Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 9, No. 4, 2021, pp. 1176-1197
DOI:10.22034/cmde.2020.35895.1621

Laguerre collocation method for solving Lane-Emden type equations

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Abstract In this paper, a Laguerre collocation method is presented in order to obtain numerical solutions for linear and nonlinear Lane-Emden type equations and their initial conditions. The basis of the present method is operational matrices with respect to modified generalized Laguerre polynomials(MGLPs) that transforms the solution of main equation and its initial conditions to the solution of a matrix equation corresponding to the system of algebraic equations with the unknown Laguerre coefficients. By solving this system, coefficients of approximate solution of the main problem will be determined. Implementation of the method is easy and has more accurate results in comparison with results of other methods.

Keywords. Modified generalized Laguerre polynomials, Numerical analysis, Collocation method, Operational matrices, Lane-Emden type equations.

2010 Mathematics Subject Classification. 65L05, 34K28, 41A10, 65M70.

1. Introduction

Most problems in science and engineering and other fields need solving the linear and nonlinear differential equations. For solving these equations many analytical methods such as the $(\frac{G'}{G})$-expansion method [2], the Exp-function method [3], the differential transformation method [4], and numerical methods such as the Jacobi operational matrix collocation [5], the finite difference, the finite element and the spectral methods have been introduced [8, 13].

The spectral methods have developed quickly in the recent two decades. They have become basic tools for numerical solving of ordinary differential equations(ODEs) and partial differential equations(PDEs). The main advantage of the spectral methods over other existing methods may be its ability in finding an accurate solution for

Received: 13 October 2019 ; Accepted: 21 July 2020.
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the problem. These methods became distinguished in the 1970s [24] and the three common spectral types are the Galerkin, collocation and Tau procedures.
Collocation methods are very accurate and efficient procedures for numerical solving of linear and nonlinear differential equations. Their fundamental idea is to consider the unknown solution $y(x)$ which can be approximated by a linear combination of certain basic functions, called the trial functions, such as orthogonal polynomials. The orthogonal polynomials can be selected matching to their specific attributes which construct them especially appropriate for the problem under investigation. We apply modified generalized Laguerre polynomials(MGLPs) to instruct Laguerre collocation method.

Lane-Emden equation was first studied by astrophysicists Jonathan Homer Lane and Robert Emden, where they investigated the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of the thermodynamics [18, 26]. This equation is generally formulated as follows:

$$y''(x) + \frac{k}{x}y'(x) + f(x)g(y(x)) = h(x), \qquad k > 0, \qquad x > 0, \tag{1.1}$$

with initial conditions

$$y(0) = a, \qquad y'(0) = b, \tag{1.2}$$

where the prime denotes the differentiation with respect to x, k is constant, $f(x)$, $g(y)$ and $h(x)$ are some given continuous functions and a, b are constants. It is one of the singular initial value problems formulated by the second-order ODE, in which singularity behavior occurs at $x = 0$. It is undeniable that an analytic solution of (1.1) is always possible in the neighborhood of the singular point for the initial conditions (1.2) [9].

Currently, many techniques used in studying Lane-Emden type equations are based on either series solutions or perturbation techniques such as, Pade approximations method [10], Quasi-Newton's method [30], Adomian decomposition method(ADM) [26, 27], variational iteration method [28], Legendre spectral method [1], Jacobi-Gauss collocation method [6], Chebyshev collocation method [7], Bessel collocation method [29], Hermite function collocation method(HFCM) [16], Hermite wavelets method [20], operational matrix of integration [21], wavelet series collocation method [22], and so on. Here, modified generalized Laguerre collocation method(MGLCM) is applied for solving some special cases of linear/nonlinear and homogenous/nonhomogenous Lane-Emden equation (1.1) and to illustrate its accuracy and efficiency in comparison with other existing numerical methods.

The paper is organized as follows. In section 2, we present an overview on MGLPs and their related features required afterward. In section 3, we explain the general procedure of generating operational matrices of derivative and product with respect to MGLPs. The creation of collocation method based on MGLPs is described in sections 4. In section 5, the suggested method is used to solve several types of Lane-Emden equation and comparisons are made between our obtained results and the existing analytical or numerical solutions that were presented in other published works. The conclusion is given in section 6.

2. Basic preparations on MGLPs

Some basic properties and results on MGLPs are introduced in this section.

