategy for various values of M. Table 6 shows that as the number of RBFs becomes larger the smaller errors are obtained, while the condition number of A becomes larger.

In Table 7, we list the E_∞ and RMSE errors and κ_2(A) resulted by IMQ-RBFs for several different strategies. Table 15 shows that for $c_{min} = 0.2$ and $c_{max} = 0.4$, EPS strategy gives the least error, although HSP strategy gives the least condition number. We plot the approximate solution and the absolute error function resulted by IQ-RBFs with $c_{min} = 0.35$, $c_{max} = 0.45$ and ILSP strategy in Fig. 1.

Table 3. The E_∞ and $C-order$ resulted by IMQ-RBFs and MQ-RBFs with $M = 16$ at $t = 1$ in Example 1 for $\alpha = 0.3$, $\beta = 0.7$ and various values of τ.

	IMQ ($c_{min} = 0.3,\ c_{max} = 0.4$)		MQ ($c_{min} = 0.35,\ c_{max} = 0.45$)	
τ	E_∞	$C-order$	E_∞	$C-order$
0.05	5.86652×10^{-6}	–	5.86735×10^{-6}	–
0.025	4.88206×10^{-7}	3.59	4.88278×10^{-7}	3.59
0.0125	3.95547×10^{-8}	3.63	3.95635×10^{-8}	3.63
0.00625	3.14875×10^{-9}	3.65	3.15241×10^{-9}	3.65
0.003125	2.42661×10^{-10}	3.70	2.45035×10^{-10}	3.69

Table 4. The E_∞, E_2 and RMSE errors using IQ-RBFs with $M = 19$, $c_{min} = 0.4$ and $c_{max} = 0.5$ in Example 1 for different values of α, $\beta = 0.65$ and $\tau = 0.0125$ at $t = 1$.

α	E_∞	E_2	RMSE
0.1	5.20578×10^{-9}	1.60702×10^{-8}	3.68676×10^{-9}
0.3	3.93740×10^{-8}	1.21321×10^{-7}	2.78330×10^{-8}
0.5	1.79679×10^{-7}	5.52454×10^{-7}	1.26742×10^{-7}
0.7	7.31865×10^{-7}	2.24455×10^{-6}	5.14936×10^{-7}
0.9	2.90753×10^{-6}	8.88856×10^{-6}	2.03918×10^{-6}

Table 5. The E_∞, E_2 and RMSE errors using GS-RBFs with $M = 21$, $c_{min} = 1$, $c_{max} = 4$ and $\tau = 0.02$ at $t = 1$ in Example 1 for $\alpha = 0.35$ and various values of β.

β	E_∞	E_2	RMSE
0.5	2.24477×10^{-7}	6.87836×10^{-7}	1.50098×10^{-7}
0.6	3.05273×10^{-7}	9.82755×10^{-7}	2.14455×10^{-7}
0.7	3.16133×10^{-7}	1.02708×10^{-6}	2.24128×10^{-7}
0.8	2.97399×10^{-7}	9.65551×10^{-7}	2.10701×10^{-7}
0.9	2.66172×10^{-7}	8.64019×10^{-7}	1.88544×10^{-7}
1	2.33202×10^{-7}	7.53738×10^{-7}	1.64479×10^{-7}

Table 6. The errors and condition number of matrix A resulted by GS-RBFs with $c_{min} = 1$ and $c_{max} = 4$ at $t = 1$ in Example 1 for $\tau = 0.005$, $\alpha = 0.15$ and $\beta = 0.75$.

