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Analytical fuzzy solution of the ventricular pressure equation and prediction of the blood pressure

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Abstract The cardiovascular system is an extremely intelligent and dynamic system which adjusts its performance depending on the individual's physical and environmental conditions. Some of these physical and environmental conditions may create slight disruptions in the cardiovascular system leading to a variety of diseases. Since prevention has always been preferable to treatment, this paper examined the Instantaneous Pressure-Volume Relation (IPVR) and also the pressure of the artery root. Fuzzy mathematics as a powerful tool is used to evaluate and predict the status of an individual's blood pressure. The arterial pressure is modeled as a first order fuzzy differential equation and an analytical solution for this equation is obtained and an example shows the behavior of the solution. The risk factors using fuzzy rules are assessed, which help diagnose the status of an individual's blood pressure. Using the outcome by drawing the individual's attention to these risk factors, the individual's health is improved. Moreover, in this study adaptive neuro-fuzzy inference system (ANFIS) models are evaluated to predict the status of an individual's blood pressure on the basis of the inputs.

Keywords. Fuzzy instant pressure volume, Generalized Hukuhara difference, Generalized Hukuhara differentiable, Characterization theorem, Fuzzy output blood flow theorem, Fuzzy blood flow back theorem, Fuzzy model of arterial pressure, Adaptive neuro-fuzzy inference system.

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1. Introduction

The function of the left ventricle (LV) is the result of the complex interaction between contractility, heart rate. How these factors interact can be understood by simulation of the relation between pressure and volume of the LV ([9]). Furthermore, simulation can support the decision as to what variables and parameters are vital in the description of the pressure and volume of the LV. This may, for example, be of use in integrative physiological modeling [19]. There are many models for the instant pressure-volume relationship in the left ventricle (IPVR)[19]. However, most of these models have been written in a nonlinear form [19]. It has been shown that the LinFree linear model can well display IPVR and the model does not need to be nonlinear and complex. On the other hand, we know that all phenomena, including IPVR, are dynamic in the real world and change under the influence of various factors such as genetic, physiological, psychological, social factors, and so on. Now, the question that arises here is whether the models that have been presented so far can cover these factors? And can a highly sensitive dynamic system, such as IPVR, be modeled with definite equations and formulas?

The nature of most of these factors are qualitative, verbal, ambiguous, and imprecise and fuzzy mathematics is a powerful tool for modeling uncertain problems. The fuzzy concept was first introduced by Professor Zadeh in [33] which discussed fuzzy sets and then fuzzy differential equations and fuzzy systems were introduced [2, 6, 8, 28]. Today, this concept is used in mathematics, engineering, medical engineering, medicine, biotechnology, genetics, etc. However, we know that most of the physiological phenomena in the body, including the circulatory system, can be well expressed as differential equations. And since mathematics has always been in the service of other sciences, we have tried to introduce an improved model for IPVR using fuzzy differential equations so that the introduced model can be closer to reality and act dynamically. We will introduce risk factors for hypertension as follows and use the fuzzy logic to control these factors so that the person is placed in the range of health (normal blood pressure) and also predict how long it takes for an individual to contract the disease if no attention is given to controlling these risk factors.

To find the relationship between inputs and outputs of a production process, artificial intelligence (AI) has drawn more attention to the relationships between input and output variables by training, and produce results without any prior assumptions [17]. During the last twenty years, fuzzy if-then rule-based systems have been used to control problems, while they are now mostly applied in classification tasks [1, 20, 24]. Many methods exist for the automatic generation and learning of the fuzzy if-then rules from numerical data for pattern classification problems [20]. Zadeh [31] proposed the concepts of a linguistic statement where it is crucial to see each attribute as a linguistic value indicated by fuzzy numbers with trapezoidal membership function [12].

Artificial neural network (ANN) models based on the studies of the human neuron can be used to overcome the non-linearity problem, analyze biophysical data, model complex relationships between inputs and outputs, find patterns in data, and capture the statistical structure in an unknown joint probability distribution between the observed variables [17, 32]. ANNs have the potential to be a more practical alternative to the traditional methods for modeling [7]. In recent years, ANN modeling technique has been employed to show the robustness of AI versus regression methods. In recent years, ANN modeling technique has been employed to show the robustness of AI versus regression methods. Many scholars have discussed the classification of inputs and their weighting [23] and produced more accurate results. The classifiers can be improved using rule weighting [32]. The classifiers can improve generating, weighting, and selecting rules. The total number of fuzzy if-then rules generated by partitioning each attribute into k fuzzy subsets in a one-dimensional pattern classification is defined as k_n[22]. The adaptive neuro-fuzzy inference system (ANFIS) is a combination of ANN and fuzzy systems which uses the learning capability of the ANN to derive the fuzzy if-then rules with appropriate membership functions based on the training pairs, which in turn leads to an inference [3].

In this paper, a fuzzy model is introduced and all fuzzy factors in the model are defined. The fuzzy model for the arterial pressure is introduced and an example is solved. In order to prove that the blood pressure is a fuzzy-valued function, 15 important factors that contribute in obtain the values of the resistance of root aortic, left ventricular pressure and the arterial pressure are identified, and it is shown that the amounts obtained are ambiguous and inaccurate. To investigate the result, we select 200 healthy and sick people randomly from hospitals in Tehran and outside of it, and we recorded all these 15 factors for them. Eventually, based on these 15 factors, we designed the ANFIS network and extracted all data for these 200 people, then we predicted the blood pressure with the adaptive neuro-fuzzy inference system (ANFIS). In fact, we translate the qualitative medical information in this paper into quantitative information with respect to fuzzy rules and the ANFIS to be able to provide a person with the necessary recommendations for maintaining health.

This paper is organized as follows: some definitions and basic notions concerning fuzzy calculus are collected in Section 2. In section 3, the model of cardiovascular system are studied in terms of fuzzy nature and related theorems are proved and the proposed model and its advantages are reviewed. The fuzzy model of the arterial pressure as a first order fuzzy differential is introduced and the generalized Hukuhara differentiable solution is obtained in section 4. In section 5, the ANFIS method is introduced and the application of fuzzy linear model with ANFIS is presented in section 6. The conclusion is drawn in section 7. Finally in section 8, a method is presented to solve first order linear fuzzy differential equation.

2. Preliminaries

In this section, we present some definitions and introduce the necessary notation, which will be used throughout the paper.

We denote by $\mathbb{R}_{\mathcal{F}}$, the set of fuzzy numbers, that is, normal, fuzzy convex, upper semi-continuous and compactly supported fuzzy sets which defined over the real line. For $0 < \alpha \leq 1$, set $[u]^\alpha = \Big\{x \in \mathbb{R}^n \Big| u(x) \geq \alpha\Big\}$, and $[u]^0 = cl\Big\{x \in \mathbb{R}^n \Big| u(x) > 0\Big\}$. We represent $[u]^\alpha = [u_l,\ u_u]$, so if $u \in \mathbb{R}_{\mathcal{F}}$, the α-level set $[u]^\alpha$ is a closed interval for all $\alpha \in [0,\ 1]$.