Definition 2.1. Let $X = [0, +\infty)$ and $w^{(\alpha,\beta)}(x) = x^{\alpha} e^{-\beta x}$, $\alpha > -1$, $\beta > 0$, be a weight function on X. The MGLP of degree n is defined by

$$L_n^{(\alpha,\beta)}(x) = \frac{1}{n!} x^{-\alpha} e^{\beta x} \partial_x^n (x^{n+\alpha} e^{-\beta x}), \qquad n = 0, 1, 2, \dots \tag{2.1}$$

They are solutions of the Sturm-Liouville problem and satisfy the following three-term recurrence formula [14, 23]:

$$\begin{aligned} &L_0^{(\alpha,\beta)}(x) = 1, \\ &L_1^{(\alpha,\beta)}(x) = 1 + \alpha - \beta x, \\ &L_{n+1}^{(\alpha,\beta)}(x) = \frac{2n + \alpha - \beta x + 1}{n+1} L_n^{(\alpha,\beta)}(x) - \frac{n+\alpha}{n+1} L_{n-1}^{(\alpha,\beta)}(x), \qquad n = 1, 2, \dots . \end{aligned} \tag{2.2}$$

Also, the $n-$th degree MGLP has the analytical form

$$L_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} (-1)^k \frac{\beta^k \Gamma(n + \alpha + 1)}{\Gamma(k + \alpha + 1)(n-k)!\ k!} x^k, \tag{2.3}$$

where $\Gamma(a) = \int_0^{+\infty} x^{a-1} e^{-x} dx$, is the Gamma function [15, 19].
Some other properties of the analytical form of MGLPs are presented as the following statements and lemmas:

$$\begin{aligned} &1 - L_n^{(\alpha,\beta)}(0) = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)}, \qquad n = 0, 1, 2, \dots \\ &2 - \frac{d^i}{dx^i} L_n^{(\alpha,\beta)}(x) = \sum_{k=i}^{n} (-1)^k \frac{\beta^k\ \Gamma(n+\alpha+1)\ x^{k-i}}{\Gamma(k+\alpha+1)\Gamma(k-i+1)(n-k)!}, \quad i \le n \\ &3 - \frac{d^i}{dx^i} L_n^{(\alpha,\beta)}(x) = 0, \qquad n = 0, 1, 2, \dots, \qquad i > n. \end{aligned} \tag{2.4}$$

Lemma 2.2. *The nth modified generalized Laguerre polynomial can be written as*

$$L_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \gamma_k^{(n)} x^k, \tag{2.5}$$

where $\gamma_k^{(n)} = (-1)^k \frac{\beta^k \Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)(n-k)!k!}$.

Proof. $\gamma_k^{(n)}$, are the Maclaurin series coefficients of $L_n^{(\alpha,\beta)}(x)$.
Thus, $\gamma_k^{(n)} = \frac{1}{k!} \frac{d^k}{dx^k} L_n^{(\alpha,\beta)}(x)\big|_{x=0}$. Now, the lemma can be proved by using properties (1-3) mentioned (2.4). □

Lemma 2.3. *If $p \ge 0$, then*

$$\int_0^{+\infty} x^p L_n^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x) dx = \sum_{l=0}^{n} \frac{(-1)^l \Gamma(n+\alpha+1)(p+l+\alpha)!}{\Gamma(l+\alpha+1)(n-l)!\ l!\ \beta^{p+\alpha+1}}. \tag{2.6}$$

Proof. From analytical form of $L_n^{(\alpha,\beta)}(x)$, in (2.3) and using integration by parts we have,

$$\int_0^{+\infty} x^p L_n^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x)dx = \int_0^{+\infty} x^p \sum_{l=0}^{n}\left[\frac{(-1)^l\beta^l\Gamma(n+\alpha+1)x^l}{\Gamma(l+\alpha+1)(n-l)!\ l!}\right]x^\alpha e^{-\beta x}dx$$
$$= \sum_{l=0}^{n}\left[\frac{(-1)^l\beta^l\Gamma(n+\alpha+1)}{\Gamma(l+\alpha+1)(n-l)!\ l!}\int_0^{+\infty} x^{p+l+\alpha}e^{-\beta x}\ dx\right]$$
$$= \sum_{l=0}^{n}\frac{(-1)^l\Gamma(n+\alpha+1)(p+l+\alpha)!}{\Gamma(l+\alpha+1)(n-l)!\ l!\ \beta^{p+\alpha+1}}.$$