M	E_∞	E_2	RMSE	$\kappa_\infty(\mathrm{A})$
9	2.0410755×10^{-3}	4.6131598×10^{-3}	1.5377199×10^{-3}	4.8254×10^{8}
13	4.7843795×10^{-5}	1.2844050×10^{-4}	3.5622986×10^{-5}	2.0748×10^{15}
17	1.1324385×10^{-7}	3.5413787×10^{-7}	8.5891050×10^{-8}	5.7438×10^{21}
21	$5.4410809 \times 10^{-10}$	1.8755918×10^{-9}	$4.0928769 \times 10^{-10}$	1.0987×10^{29}

5.1.2. *Example 2.* Consider [21, 24, 25]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \Big[-\frac{\partial^\beta}{\partial x^\beta}\Big(\frac{4u(x,t)}{x} - \frac{x}{3}\Big) + \frac{\partial^{2\beta}}{\partial x^{2\beta}}\Big(u(x,t)\Big)\Big] u(x,t),$$

with the initial and boundary conditions,

$$u(x,0) = x^2, \ 0 \le x \le 1,$$
$$u(0,t) = 0, \ t \ge 0,$$
$$u(1,t) = e^t, \ t \ge 0.$$

We apply the present method and solve the problem for the case $\alpha = \beta = 1$. In this case, the exact solution is $u(x,t) = x^2 e^t$. The solution of the above problem has been

TABLE 7. The errors and condition number of matrix A using IMQ-RBFs, $M = 13$, $c_{min} = 0.2$, $c_{max} = 0.4$, $\alpha = 0.5$, $\beta = 0.6$ and $\tau = 0.01$ at $t = 1$ in Example 1 for some different choices of the shape parameter.

Shape parameter	E_∞	RMSE	κ_2(A)
HSP	8.275332×10^{-8}	5.769436×10^{-8}	6.752×10^{18}
SSP	8.215845×10^{-8}	5.715384×10^{-8}	5.935×10^{20}
DLSP	8.326350×10^{-8}	5.813326×10^{-8}	2.909×10^{21}
ILSP	8.254839×10^{-8}	5.751901×10^{-8}	2.909×10^{21}
EPS	8.015564×10^{-8}	5.546930×10^{-8}	2.214×10^{22}
CR	8.270443×10^{-8}	5.765299×10^{-8}	4.806×10^{28}
SR	8.271487×10^{-8}	5.766176×10^{-8}	1.224×10^{37}
0.3	8.282991×10^{-8}	5.776044×10^{-8}	7.249×10^{24}
0.4	8.291937×10^{-8}	5.788042×10^{-8}	8.434×10^{21}
0.5	8.367091×10^{-8}	5.905132×10^{-8}	4.802×10^{19}

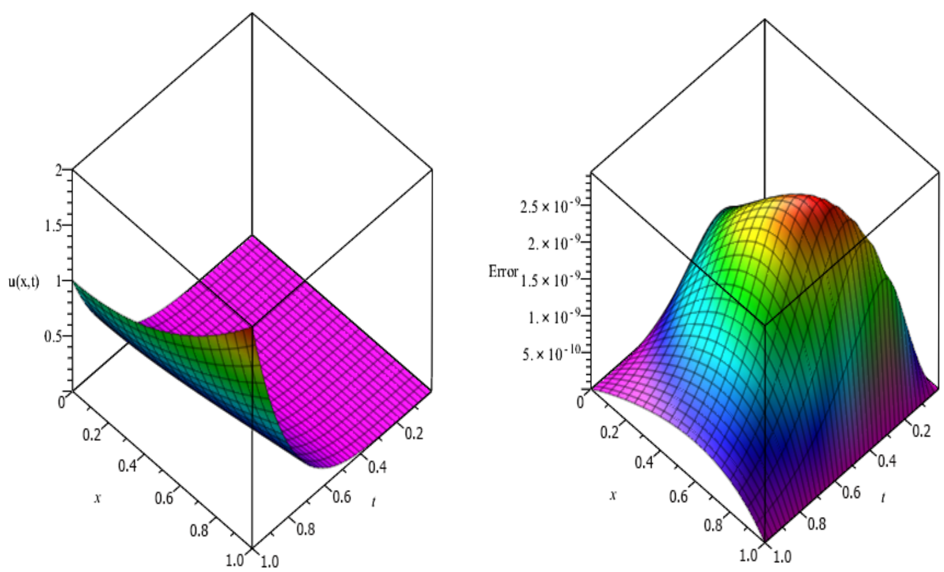

FIGURE 1. Plot of approximate solution (left) and plot of the absolute error function (right) obtained by IQ-RBFs with $M = 21$ and $\tau = 0.005$ at $t = 1$ in Example 1 for $\alpha = 0.2$ and $\beta = 0.75$.