Definition 2.1. (See [13]) Consider $u, v \in \mathbb{R}_{\mathcal{F}}$. If there exists $w \in \mathbb{R}_{\mathcal{F}}$ so that $u = v + w$, then w is called the Hukuhara difference of u and v, and it is denoted by $u \ominus_H v$ and an important property of $\ominus_H$ is that $u \ominus_H u = 0$. If $u \ominus_H v$ exists, it is unique and its $\alpha-$cut's are $[u \ominus_H v]^\alpha = [u_l - v_l, u_u - v_u]$.

Definition 2.2. (See [15]) A triangular fuzzy number defined as a fuzzy set in $\mathbb{R}_{\mathcal{T}}$, that is specified by an ordered triple $u = (u_1, u_2, u_3)$ with $u_1 \leq u_2 \leq u_3$. Let $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ are two triangular fuzzy numbers, so

1: $[u]^\alpha = [(u_1, u_2, u_3)]^\alpha = [u_1 + (u_2 - u_1)\alpha, u_3 - (u_3 - u_2)\alpha]$ for all $\alpha \in [0, 1]$.

2: For all $\lambda \in \mathbb{R}$

$$\lambda \odot u = \begin{cases} (\lambda u_1, \lambda u_2, \lambda u_3), & \text{If } \lambda \geq 0; \\ (\lambda u_3, \lambda u_2, \lambda u_1), & \text{If } \lambda < 0. \end{cases}$$

3: $u \odot v = \Big(\min\{u_1v_1, u_1v_3, u_3v_1, u_3v_3\}, u_2v_2, \max\{u_1v_1, u_1v_3, u_3v_1, u_3v_3\} \Big)$.

4: The triangular fuzzy number u is said to be positive if $u_1 > 0$. We denote by $\mathbb{R}^+_T$, the set of all positive triangular fuzzy numbers.

Definition 2.3. (See [6]) The generalized Hukuhara difference of two fuzzy numbers $u,\ v \in \mathbb{R}_{\mathcal{F}}$ is defined as follows

$$u \ominus_{gH} v = w \Longleftrightarrow \begin{cases} \quad (i).\ u = v + w; \\ or\ (ii).\ v = u + (-1)w. \end{cases}$$

In terms of α-levels we have $[u \ominus_{gH} v]^\alpha = [\min\{u_l - v_l, u_u - v_u\}\ ,\ \max\{u_l - v_l, u_u - v_u\}]$. Now consider $a, b \in \mathbb{R}_T$, then

$$a \ominus_{gH} b = c \Longleftrightarrow\ (i).\ c = (a_1 - b_1, a_2 - b_2, a_3 - b_3),\ or\ (ii).\ c = (a_3 - b_3, a_2 - b_2, a_1 - b_1).$$

Provided that c is a triangular fuzzy number and $[a \ominus_{gH} b]^\alpha = [\min\{a_1 - b_1, a_3 - b_3\}\ ,\ a_2 - b_2,\ \max\{a_1 - b_1, a_3 - b_3\}]$.

Remark 2.4. Throughout the rest of this paper, we assume that $u \ominus_{gH} v \in \mathbb{R}_{\mathcal{F}}$.

Remark 2.5. Consider a and b are triangular fuzzy numbers. It is easy to prove that

1: $0 \ominus_{gH} a = (-1)a$ provided that $(-1)a$ is a triangular fuzzy number.
2: $a \ominus_{gH} b \neq a \oplus (-1)b$.
3: $a \ominus_{gH} (-1)b \neq a \oplus b$.

In this paper, for the ranking concept, we will use a partial ordering introduced in [14].

Definition 2.6. Let $\preceq$ be the partial ordering in $\mathbb{R}_{\mathcal{F}}$ defined by

$$u \preceq v \text{ if and only if } u_l \leq v_l \text{ and } u_u \leq v_u$$

and the strict inequality $\prec$ in $\mathbb{R}_{\mathcal{F}}$ defined by

1: $u \prec v$ if and only if $u_l < v_l$ and $u_u < v_u$.
2: $u \succ 0$ if and only if $0 < u_l \leq u_u$.
3: $u \prec 0$ if and only if $u_l \leq u_u < 0$.

$\forall \alpha \in [0, 1]$, where $[u]^\alpha = [u_l, u_u]$, $[v]^\alpha = [v_l, v_u]$.

Definition 2.7. (See [10]) A fuzzy-valued function $f : [a,\ b] \to \mathbb{R}_{\mathcal{F}}$ is said to be continuous at $t_0 \in [a,\ b]$ if for each $\epsilon > 0$ there is $\delta > 0$ such that $D_{\mathcal{F}}\Big(f(t), f(t_0)\Big) < \epsilon$, whenever $t \in [a,\ b]$ and $|t - t_0| < \delta$. We say that f is fuzzy continuous on $[a, b]$ if f is continuous at each $t_0 \in [a,\ b]$.

3. The Cardiovascular Fuzzy Model and Advantages

3.1. **The Fuzzy Linear Model.** In this section, the equations used extensively in the cardiovascular system are studied in terms of fuzzy nature and the all needed theorems are proved. In this paper, we will use all the constants and functions presented in Table (1).

Remark 3.1. Note that, all the constants or fuzzy functions in Table (1) are related to the left ventricular and the cardiovascular system, so their values are naturally positive.

Table 1. Parameter and Location

Parameter	Location
T	Duration of a heart beat
$\frac{1}{T}$	Heart rate (or pulse)
V	Fuzzy Volume
V_{lv}	Fuzzy left ventricular volume
$V_{d,lv}$	Fuzzy left ventricular volume at zero pressure
V_{a1}	Fuzzy arterial volume,
P	Fuzzy pressure
P_s	Fuzzy systolic pressure
P_{lv}	Fuzzy left ventricular pressure
P_{a1}	Fuzzy arterial pressure
P_{as}	Fuzzy root aortic pressure
P_{ve}	Fuzzy venous pressure
E_{lv}	Fuzzy left ventricular elastance function
$E_{min,lv}$	Minimal diastolic fuzzy value of E_{lv}
$E_{max,lv}$	Maximal systolic fuzzy value of E_{lv}
R_{0s}	Fuzzy resistance of root aortic
R_{si}	systemic resistance
Q_{lv}	Fuzzy left ventricular outflow
Q_s	Fuzzy systemic blood flow
C_{a1}	Compliance

We know that the amount of blood volume entering the ventricle of each person is a function of complicity, so the pressure caused by it is also different, and these two have an organism relationship with each other.