□

Lemma 2.4. *If $L_j^{(\alpha,\beta)}(x)$ and $L_k^{(\alpha,\beta)}(x)$ are $j-th$ and $k-th$ MGLPs respectively, then their product can be written as*

$$Q_{j+k}^{(\alpha,\beta)}(x) = \sum_{r=0}^{j+k}\lambda_r^{(j,k)}\ x^r, \tag{2.7}$$

where

$$\lambda_r^{(j,k)} = \sum_{l=0}^{r}\frac{(-1)^r\beta^r\Gamma(j+\alpha+1)\Gamma(k+\alpha+1)}{\Gamma(r-l+\alpha+1)\Gamma(l+\alpha+1)(j-r+l)!(r-l)!(k-l)!\ l!}.$$

Proof. The polynomial $Q_{j+k}^{(\alpha,\beta)}(x) = L_j^{(\alpha,\beta)}(x)L_k^{(\alpha,\beta)}(x)$ will be a polynomial of degree $j+k$ and considering lemma 2.2 it can be written as:

$$Q_{j+k}^{(\alpha,\beta)}(x) = (\sum_{m=0}^{j}\gamma_m^{(j)}\ x^m)(\sum_{n=0}^{k}\gamma_n^{(k)}\ x^n) = \sum_{r=0}^{j+k}\lambda_r^{(j,k)}x^r.$$

where

$$\lambda_r^{(j,k)} = \sum_{l=0}^{r}\gamma_{r-l}^{(j)}\gamma_l^{(k)}$$
$$= \sum_{l=0}^{r}\left[\frac{(-1)^{r-l}\beta^{r-l}\ \Gamma(j+\alpha+1)}{\Gamma(r-l+\alpha+1)(j-r+l)!(r-l)!}\cdot\frac{(-1)^l\beta^l\ \Gamma(k+\alpha+1)}{\Gamma(l+\alpha+1)(k-l)!\ l!}\right]$$
$$= \sum_{l=0}^{r}\frac{(-1)^r\beta^r\ \Gamma(j+\alpha+1)\Gamma(k+\alpha+1)}{\Gamma(r-l+\alpha+1)\Gamma(l+\alpha+1)(j-r+l)!(r-l)!(k-l)!\ l!}.$$

The relation between coefficients $\lambda_r^{(j,k)}$ with coefficients $\gamma_m^{(j)}$ and $\gamma_n^{(k)}$ is in an algorithm as follows.

If $j \geq k$ **:**
$r = 0, 1, \ldots, j+k,$
$if \;\; r > j \;\; then$
$\quad \lambda_r^{(j,k)} = \sum_{l=r-j}^{k} \gamma_{r-l}^{(j)} \; \gamma_l^{(k)},$
$else$
$\quad r_1 = min\{r, k\},$
$\quad \lambda_r^{(j,k)} = \sum_{l=0}^{r_1} \gamma_{r-l}^{(j)} \; \gamma_l^{(k)},$
$end.$

If $j < k$ **:**
$r = 0, 1, \ldots, j+k,$
$if \;\; r \leq j \;\; then$
$\quad r_1 = min\{r, j\},$
$\quad \lambda_r^{(j,k)} = \sum_{l=0}^{r_1} \gamma_{r-l}^{(j)} \; \gamma_l^{(k)},$
$else$
$\quad r_2 = min\{r, k\},$
$\quad \lambda_r^{(j,k)} = \sum_{l=r-j}^{r_2} \gamma_{r-l}^{(j)} \; \gamma_l^{(k)},$
$end.$

Thus, the coefficients $\lambda_r^{(j,k)}$ are determined. □

Lemma 2.5. *If $L_i^{(\alpha,\beta)}(x)$, $L_j^{(\alpha,\beta)}(x)$ and $L_k^{(\alpha,\beta)}(x)$ are $i-$, $j-$ and $k-$th MGLPs respectively, then*

$$\int_0^{+\infty} L_i^{(\alpha,\beta)}(x) L_j^{(\alpha,\beta)}(x) L_k^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x) dx = q_{ijk}, \tag{2.8}$$

where

$$q_{ijk} = \sum_{l=0}^{k} \sum_{r=0}^{i+j} \frac{(-1)^l \lambda_r^{(i,j)} \Gamma(k+\alpha+1) \Gamma(r+l+\alpha+1)}{\Gamma(l+\alpha+1)(k-l)! \; l! \; \beta^{r+\alpha+1}}, \tag{2.9}$$

and $\lambda_r^{(j,k)}$ has been introduced in Lemma 2.4.