approximated by Adomian method (ADM) in [21], by q-homotopy analysis method (q-HAM) in [24] and by both fractional reduced differential transform method (FRDTM) and fractional variational iteration method (FVIM) in [25]. However, their methods differ from our method and we can use this example as a basis for comparison. Table

8 presents the results of our method with $M = 5$ MQ-RBFs, $c = 0.15$ and $\tau = 0.02$, and the methods proposed in [21, 24, 25].
Tables 9 and 10 depict our results for different values of τ and various values of M, respectively. Table 9 shows that the $C - order$ is approximately equal to 3, that is $4 - \alpha$ for $\alpha = 1$.

Table 8. The comparison of numerical solutions obtained by the methods in [21, 24, 25] and the present method with the exact solution for $\alpha = 1$ and $\beta = 1$, in Example 2.

x	t	ADM [21]	q- HATM [24]	FRDTM [25]	FVIM [25]	MQ (M=5)	Exact
	0.2						
0.25		0.076333	0.0762	0.0762	0.0761	0.076346	0.076338
0.50		0.305333	0.3050	0.3050	0.3050	0.305361	0.305351
0.75		0.687000	0.6863	0.6863	0.6860	0.687049	0.687039
1		1.221333	1.2200	1.2200	1.2190	1.221403	1.221403
	0.4						
0.25		0.093167	0.0925	0.0925	0.0924	0.093242	0.093239
0.50		0.372667	0.3700	0.3700	0.3670	0.372960	0.372956
0.75		0.838500	0.8325	0.8325	0.8319	0.839159	0.839151
1		1.490667	1.4800	1.4800	1.4770	1.491825	1.491825

Table 9. The E_∞ error and $C - order$ for different values of τ resulted by MQ-RBFs and GS-RBFs with $M = 21$ at $t = 1$ in Example 2 for $\alpha = 1$ and $\beta = 1$.

	MQ ($c = 0.45$)		GS ($c = 1$)	
τ	E_∞	$C - order$	E_∞	$C - order$
0.04	4.814008×10^{-7}	–	4.813946×10^{-7}	–
0.02	6.218827×10^{-8}	2.95	6.218675×10^{-8}	2.95
0.01	8.021933×10^{-9}	2.95	8.021555×10^{-9}	2.95
0.005	1.048054×10^{-9}	2.94	1.047949×10^{-9}	2.94
0.0025	1.419136×10^{-10}	2.88	1.418750×10^{-10}	2.88

5.2. **Reimann-Liouville case.** In this section, we obtain the numerical solutions of a time- and space-FFPE with Riemann-Liouville space fractional derivative.

5.2.1. *Example 3.* Consider

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \Big[- \frac{\partial^\beta}{\partial x^\beta}\Big(\frac{1}{6}u(x,t)\Big) + \frac{\partial^{2\beta}}{\partial x^{2\beta}}\Big(\frac{1}{12}u(x,t)\Big)\Big]u(x,t) + f(x,t),$$

Table 10. The errors and condition number of matrix A for various values of M with IMQ-RBFs, $c = 0.65$ and $\tau = 0.0125$ at $t = 1$ in Example 2 for the case $\alpha = 1$ and $\beta = 1$.