Definition 3.2. (FIPVR equation) Let $P_{lv}(t), E_{lv}(t), V_{lv}(t)$, and $V_{d.lv}(t)$ are fuzzy-valued functions. The fuzzy instant pressure volume relationship in the left ventricle (FIPVR) is equal

$$P_{lv} \quad = \quad E_{lv} \odot (V_{lv} \ominus_{gH} V_{d.lv}) = g(t, E_{lv}, V_{lv}, V_{d.lv}), \tag{3.1}$$

where $t \in [t_0, T] \subseteq \mathbb{R}^+$, g is a continuous mapping and $g : \mathbb{R}^+ \times \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$.

Lemma 3.3. *Suppose that $P_{lv}(t), E_{lv}(t), V_{lv}(t)$, and $V_{d.lv}(t)$ are fuzzy-valued functions, then*

i: $E_{lv} \odot (V_{lv} \ominus_{gH} V_{d.lv}) = (E_{lv} \odot V_{lv}) \ominus_{gH} (E_{lv} \odot V_{d.lv})$.
ii: $E_{lv} \odot (V_{lv} \ominus_{H} V_{d.lv}) = (E_{lv} \odot V_{lv}) \ominus_{H} (E_{lv} \odot V_{d.lv})$.

iii: $E_{lv} \odot (V_{d.lv} \ominus_H V_{lv}) = \Big((E_{lv} \odot V_{d.lv}) \ominus_H (E_{lv} \odot V_{lv})\Big)$.

Proof. Let $E_{lv} \odot (V_{lv} \ominus_{gH} V_{d.lv})$. Then by considering Definition (2.3), we have

$$\begin{aligned}
&[E_{lv} \odot (V_{lv} \ominus_{gH} V_{d.lv})]^{\alpha} \\
&= [E_{l,lv}, E_{u,lv}].\Big[\min\Big\{(V_{l,lv} - V_{l,d.lv}), (V_{u,lv} - V_{u,d.lv})\Big\} \\
&\quad , \max\Big\{(V_{l,lv} - V_{l,d.lv}), (V_{u,lv} - V_{u,d.lv})\Big\}\Big] \\
&= \Big[\min\Big\{E_{l,lv}(V_{l,lv} - V_{l,d.lv}), E_{l,lv}(V_{u,lv} - V_{u,d.lv}), E_{u,lv}(V_{l,lv} - V_{l,d.lv}) \\
&\quad , E_{u,lv}(V_{u,lv} - V_{u,d.lv})\Big\} \\
&\quad , \ \max\Big\{E_{l,lv}(V_{l,lv} - V_{l,d.lv}), E_{l,lv}(V_{u,lv} - V_{u,d.lv}), E_{u,lv}(V_{l,lv} - V_{l,d.lv}) \\
&\quad , E_{u,lv}(V_{u,lv} - V_{u,d.lv})\Big\}\Big] \\
&= \Big[\min\Big\{E_{l,lv}V_{l,lv} - E_{l,lv}V_{l,d.lv}, E_{l,lv}V_{u,lv} - E_{l,lv}V_{u,d.lv}, E_{u,lv}V_{l,lv} - E_{u,lv}V_{l,d.lv} \\
&\quad , E_{u,lv}V_{u,lv} - E_{u,lv}V_{u,d.lv}\Big\} \\
&\quad , \max\Big\{E_{l,lv}V_{l,lv} - E_{l,lv}V_{l,d.lv}, E_{l,lv}V_{u,lv} - E_{l,lv}V_{u,d.lv}, E_{u,lv}V_{l,lv} - E_{u,lv}V_{l,d.lv} \\
&\quad , E_{u,lv}V_{u,lv} - E_{u,lv}V_{u,d.lv}\Big\}\Big]
\end{aligned}$$

Since $V_{lv}, V_{d.lv}, E_{lv} \succ 0$, we obtain

$$\begin{aligned}
&\min\Big\{E_{l,lv}V_{l,lv} - E_{l,lv}V_{l,d.lv}, E_{l,lv}V_{u,lv} - E_{l,lv}V_{u,d.lv}, E_{u,lv}V_{l,lv} - E_{u,lv}V_{l,d.lv} \\
&\quad , E_{u,lv}V_{u,lv} - E_{u,lv}V_{u,d.lv}\Big\} \\
&= E_{l,lv}V_{l,lv} - E_{l,lv}V_{l,d.lv} \\
&\quad , \max\Big\{E_{l,lv}V_{l,lv} - E_{l,lv}V_{l,d.lv}, E_{l,lv}V_{u,lv} - E_{l,lv}V_{u,d.lv}, E_{u,lv}V_{l,lv} - E_{u,lv}V_{l,d.lv} \\
&\quad , E_{u,lv}V_{u,lv} - E_{u,lv}V_{u,d.lv}\Big\} \\
&= E_{u,lv}V_{u,lv} - E_{u,lv}V_{u,d.lv}.
\end{aligned}$$

Therefore,

$$[E_{lv} \odot (V_{lv} \ominus_{gH} V_{d.lv})]^{\alpha} = [E_{l,lv}V_{l,lv} - E_{l,lv}V_{l,d.lv}, E_{u,lv}V_{u,lv} - E_{u,lv}V_{u,d.lv}]. \tag{3.2}$$

Now let $(E_{lv} \odot V_{lv}) \ominus_{gH} (E_{lv} \odot V_{d.lv})$, then

$$\begin{aligned}
[(E_{lv} \odot V_{lv})]^{\alpha} &= [E_{l,lv}, E_{u,lv}].[V_{l,lv}, V_{u,lv}] \\
&= \Big[\min\Big\{E_{l,lv}V_{l,lv}, E_{l,lv}V_{u,lv}, E_{u,lv}V_{l,lv}, E_{u,lv}V_{u,lv}\Big\} \\
&\quad , \max\Big\{E_{l,lv}V_{l,lv}, E_{l,lv}V_{u,lv}, E_{u,lv}V_{l,lv}, E_{u,lv}V_{u,lv}\Big\}\Big].
\end{aligned}$$

$$
\begin{aligned}
(E_{lv} \odot V_{d.lv})]^{\alpha} &= [E_{l,lv}, E_{u,lv}].[V_{l,d.lv}, V_{u,d.lv}] \\
&= \Big[\min \Big\{ E_{l,lv}V_{l,d.lv}, E_{l,lv}V_{u,d.lv}, E_{u,lv}V_{l,d.lv}, E_{u,lv}V_{u,d.lv} \Big\} \\
&\quad , \max \Big\{ E_{l,lv}V_{l,d.lv}, E_{l,lv}V_{u,d.lv}, E_{u,lv}V_{l,d.lv}, E_{u,lv}V_{u,d.lv} \Big\} \Big].
\end{aligned}
$$