Proof. Simply, the lemma can be proved by setting $L_j^{(\alpha,\beta)}(x) L_k^{(\alpha,\beta)}(x) = Q_{j+k}^{(\alpha,\beta)}(x)$, and applying (2.3) and Lemmas 2.3 − 2.4. □

3. Operational matrix

Here, the operational matrices of derivatives and product of MGLPs are expressed. Firstly, the concept of function approximation is introduced.

3.1. **Function approximation.** The set of MGLPs is the $L^2_{w^{(\alpha,\beta)}}(X)$-orthogonal system, namely

$$\begin{aligned} \left\langle L_j^{(\alpha,\beta)}(x), L_k^{(\alpha,\beta)}(x) \right\rangle_{w^{(\alpha,\beta)},X} &= \int_0^{+\infty} L_j^{(\alpha,\beta)}(x) L_k^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x) dx \\ &= h_k^{(\alpha,\beta)} \delta_{jk}, \qquad j,k = 0,1,2,\ldots, \end{aligned} \tag{3.1}$$

where δ_{jk} is the Kronecker function, $h_k^{(\alpha,\beta)} = \frac{\Gamma(k+\alpha+1)}{\beta^{\alpha+1} k!}$ and $L^2_{w^{(\alpha,\beta)}}(X)$ is a Hilbert space.

A function $y(x) \in L^2_{w^{(\alpha,\beta)}}(X)$ can be expanded in terms of MGLPs as follows:

$$\begin{aligned} y(x) &= \sum_{j=0}^{+\infty} c_j L_j^{(\alpha,\beta)}(x), \\ c_j &= \frac{1}{h_j^{(\alpha,\beta)}} \int_0^{+\infty} y(x) L_j^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x) dx, \quad j = 0, 1, 2, \dots . \end{aligned} \tag{3.2}$$

In implementation, just the first $(N+1)$ terms of MGLPs are considered. Then we have

$$y(x) \simeq y_N(x) = \sum_{j=0}^{N} c_j L_j^{(\alpha,\beta)}(x) = C^T \Phi(x), \tag{3.3}$$

where, the MGLP coefficient vector C and the MGLP vector $\Phi(x)$ respectively, are defined by

$$C = [c_0, c_1, \dots, c_N]^T, \qquad \Phi(x) = [L_0^{(\alpha,\beta)}(x), L_1^{(\alpha,\beta)}(x), \dots, L_N^{(\alpha,\beta)}(x)]^T. \tag{3.4}$$

3.2. **Operational matrix of derivative.** The basic purpose of current subsection is to extract the operational matrix of derivative from the MGLP vector $\Phi(x)$ defined in (3.4).

Definition 3.1. Suppose $\Phi(x)$ be a MGLP vector, the matrix $D_{(N+1)\times(N+1)}$ is named as the operational matrix of derivative if and only if

$$\frac{d\Phi(x)}{dx} = D \ \Phi(x). \tag{3.5}$$

Theorem 3.2. *The operational matrix of derivation D is defined by:*

$$D = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ -\beta & 0 & 0 & \cdots & 0 & 0 \\ -\beta & -\beta & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ -\beta & -\beta & -\beta & \cdots & 0 & 0 \\ -\beta & -\beta & -\beta & \cdots & -\beta & 0 \end{pmatrix}. \tag{3.6}$$

Proof. MGLPs are satisfy the recurrence relation

$$\frac{d}{dx} L_n^{(\alpha,\beta)}(x) = \frac{d}{dx} L_{n-1}^{(\alpha,\beta)}(x) - \beta L_{n-1}^{(\alpha,\beta)}(x), \qquad n = 1, 2, \dots$$

thus,

$$\frac{d}{dx} L_n^{(\alpha,\beta)}(x) = \frac{d}{dx} L_{n-2}^{(\alpha,\beta)}(x) - \beta L_{n-2}^{(\alpha,\beta)}(x) - \beta L_{n-1}^{(\alpha,\beta)}(x), \quad n = 2, 3, \dots .$$

Using the above relation sequentially, yields

$$\frac{d}{dx} L_n^{(\alpha,\beta)}(x) = \frac{d}{dx} L_{n-n}^{(\alpha,\beta)}(x) - \beta \sum_{m=0}^{n-1} L_m^{(\alpha,\beta)}(x), \qquad n = 0, 1, 2, \dots,$$

whereof, $L_0^{(\alpha,\beta)}(x) = 1$, we give

$$\frac{d}{dx} L_n^{(\alpha,\beta)}(x) = -\beta \sum_{m=0}^{n-1} L_m^{(\alpha,\beta)}(x), \qquad n = 0, 1, 2, \ldots .$$