M	E_∞	E_2	RMSE	κ_2(A)
6	4.10836×10^{-3}	4.96741×10^{-3}	2.02794×10^{-3}	6.3036×10^{6}
9	1.23098×10^{-4}	1.67750×10^{-4}	5.59167×10^{-5}	1.5984×10^{11}
12	6.13871×10^{-6}	9.75654×10^{-6}	2.81647×10^{-6}	4.2237×10^{15}
15	1.73399×10^{-7}	3.23292×10^{-7}	8.34736×10^{-8}	1.1170×10^{20}
18	1.93304×10^{-8}	5.82034×10^{-8}	1.37187×10^{-8}	2.9897×10^{24}
21	1.55920×10^{-8}	4.87087×10^{-8}	1.06291×10^{-8}	7.9815×10^{28}

with the initial and boundary conditions,

$$u(x,0) = 0,\ 0 \le x \le 1,$$
$$u(0,t) = 0,\ t \ge 0,$$
$$u(1,t) = t^{4+\alpha},\ t \ge 0,$$

and

$$f(x,t) = \frac{\Gamma(5+\alpha)}{\Gamma(5)} x^2 t^4 + \Big[\frac{4}{\Gamma(5-\beta)} x^{4-\beta} - \frac{2}{\Gamma(5-2\beta)} x^{4-2\beta}\Big] t^{8+2\alpha}.$$

The exact solution is $u(x,t) = t^{4+\alpha} x^2$.
In Table 11, we report the E_∞ errors and $C-order$ of the method using IMQ and GS-RBFs with SSP strategy for different values of τ. Table 11 shows that the $C-order$ is approximately $4-\alpha$ and, furthermore, the method has a good stability.
Table 12 presents the E_∞, E_2 and RMSE errors obtained by our method using IQ-RBFs and HSP strategy for various values of β. Table 12 shows that as β becomes larger, the smaller errors are obtained.
Table 13 depicts the E_∞, E_2 and RMSE errors resulted by MQ-RBFs and EPS strategy, for different values of α. Table 13 shows that as α becomes smaller, the more accurate approximations are obtained.
In Table 14, we list the E_∞, E_2, RMSE errors and κ_2(A) for different values of M. Table 14 shows that as the number of RBFs becomes larger the more accurate approximations are obtained, while the condition number of A becomes larger.
Table 15 presents the E_∞, E_2, RMSE errors and κ_∞(A) resulted by IQ-RBFs for several different strategies. As Table 15 shows, for $c_{min} = 0.4$ and $c_{max} = 0.5$, the least errors are obtained by EPS strategy while the least condition number is given by HSP strategy.
We plot the approximate solution and the absolute error function resulted by IMQ-RBFs with $c_{min} = 0.3$, $c_{max} = 0.5$ and SSP strategy in Fig. 2.

Acknowledgment

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TABLE 11. The E_∞ error and $C-order$ resulted by IMQ-RBFs and GS-RBFs with $M = 21$ and different values of τ for $\alpha = 0.2$ and $\beta = 0.65$ at $t = 1$ in Example 3.

	IMQ $(c_{min} = 0.3,\ c_{max} = 0.5)$		GS $(c_{min} = 1,\ c_{max} = 2)$	
τ	E_∞	$C-order$	E_∞	$C-order$
0.04	2.8019738×10^{-6}	–	2.8019737×10^{-6}	–
0.02	2.1311712×10^{-7}	3.72	2.1311685×10^{-7}	3.72
0.01	1.6019491×10^{-8}	3.73	1.6019194×10^{-8}	3.73
0.005	1.1938904×10^{-9}	3.75	1.1935909×10^{-9}	3.75
0.0025	$8.8402097 \times 10^{-11}$	3.76	$8.8102409 \times 10^{-11}$	3.76
0.00125	$6.5129368 \times 10^{-12}$	3.76	$6.2132373 \times 10^{-12}$	3.83

TABLE 12. The E_∞, E_2 and RMSE errors using IQ-RBFs with $M = 16$, $c_{min} = 0.4$, $c_{max} = 0.6$ and $\tau = 0.02$ for $\alpha = 0.3$ and various values of β at $t = 1$ in Example 3.