Since $V_{lv}, E_{lv}, V_{d.lv} \in \mathbb{R}^{+}_{\mathcal{F}}$, then we have

$$
\begin{aligned}
&\min \Big\{ E_{l,lv}V_{l,lv}, E_{l,lv}V_{u,lv}, E_{u,lv}V_{l,lv}, E_{u,lv}V_{u,lv} \Big\} = E_{l,lv}V_{l,lv}, \\
&\max \Big\{ E_{l,lv}V_{l,lv}, E_{l,lv}V_{u,lv}, E_{u,lv}V_{l,lv}, E_{u,lv}V_{u,lv} \Big\} = E_{u,lv}V_{u,lv}, \\
&\min \Big\{ E_{l,lv}V_{l,d.lv}, E_{l,lv}V_{u,d.lv}, E_{u,lv}V_{l,d.lv}, E_{u,lv}V_{u,d.lv} \Big\} = E_{l,lv}V_{l,d.lv}, \\
&\max \Big\{ E_{l,lv}V_{l,d.lv}, E_{l,lv}V_{u,d.lv}, E_{u,lv}V_{l,d.lv}, E_{u,lv}V_{u,d.lv} \Big\} = E_{u,lv}V_{u,d.lv}.
\end{aligned}
$$

According to Definition (2.1), we get

$$
\begin{aligned}
&[(E_{lv} \odot V_{lv}) \ominus_{gH} (E_{lv} \odot V_{d.lv})]^{\alpha} = \\
&= [E_{l,lv}V_{l,lv} - E_{l,lv}V_{l,d.lv}, E_{u,lv}V_{u,lv} - E_{u,lv}V_{u,d.lv}].
\end{aligned} \tag{3.3}
$$

By attention to Eq. (3.2) and Eq. (3.3), the required result is obtain. Similarly by using Definition 2.1, the rest of the cases can be proved. □

Proposition 3.4. *If $P_{lv}, E_{lv}, V_{lv}, V_{d.lv}$ are fuzzy-valued functions. Then one of the cases may occurs:*

- **i:** $P_{lv} = E_{lv} \odot (V_{lv} \ominus_H V_{d.lv}) = g(t, v(t))$.
- **ii:** $P_{lv} = (-1)\Big(E_{lv} \odot (V_{d.lv} \ominus_H V_{lv})\Big) = q(t, v(t))$,

where $g(t, v(t))$ and $q(t, v(t))$ denote by $g(t, v(t)) := g(t, E_{lv}, V_{lv}, V_{d.lv})$ and $q(t, v(t)) := q(t, E_{lv}, V_{lv}, V_{d.lv})$.

Proof. (i) By applying Lemma 3.3 we have $P_{lv} = (E_{lv} \odot V_{lv}) \ominus_{gH} (E_{lv} \odot V_{d.lv})$, so by using Definition 2.3 (i) we get $P_{lv} \oplus (E_{lv} \odot V_{d.lv}) = (E_{lv} \odot V_{lv})$, therefore $P_{lv} = (E_{lv} \odot V_{lv}) \ominus_H (E_{lv} \odot V_{d.lv}) = E_{lv} \odot (V_{lv} \ominus_H V_{d.lv})$.
(ii) Using Lemma 3.3 and Definition 2.3 (ii) we get $(E_{lv} \odot V_{lv}) \oplus (-1)P_{lv} = (E_{lv} \odot V_{d.lv})$, therefore $(-1)P_{lv} = (E_{lv} \odot V_{d.lv}) \ominus_H (E_{lv} \odot V_{lv})$, so $P_{lv} = (-1)\Big((E_{lv} \odot V_{d.lv}) \ominus_H (E_{lv} \odot V_{lv})\Big) = (-1)\Big(E_{lv} \odot (V_{d.lv} \ominus_H V_{lv})\Big)$. □

Proposition 3.5. *Suppose that $P_{lv}, E_{lv}, V_{lv}, V_{d.lv}$ are fuzzy-valued functions. If $P_{lv} = E_{lv} \odot (V_{lv} \ominus_H V_{d.lv}) = g(t, v(t))$ or $(-1)P_{lv} = E_{lv} \odot (V_{d.lv} \ominus_H V_{lv}) = q(t, v(t))$ there exist ($g(t, v(t)) := g(t, E_{lv}, V_{lv}, V_{d.lv})$ and $q(t, v(t)) := q(t, E_{lv}, V_{lv}, V_{d.lv})$), then one of the following statements may happen:*

$$
(i) \quad \begin{cases} P_{l,lv} = E_{l,lv}.(V_{l,lv} - V_{l,d.lv}) = g_l\left(t, v_l, v_u\right), \\ P_{u,lv} = E_{u,lv}.(V_{u,lv} - V_{u,d.lv}) = g_u\left(t, v_l, v_u\right). \end{cases} \tag{3.4}
$$

$$(ii)\ \begin{cases} (-1)P_l = E_{u,lv}.(V_{u,lv} \ - \ V_{u,d.lv}) = (-1)g_u(t, v_l, v_u) = q_l\,(t, v_l, v_u)\,, \\ \\ (-1)P_u = E_{l,lv}.(V_{l,lv} \ - \ V_{l,d.lv}) = (-1)g_l(t, v_l, v_u) = q_u\,(t, v_l, v_u)\,. \end{cases} \tag{3.5}$$

Proof. Let $P_{lv} = E_{lv} \odot (V_{lv} \ominus_H V_{d.lv}) = g(t, v(t))$ there exist, so by applying Lemma 3.3 and Definition (2.1) we have

$$\begin{array}{rcl} P_{l,lv} & = & E_{l,lv}.(V_{l,lv} - V_{l,d.lv}) = g_l(v_l, v_u), \\ P_{u,lv} & = & E_{u,lv}.(V_{u,lv} - V_{u,d.lv}) = g_u(t, v_l, v_u). \end{array}$$

Similarly, the case (ii) can be proved. □

Then to show when equations (3.1) and (3.4), (3.5) are equivalent, we express and prove the following theorem.

Theorem 3.6. *(Characterization Theorem) Let $P_{lv}, E_{lv}, V_{lv}, V_{d.lv}$ are fuzzy-valued functions, $t \in [t_0, T]$ and $g : [t_0, T] \times \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is a continuous g-process such that:*

$$1.[g(t, v)]^\alpha = [g_l(t, v_l, v_u), g_u(t, v_l, v_u].$$
$$2.[q(t, v)]^\alpha = (-1)[g(t, v)]^\alpha = [q_l(t, v_l, v_u), q_u(t, v_l, v_u)].$$

Then the equation (3.1) and the system of ((3.4)),((3.5)) are equivalent.

Proof. According to Definition (3.2) and by using Propositions (3.4) and (3.5) the proof is clear. □

In the next section, we will review the proposed model and its advantages.