Generally, for the vector $\Phi(x) = [L_0^{(\alpha,\beta)}(x), L_1^{(\alpha,\beta)}(x), \ldots, L_N^{(\alpha,\beta)}(x)]^T$ we have matrix notation with the following form

$$\frac{d}{dx}\Phi(x) = \left[\ \tfrac{d}{dx}L_0^{(\alpha,\beta)}(x), \tfrac{d}{dx}L_1^{(\alpha,\beta)}(x), \ldots, \tfrac{d}{dx}L_{N-1}^{(\alpha,\beta)}(x), \tfrac{d}{dx}L_N^{(\alpha,\beta)}(x)\ \right]^T$$

$$= \left[\ 0, -\beta L_0^{(\alpha,\beta)}(x), \ldots, -\beta \textstyle\sum_{m=0}^{N-2} L_m^{(\alpha,\beta)}(x), -\beta \sum_{m=0}^{N-1} L_m^{(\alpha,\beta)}(x)\ \right]^T$$

$$= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ -\beta & 0 & 0 & \cdots & 0 & 0 \\ -\beta & -\beta & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ -\beta & -\beta & -\beta & \cdots & 0 & 0 \\ -\beta & -\beta & -\beta & \cdots & -\beta & 0 \end{bmatrix} \begin{bmatrix} L_0^{(\alpha,\beta)}(x) \\ L_1^{(\alpha,\beta)}(x) \\ L_2^{(\alpha,\beta)}(x) \\ \vdots \\ L_{N-1}^{(\alpha,\beta)}(x) \\ L_N^{(\alpha,\beta)}(x) \end{bmatrix} = D\Phi(x).$$

□

Clearly, by using (3.5) we have $\frac{d^n\Phi(x)}{dx^n} = (D^{(1)})^n\Phi(x)$, where $n \in \mathbb{N}$, and the superscript in $D^{(1)}$ expresses matrix powers. Therefore $D^{(n)} = (D^{(1)})^n$.

3.3. **Operational matrix of product.** In this subsection, a general process is introduced to find the $(N+1) \times (N+1)$ operational matrix of product of MGLPs.

Definition 3.3. Suppose $C = [c_0, c_1, \ldots, c_N]^T$, $\tilde{C}$ is named as operational matrix of product if and only if

$$\Phi(x)\Phi^T(x)C \simeq \tilde{C}\Phi(x). \tag{3.7}$$

The elements of matrix $\tilde{C}$ are determined using the following theorem.

Theorem 3.4. *The elements of the matrix $\tilde{C}$ in (3.7) are calculated as:*

$$\tilde{C}_{jk} = \frac{1}{h_k^{(\alpha,\beta)}} \sum_{i=0}^{N} C_i q_{ijk}, \qquad j, k = 0, 1, \ldots, N, \tag{3.8}$$

where

$$q_{ijk} = \int_0^{+\infty} L_i^{(\alpha,\beta)}(x) L_j^{(\alpha,\beta)}(x) L_k^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x)dx, \quad i, j, k = 0, 1, \ldots, N,$$

are computed by means of lemma 2.5 *and C_i are the elements of the vector C in* (3.7).

Proof. The left side of (3.7) is as follows,

$$\Phi(x)\Phi^T(x)C = \begin{bmatrix} \sum_{i=0}^{N} C_i L_0^{(\alpha,\beta)}(x) L_i^{(\alpha,\beta)}(x) \\ \sum_{i=0}^{N} C_i L_1^{(\alpha,\beta)}(x) L_i^{(\alpha,\beta)}(x) \\ \vdots \\ \sum_{i=0}^{N} C_i L_N^{(\alpha,\beta)}(x) L_i^{(\alpha,\beta)}(x) \end{bmatrix}. \quad (3.9)$$

One puts,

$$L_j^{(\alpha,\beta)}(x) L_i^{(\alpha,\beta)}(x) = \sum_{k=0}^{N} a_k L_k^{(\alpha,\beta)}(x), \qquad i,j = 0,1,\dots,N. \quad (3.10)$$

Multiplying both side of (3.10) by $L_m^{(\alpha,\beta)}(x)$, $m = 0,1,\dots,N$, and integrating from 0 to $+\infty$ results,

$$\int_0^{+\infty} L_j^{(\alpha,\beta)}(x) L_i^{(\alpha,\beta)}(x) L_m^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x)dx$$
$$= \sum_{k=0}^{N} a_k \int_0^{+\infty} L_k^{(\alpha,\beta)}(x) L_m^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x)dx = a_m h_m^{(\alpha,\beta)}.$$