β	E_∞	E_2	RMSE
0.5	7.08412×10^{-7}	1.52831×10^{-6}	3.82078×10^{-7}
0.6	6.18446×10^{-7}	1.38747×10^{-6}	3.46867×10^{-7}
0.7	5.24731×10^{-7}	1.23437×10^{-6}	3.08591×10^{-7}
0.8	4.36537×10^{-7}	1.07849×10^{-6}	2.69623×10^{-7}
0.9	3.58089×10^{-7}	9.26213×10^{-7}	2.31553×10^{-7}
1	2.92288×10^{-7}	7.84399×10^{-7}	1.96100×10^{-7}

TABLE 13. The E_∞, E_2 and RMSE errors using MQ-RBFs with $M = 21$, $c_{min} = 0.2$, $c_{max} = 0.5$ and $\tau = 0.01$ for various values of α and $\beta = 0.7$ at $t = 1$ in Example 3.

α	E_∞	E_2	RMSE
0.1	3.87806×10^{-9}	1.09294×10^{-8}	2.38499×10^{-9}
0.3	4.16981×10^{-8}	1.13520×10^{-7}	2.47721×10^{-8}
0.5	2.48802×10^{-7}	6.51157×10^{-7}	1.42094×10^{-7}
0.7	1.20715×10^{-6}	3.02197×10^{-6}	6.59447×10^{-7}
0.9	5.04103×10^{-6}	1.22246×10^{-5}	2.66762×10^{-6}

6. CONCLUSION

A new method for solving FFPE using RBFs is proposed. The equation is solved for $0 \leq x \leq 1$. So, for the quadric RBFs, the shape parameter c must be selected in the interval $[0, 1]$. Several well-known strategies introduced in the literature are used to find the suitable shape parameter. The Caputo and Riemann-Liouville fractional derivatives of RBFs are obtained to approximate the spatial derivatives of the unknown function. For discretizing on time variable, the high order formulas introduced

Table 14. The errors and condition number of matrix A using GS-RBFs, $c_{min} = 1.5$, $c_{max} = 3$ and $\tau = 0.01$ at $t = 1$ in Example 3 for $\alpha = 0.15$ and $\beta = 0.55$.

M	E_∞	E_2	RMSE	κ_2(A)
7	3.33134×10^{-4}	3.34419×10^{-4}	1.26399×10^{-4}	1.1371×10^{6}
10	1.99662×10^{-4}	2.03376×10^{-4}	6.43132×10^{-5}	8.0484×10^{10}
13	5.29971×10^{-6}	5.62392×10^{-6}	1.55980×10^{-6}	4.6552×10^{14}
15	4.50182×10^{-7}	4.89021×10^{-7}	1.26265×10^{-7}	7.2938×10^{18}
18	1.90638×10^{-8}	3.49428×10^{-8}	8.23611×10^{-9}	1.5893×10^{24}
21	1.03165×10^{-8}	2.67146×10^{-8}	5.82960×10^{-9}	5.2082×10^{29}

Table 15. The errors and condition number of matrix A resulted by IQ-RBFs with $M = 16$, $c_{min} = 0.4$ and $c_{max} = 0.5$ at $t = 1$ in Example 3 for $\tau = 0.0125$, $\alpha = 0.45$ and $\beta = 0.7$.