3.2. **Advantages of the model (Fuzzy LinFree).** We know that thousands of chemical reactions occur and thousands of changes take place every second in the human body, which the reactions are created on the basis of different physical, psychological, and environmental conditions of the individual. The human body smartly organizes our activities under all circumstances to maintain our health. Therefore, it can be said that all of the phenomena and activities occurring in the body, including the instant pressure-volume relationship in the left ventricle, are dynamic and change under the influence of various factors such as genetic, physiological, psychological and social factors. Considering the importance of modeling the instant pressure-volume relationship in the left ventricle, many scientists have been working on it [19, 25] and it has been tried that the best model is provided to show the IPVR relationship. Among all models, the model in [19] has been considered more and better results have been obtained by it. In this modeling, a linear relationship has been established between the pressure and volume and the elastance function. Now, look at the following conditions.

A: Arterial blood pressure is a function of cardiac output and vascular resistance, which varies according to gender, age, metabolic rate, emotional states, and various diseases, including cardiovascular disease.

B: Left ventricular volume varies in different people depending on the gender, the bulk of the body, history of the disease, age, and even high or low blood pressure.

C: When an artery suffers a pathological process such as artery diseases, the elasticity of the arterial wall decreases and the stiffness increases due to changes that the disease creates in the arterial wall, such as the loss of elastic fibers and increased collagen and fat deposits and increased wall thickness. As a result, the lateral pressure imposed on the arterial wall increases abnormally during systole and decreases slightly during diastole.

An overview of three above-mentioned definitions leads to this question whether the models that have been presented so far for IPVR relationships can cover these factors? And are they consistent with the essence of the system that is smart and dynamic?

Do the environmental, genetic, and social factors, history of the disease, smoking, age, gender, diet, physical activity, stress, heart rate, and BMI index affect these factors?

Have these factors been considered or ignored in the proposed models or have the models been designed in a normal mode?

In general, is it appropriate to consider the definitive models for a very sensitive dynamic system, such as IPVR (which is subject to change, although partial, at any moment)?

In response to all of these questions, we have tried to provide a fuzzy model for IPVR that not only matches the essence of the problem, namely its dynamism, but also allows for environmental and individual factors to be considered and shows when the individual is in perfect health, when he enters a stage that is associated with risk and ultimately when he is in the stage of disease (blood pressure).

The advantage of this model is that the health status of a person can be determined and, given the factors that are considered during the model, the prevention of disease can always be achieved with minor changes in the life process of the individual.

The cardiac contractile properties of the two ventricles are assumed to be defined by a pair of time-varying elastance functions [25]. The relation between the fuzzy left ventricular pressure p_{lv} and the fuzzy ventricular volume V_{lv} is described by

$$P_{lv} = E_{lv} \odot (V_{lv} \ominus_{gH} V_{d.lv}) \Longleftrightarrow \begin{cases} (i) P_{lv} = E_{lv} \odot (V_{lv} \ominus_H V_{d.lv}), \\ or \\ (ii)(-1)P_{lv} = E_{lv} \odot (V_{d.lv} \ominus_H V_{lv}). \end{cases} \tag{3.6}$$

where $V_{d.lv}$ is the fuzzy left ventricular volume at zero pressure. The fuzzy elastance function $E_{lv}(t)$ in (3.6) is given by

$$E_{lv} = E_{min,lv}(1-\phi(t)) \oplus E_{max,lv}\phi(t),$$

where

$$\phi(t) = \begin{cases} a_\phi sin(\frac{\pi t}{t_{ce}}) - b_\phi sin(\frac{2\pi t}{t_{ce}}) & \text{For } 0 \le t \le t_{ce}, \\ 0 & \text{For } t_{ce} \le t \le t_h. \end{cases}$$

The parameters $E_{min,lv}$ and $E_{max,lv}$ are minimal diastolic and maximal systolic fuzzy values of the fuzzy left ventricular elastance function, respectively, t_h is the heart period and t_{ce} the time for onset of real constant elastance. The relation between heart period t_h and t_{ce} is given by

$$t_{ce} = k_0 + k_1 t_h,$$

where k_0 and k_1 are real constant parameters and P_{as} is the fuzzy root aortic pressure and is given by

$$P_{as} = R_{0s} \odot Q_{lv} \oplus P_{a1}.$$

In the crisp models only left ventricular pressure is achieved [25]. But the advantage of this fuzzy model over other models is that it shows not only the fuzzy left ventricular pressure (Theorem 3.7) but also the fuzzy flow back (Theorem 3.8). In addition, suitable fuzzy numbers with suitable qualities can be considered for all the factors, which the issue of the multi-dimensional fuzzy numbers arises in this case.

Theorem 3.7. *(Fuzzy output blood flow Theorem) Let $P_{lv}(t), E_{lv}(t), V_{lv}(t)$ and $V_{d.lv}(t)$ are fuzzy valued functions and $V_{lv} \succ V_{d.lv}$ then FIPVR represents the fuzzy flow of the output from the left ventricle and is expressed in equation (3.7).*

$$P_{lv} = (E_{lv} \odot V_{lv}) \ominus_H (E_{lv} \odot V_{d.lv}). \tag{3.7}$$

Proof. We know that the positive pressure expresses the flow of blood output from the left ventricle. Then by using Definition 3.2, Proposition 3.4 (i) and Lemma 3.3 (ii), the proof is completed. □

Theorem 3.8. *(Fuzzy blood flow back Theorem) If $P_{lv}(t), E_{lv}(t), V_{lv}(t)$ and $V_{d.lv}(t)$ are fuzzy valued functions and $V_{lv} \prec V_{d.lv}$ then FIPVR represents the fuzzy flow back to the left atrium and is expressed in equation (3.8).*

$$(-1)P_{lv} = (E_{lv} \odot V_{d.lv}) \ominus_H (E_{lv} \odot V_{lv}). \tag{3.8}$$

Proof. The negative pressure expresses the flow back from the left ventricle to the left atrium. Then by applying Definition 3.2, Proposition 3.4 (ii) and Lemma 3.3 (iii), the proof is completed. □

4. Fuzzy Model of the Arterial Pressure

In this section, we will try to obtain the fuzzy model of the arterial pressure, $P_{a_1}(t)$, during a single beat of the heart.

Let the blood ejected from the left ventricle into the aorta. If T is the duration of a heart beat, so $\frac{1}{T}$ is the heart rate (or pulse) in beats/min. We have the maximum fuzzy pressure of the blood in the aorta, when the left ventricle completes pumping the blood into the aorta and the valve close. This maximum fuzzy pressure is denoted by P_s and is equal to the fuzzy systolic pressure.

If R_{si} is the systemic resistance, then we can obtain the value of the fuzzy systemic blood flow, $Q_s(t)$ by the following equation

$$Q_s(t) = \frac{1}{R_{si}} \odot \Big(P_{a1}(t) \ominus_{gH} P_{ve}(t)\Big). \tag{4.1}$$

where $\Big(P_{a1}(t) \ominus_{gH} P_{ve}(t)\Big)$ is the difference between the arterial and venous fuzzy pressure. By attention to this fact that fuzzy venous pressure are very low, we can approximate the fuzzy systemic flow by the following fuzzy equation

$$Q_s(t) = \frac{1}{R_{si}} \odot P_{a1}(t). \tag{4.2}$$

Moreover, the fuzzy arterial volume, $V_{a1}(t)$ and the fuzzy arterial pressure have the following relation

$$V_{a1}(t) = C_{a1} \odot P_{a1}(t), \tag{4.3}$$

where $C_{a1} \neq 0$ is the compliance.

Using this fact that the change in the fuzzy arterial volume is given by the difference between the rate of fuzzy flow entering the aorta and the rate of fuzzy flow from the aorta, we have the following fuzzy differential equation

$$V'_{a1}(t) = 0 \ominus_{gH} Q_s(t). \tag{4.4}$$

In this equation, the rate of fuzzy flow entering the aorta is equal to zero, because the aortic valve is closed during systole and no blood is entering the aorta. Hence by Eq.(4.2) and Eq.(4.4) and Remark 2.5 we have

$$V'_{a1}(t) = \ominus_{gH} \frac{1}{R_{si}} \odot P_{a1}(t). \tag{4.5}$$

But from Eq.(4.3), we can write

$$V'_{a1}(t) = C_{a1} \odot P'_{a1}(t). \tag{4.6}$$

Thus, by Eqs.(4.5) and (4.6) we can obtain a linear first order fuzzy differential describing the fuzzy arterial pressure

$$P'_{a1}(t) = \ominus_{gH} \frac{1}{C_{a1} R_{si}} \odot P_{a1}(t). \tag{4.7}$$

with the fuzzy initial value $P_{a1}(0) = P_s$.

Using the method described in Appendix 8, applying the fuzzy Laplace transform to equation (4.7) and by Lemma (8.1) and initial condition P_s, Eq.(4.7) has the following $[(i) - gH]-$differentiable solution (or $[(ii) - gH]-$differentiable solution)

$$P_{a1}(t) = P_s C_{a1} R_{si} \odot e^{-C_{a1} R_{si} t}.$$

This solution is valid for time t between 0 and T.

Example 4.1. For a normal person, we have a pulse approximately 70 beats/ min. The systemic resistance and compliance for this person are 17.6165 and 0.002, respectively. If the fuzzy systolic pressure is $P_s = (118.8, 120, 121.2)$, then the fuzzy arterial pressure for a normal person obtain by the following equation

$$P_{a1}(t) = (4.18568, 4.22796, 4.27024)e^{-0.035233t}.$$

The fuzzy arterial pressure is presented in Fig. 1 for $t \in [0, \frac{1}{70}]$ and all $\alpha \in [0, 1]$.

Now we want to examine the cardiovascular equations from a different perspective. we will introduce risk factors for hypertension and will show that the person is placed in the range of health (normal blood pressure) by using the fuzzy logic and ANFIS network. Moreover, we will predict how long it takes for an individual to contract

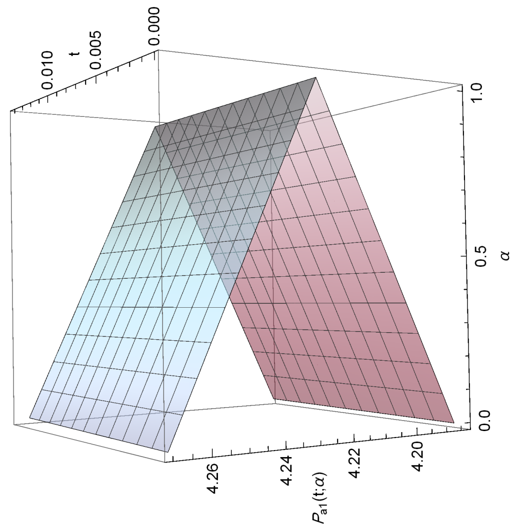

FIGURE 1. Graph of $P_{a1}(t)$ for all $t \in [0, \frac{1}{70}]$ for Example 4.1

the disease if no attention is given to controlling these risk factors. The advantages of this method include high accuracy, prediction of blood pressure and practicality of the procedure.

5. Methods:Adaptive neuro-fuzzy inference system (ANFIS)

In the following sections we recall the structure of the ANFIS model and the topology of inputs variables.

5.1. **The structure of the ANFIS model.** A fuzzy inference algorithm is a method where fuzzy rules are used to deduce a new approximate fuzzy set conclusion while taking the fuzzy set as a premise [17]. A fuzzy inference system (FIS) is used in cases where either the systems cannot be easily modeled or where the description is ambiguous [30]. An ANFIS is used to map input features to input membership functions (MFs), input MF to a set of if-then rules, rules to a set of output features, output features to output MFs, and the output MFs to a single valued output or a decision related to the output [27]. A typical $ANFIS$ structure, which can be seen in Fig. 2, includes 5 layers. Layer 1: Every node J in this layer is an adaptive node with a node function,

$$Q_j^1 = \mu_{A_j}(x),$$

where x is the input to node j, A_j represents the linguistic label related to this node function, and Q_j^1 is the membership function of A_j that shows the degree to which the given x satisfies A_j. To the input y, the node functions in the same layer as the same function family as x. The most common MFs are triangular and bell-shaped.

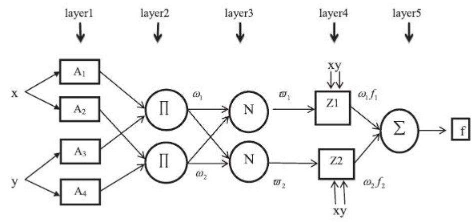

Figure 2. Adaptive neuro-fuzzy inference system structur

Bell-shaped MF with maximally equals 1 and minimally equals 0 calculated as follows:

$$\mu(x) = \frac{1}{1 + \left|(x-c)/a\right|^{2b}}$$

Layer 2: in this layer every node is a fixed node which serves as a simple multiplier. The outputs of these nodes are calculated by

$$Q_j^2 = \psi_j = \mu_{A_j}(x) \times \mu_{B_j}(y), \quad j = 1, 2, \ldots$$

which are the firing strengths of the rules. Layer 3: In this layer, each node is an adaptive node labeled as N. The jth node determines the ratio of the jth rule's firing strength to the sum of all rules' firing strengths

$$Q_j^3 = \overline{\psi_j} = \frac{\psi_j}{\psi_1 + \psi_2}, \quad j = 1, 2, \ldots$$

For the sake of convenience, outputs of this layer are called normalized firing strengths. Layer 4: In this layer every node is an adaptive node with a function

$$Q_j^4 = \overline{\psi_j} f_i = \overline{\psi_j}(p_j x + q_j y + r_j), \quad j = 1, 2, \ldots$$

where $\overline{\psi_j}$ is the output of layer 3, and $\{p_j, q_j, r_j\}$ are called consequent parameters. Layer 5: here, the single node is a fixed node labeled as $\sum$, which computes the overall output as the sum of all incoming signals as follows:

$$Q_j^5 = \sum_{j=1}^{2} \overline{\psi_j} f_j = \frac{\sum_{j=1}^{2} \psi_j f_j}{\sum_{j=1}^{2} \psi_j}.$$

It is seen that there are two modifiable parameter sets, $\{a_j, b_j, c_j\}$ labeled as premise parameters and $\{p_j, q_j, r_j\}$ labeled as consequent parameters. The aim of the training algorithm for this architecture is to tune the two parameter sets above to match the ANFIS output with the training data [26, 27]. ANFIS only supports Sugeno-type systems with the following properties [21]:

- Are first or zero order Sugeno-type systems.
- Have a single output, obtained using weighted average defuzzification. All output MFs must be of the same type, either linear or constant.
- Have no rule sharing. Different rules cannot share the same output MF, namely the number of output MFs must be equal to the number of rules.
- Have unity weight for each rule

The main restriction of the ANFIS model is the number of input variables. If ANFIS inputs go beyond five, the computational time and the number of rules will increase, so ANFIS will not be able to model output with respect to inputs.