Shape parameter	E_∞	E_2	RMSE	κ_∞(A)
HSP	3.563947×10^{-7}	8.191995×10^{-7}	2.047999×10^{-7}	1.1321×10^{22}
SSP	3.565245×10^{-7}	8.199086×10^{-7}	2.049771×10^{-7}	1.5952×10^{23}
DLSP	3.563872×10^{-7}	8.191643×10^{-7}	2.047911×10^{-7}	5.8925×10^{24}
ILSP	3.563901×10^{-7}	8.191866×10^{-7}	2.047966×10^{-7}	5.8925×10^{24}
EPS	3.563551×10^{-7}	8.189983×10^{-7}	2.047496×10^{-7}	1.0917×10^{25}
CR	3.563954×10^{-7}	8.191915×10^{-7}	2.047979×10^{-7}	8.3336×10^{28}
SR	3.563954×10^{-7}	8.192106×10^{-7}	2.048026×10^{-7}	1.5524×10^{35}
0.3	3.563945×10^{-7}	8.192008×10^{-7}	2.048002×10^{-7}	3.8404×10^{30}
0.4	3.564091×10^{-7}	8.192834×10^{-7}	2.048208×10^{-7}	8.1911×10^{26}
0.45	3.564623×10^{-7}	8.195701×10^{-7}	2.048925×10^{-7}	2.6610×10^{25}
0.5	3.566473×10^{-7}	8.205630×10^{-7}	2.051407×10^{-7}	1.2694×10^{24}
0.6	3.587414×10^{-7}	8.320082×10^{-7}	2.080021×10^{-7}	6.9984×10^{21}

in [5] are applied. At each time step, by using a collocation method, the computations of FFPE are reduced to a system of nonlinear algebraic equations. These systems can be solved by the Newton iteration method. Our method is relatively simple and computationally attractive, although selecting a proper shape parameter c as well as choosing a suitable initial guess Λ_0^{n+1} may be somewhat difficult. The present method provides a closed form approximate solution, in each time step. The numerical examples show that the method has a good efficiency, accuracy and stability. Also, the experimental order of convergence for the new method is approximately $4 - \alpha$ where α is the order of time derivative.

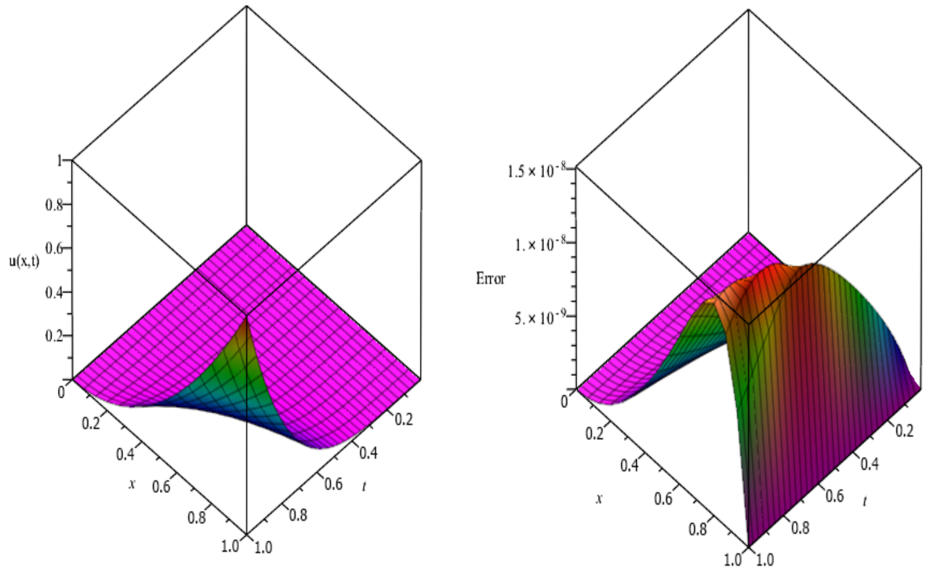

FIGURE 2. Plot of the approximate solution (left) and plot of the absolute error function (right) obtained by IMQ-RBFs with $M = 16$ and $\tau = 0.00625$ at $t = 1$ in Example 3 for $\alpha = 0.35$ and $\beta = 0.6$.

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