5.2. **Topology of input variables.** In this study, we consider 15 energy inputs including systolic and diastolic blood pressure, age, smoking, exercise, BMI, weight, height, sex, stress, background, and diet. To investigate which combination of input parameters can produce the best ANFIS results with the highest accuracy, three main schemes were developed. The first topology can be observed in Fig. 3

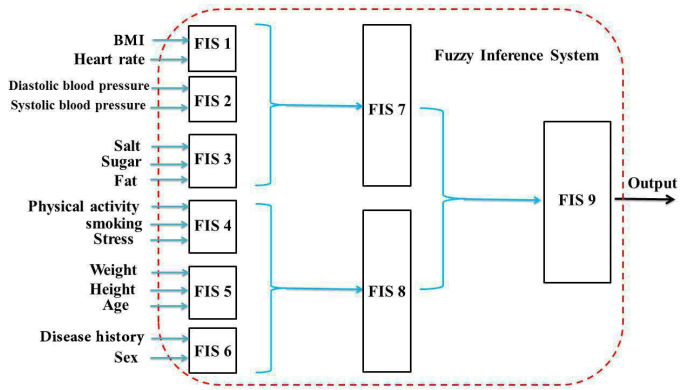

FIGURE 3. The topology of ANFIS model to predict blood pressure

Now, we want to consider our claim that the model is fuzzy and the qualities mentioned in the fuzzy model.

6. Application of fuzzy linear model with ANFIS

This section, with a real sampling of 200 healthy and sick people, demonstrates that blood pressure is a fuzzy function that depends on a variety of factors, in order to prove that all the formulas and Theorems in Sections 3 and 4 are correct. We have the following steps

Step 1: Fuzzy Equation of Aortic Root Pressure and all Fuzzy Factors Consider the following fuzzy model.

$$P_{as} = R_{0s} \odot Q_{lv} \oplus P_{a1}, \tag{6.1}$$

In this model, three basic functions including pressure, volume, and blood flow can be seen that are introduced as a fuzzy functions due to the influence of the various factors mentioned below.

Aortic root resistance (R_{0s}):

- Age: as time elapses and the age increases, the elasticity of the vascular wall decreases and the wall gets harder. As a result, the vascular wall resistance increases with the increase in age.

• Diet: another factor affecting the vascular wall is the high intake of salt, sugar (glycosylated vascular wall), and fat, which increases the vascular wall resistance.
• Smoking: smoking increases the vascular wall resistance.
• Exercise: aerobic exercises can prevent arteriosclerosis.
• Stress: stress stimulates the sympathetic nerves and increases the pressure of peripheral vascular resistance.

Left ventricular volume (V_{lv}):

• Men have larger left ventricles than women.
•The height, weight, BMI of a person has a direct impact on the left ventricular volume.

Arterial blood pressure (P_{a1}):

• The higher the systolic and diastolic blood pressure, the greater the blood pressure in the arteries.

Step 2: Introduction of ANFIS network

Step 2-1: Input variables

• Systolic and diastolic blood pressure (in mmHg): Different values of blood pressure change the result easily. We use systolic and diastolic BP. Generally, diastolic blood pressure is more important but systolic BP is more important above 50 years age. This input variable is divided into 7 fuzzy sets: Normal, Above Normal, Moderate, Above Moderate, Little High, High and Very high sets (Table 2).
• Age: This input field is classified into 6 fuzzy sets. The fuzzy sets with their ranges are given in the Table 3.

Table 2. Blood pressure

	Systolic BP	Diastolic
normal(n)	< 120	< 80
above normal (an)	120-129	80-85
moderate(m)	130-139	86-91
above moderate(am)	140-149	92-97
little high(lh)	150-159	98-103
high (h)	160-169	104-109
very high(vh)	> 170	> 110

Table 3. Age

	Age
young(y)	< 25
adult(a)	25-45
mid aged(m)	45-55
aged(ag)	55-65
old(o)	65-85
very old(vo)	> 85

• Aerobic exercise (Physical activity): This input field is classified into 6 fuzzy sets. The fuzzy sets with their ranges are shown in the Table 4.
• Sex: This input field just has 2 values 0 and 1 and the sets are male and female.
• Disease history: This input field just has 2 values 0 and 1 and the sets are with a history of disease and no history of the disease.

• Stress: This input field is classified into 6 fuzzy sets. The fuzzy sets with their range are shown in the Table 5.

TABLE 4. Physical activity

	Aerobic exercise
very less effective(vle)	< 10 min
less effective(le)	10-20 min
moderate(m)	20-45 min
above moderate(am)	45-60 min
highly effective(he)	60-90 min
very highly effective(vhe)	> 90 min

• Smoking: This input field is classified into 5 fuzzy sets. The fuzzy sets with their range are shown in the Table 6.
• Diet: This input field is classified into 9 fuzzy sets. The fuzzy sets with their ranges are shown in the Table 7. We consider three classes of salt, sugar and fat each having three fuzzy sets.
• Heart Rate: In this field, we have 5 fuzzy sets. In the Table 8, we have defined these fuzzy sets.

TABLE 5. Stress

	Stress
very less (vl)	< 3 min
less (l)	3-6 min
moderate(m)	6-10 min
above moderate(am)	10-20 min
highly (he)	20-30 min
very highly (vh)	> 30 min

TABLE 6. Smoking

	Smoking
Low smoking	< 3 cigarettes
Moderate	3-6 cigarettes
Above moderate	6-10 cigarettes
High	10-15 cigarettes
Very high	> 15 cigarettes

TABLE 7. Diet

	Fat	Salt	Sugar
very less (vl)	< 70 gr	< 2 gr	< 10 gr
less (l)	70-80 gr	2-4 gr	10-30 gr
very highly (vh)	> 80 gr	> 4 gr	> 30 gr

• Body Mass:(BMI) This input field is classified into 7 fuzzy sets. The fuzzy sets with their ranges are shown in Table 9 (Considering the height and weight).

Table 8. Heart Rate

	Heart rate
very less (vle)	< 55 beat
less (le)	55-60 beat
moderate(m)	60-90 beat
above moderate(am)	90-110 beat
very highly (vhe)	> 110 beat

Table 9. Body Mass

	BMI
very less (vle)	< 20
less (le)	20-25
moderate(m)	25-30
above moderate(am)	30-35
less highly (lhe)	35-40
highly (he)	40-45
very highly (vhe)	> 45

Step 2-2: Combination of input variables.
In this regard, 200 individuals have been selected as samples and the 15 factors have been evaluated and recorded for them. Then to study which combination of input variables can produce the best ANFIS results with the highest accuracy, three important schemes were developed. The topology can be apperceived in Fig 3.

Step 2-3: Prediction of blood pressure (Output variable) and evaluation of ANFIS model.
Then, considering the ANFIS network, the model has been designed and 70% of the data have been used as training data and 30% of them have been used as check data (Figure 4, Figure 5, Figure 6) and finally, the error Table 10 has been obtained as below, indicating that ANFIS has done the modeling with a high confidence. To clearly see the relationship between inputs and outputs, in Figure 7 we show the relationship between sugar and salt with blood pressure, respectively.

Table 10. Error Table

Data	RMES Error	Correlation
Train Data	0.000046	0.99
Test Data	0.1212	0.94
All Data	0.0363	0.93

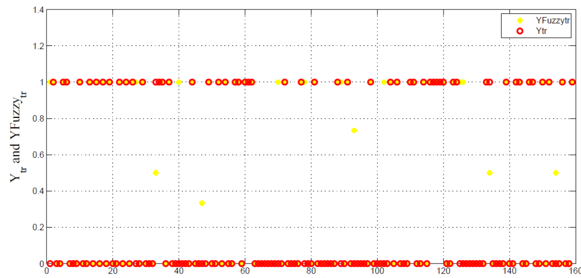

FIGURE 4. Train Data

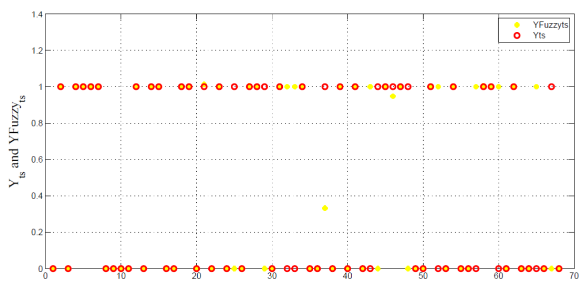

FIGURE 5. Test Data

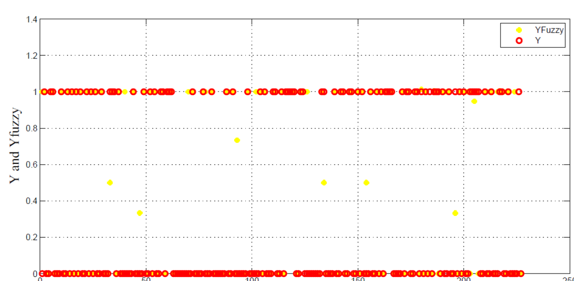

FIGURE 6. All data

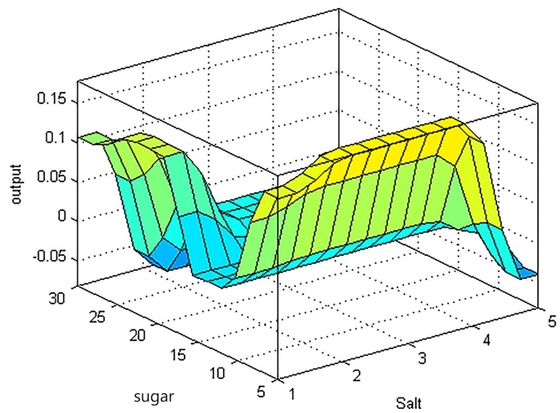

FIGURE 7. The relationship between sugar and salt with blood pressure

7. Conclusion

In this paper, the cardiovascular fuzzy model and advantages are introduced and fuzzy instantaneous pressure-volume relation is presented and it has been shown that the fuzzy acquisition of the factors of this model can also provide fuzzy blood flow back. A first order fuzzy differential equation for modeling the arterial pressure is investigated and all types of solutions of this equation are obtained. Finally, in order to prove the claim that pressure, volume, and left ventricular outflow are dynamic and fuzzy factors, 15 factors were considered and according to the ANFIS network, it was shown that these inputs affect the pressure. In this regard,70% of the data was used as train data and 30% as check data, the results of which are shown in Table 10, which shows that the model has been very confident.

8. Appendix

Let f is a fuzzy-valued function of the variable $t > 0$ and s is a real parameter . The fuzzy Laplace transform is defined as following [16]

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} \odot f(t)dt = \lim_{\tau\to\infty} \int_0^\tau e^{-st} \odot f(t)dt. \tag{8.1}$$

whenever the limit exist (as a finite fuzzy number).

Lemma 8.1. *(See [16]) Consider $f(t)$ and $g(t)$ are fuzzy-valued functions. Let a, b are two real constant such that $a, b \geq 0$ (or $a, b \leq 0$). If the fuzzy Laplace transform f exists for $Re(s) > \alpha$ and the fuzzy Laplace transform g exists for $Re(s) > \beta$, hence the fuzzy Laplace transform of $af(t) \oplus bg(t)$ and $af(t) \ominus_{gH} bg(t)$ exist for $Re(s) > Max\{\alpha, \beta\}$, and*

$$a\mathcal{L}[f(t)] \ominus_{gH} b\mathcal{L}[g(t)] = \mathcal{L}[af(t) \ominus_{gH} bg(t)].$$

Theorem 8.2. *(See [16]) Let us consider*

1: *f be a fuzzy-valued function that is fuzzy continuous for $t \geq 0$ and of exponential α.*

2: *$f'_{gH}(t)$ be piecewise continuous in every finite closed interval $\mathbb{I} = [0, b]$.*

Moreover, assume that $f(t)$ is $gH-$differentiable in $\mathbb{I}$ provided that the type of $gH-$differentiability doesn't change in interval $\mathbb{I}$. If $Re(s) > \beta$ then the fuzzy Laplace transform of $f(t)$ by considering the type of gH-differentiability is

1: *If $f(t)$ is $[i - gH]-$differentiable in $\mathbb{I}$, then*

$$\mathcal{L}[f'_{i.gH}(t)] = sF(s) \ominus_H f(0). \tag{8.2}$$

2: *If $f(t)$ is $[ii - gH]-$differentiable in $\mathbb{I}$, then*

$$\mathcal{L}[f'_{ii.gH}(t)] = (-1)f(0) \ominus_{gH} (-1)sF(s). \tag{8.3}$$

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