Therefore,

$$a_m = \frac{1}{h_m^{(\alpha,\beta)}} \int_0^{+\infty} L_j^{(\alpha,\beta)}(x) L_i^{(\alpha,\beta)}(x) L_m^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x)dx.$$

Now suppose,

$$\int_0^{+\infty} L_j^{(\alpha,\beta)}(x) L_i^{(\alpha,\beta)}(x) L_m^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x)dx = q_{ijm}, \qquad i,j,m = 0,1,\dots,N.$$

So, one gets,

$$a_m = \frac{q_{ijm}}{h_m^{(\alpha,\beta)}}, \qquad i,j,m = 0,1,\dots,N.$$

Substituting a_m into (3.10) one has

$$L_j^{(\alpha,\beta)}(x) L_i^{(\alpha,\beta)}(x) = \sum_{k=0}^{N} \frac{q_{ijk}}{h_k^{(\alpha,\beta)}} L_k^{(\alpha,\beta)}(x), \qquad i,j = 0,1,\dots,N.$$

Thus, each element of the vector $\Phi(x)\Phi^T(x)C$ in (3.9), is as

$$\begin{aligned} \sum_{i=0}^{N} C_i L_j^{(\alpha,\beta)}(x) L_i^{(\alpha,\beta)}(x) &= \sum_{i=0}^{N} C_i \left(\sum_{k=0}^{N} \frac{q_{ijk}}{h_k^{(\alpha,\beta)}} L_k^{(\alpha,\beta)}(x) \right) \\ &= \sum_{k=0}^{N} \left[\frac{1}{h_k^{(\alpha,\beta)}} \sum_{i=0}^{N} C_i q_{ijk} \right] L_k^{(\alpha,\beta)}(x) \quad (3.11) \\ &= \sum_{k=0}^{N} \tilde{C}_k L_k^{(\alpha,\beta)}(x). \end{aligned}$$

Using (3.11) for other elements of the vector $\Phi(x)\Phi^T(x)C$ leads to the desired result. □

Forthcoming theorem offers the public relation for approximating nonlinear term $v^r(x)u^s(x)$ which may appear in nonlinear equations.

Theorem 3.5. *If*

$$\begin{aligned} u(x) &\simeq C^T\Phi(x) = \Phi(x)^T C, & v(x) &\simeq V^T\Phi(x) = \Phi^T(x)V, \\ \Phi(x)\Phi^T(x)C &\simeq \tilde{C}\Phi(x), & \Phi(x)\Phi^T(x)V &\simeq \tilde{V}\Phi(x), \end{aligned} \tag{3.12}$$

where C and V are the $(N+1)$ vectors and $\tilde{C}$ and $\tilde{V}$ are the $(N+1)\times(N+1)$ operational matrices of product, then the following relation is valid:

$$v^r(x)u^s(x) \simeq V^T(\tilde{V})^{r-1}\tilde{B}_{s-1}\Phi(x), \quad B_{s-1} = (\tilde{C}^T)^{s-1}C, \quad r,s = 1,2,\dots .$$

Proof. One has

$$u^2(x) \simeq (\Phi^T(x)C)^2 = C^T\Phi(x)\Phi^T(x)C = C^T\tilde{C}\Phi(x),$$

So, by use of induction $u^s(x)$ will be approximated as

$$u^s(x) \simeq C^T(\tilde{C})^{s-1}\Phi(x), \qquad s = 1,2,\dots .$$

To similar way, $v^r(x)$ is approximated as

$$v^r(x) \simeq V^T(\tilde{V})^{r-1}\Phi(x), \qquad r = 1,2,\dots .$$

By applying the declared formulas and induction is easily observed,

$$v^r(x)u^s(x) \simeq V^T(\tilde{V})^{r-1}\tilde{B}_{s-1}\Phi(x), \quad B_{s-1} = (\tilde{C}^T)^{s-1}C, \quad r,s = 1,2,\dots .$$

□

4. Modified generalized Laguerre collocation method

This section includes two subsections. The first one describes the concept of spectral methods and modified generalized Laguerre collocation method (MGLCM). The existence, uniqueness, convergence and stability of approximation and related theorems are introduced in the second subsection.

4.1. Explaining the method.

Spectral methods in the subject of numerical procedures for solving differential equations commonly belong to the category of weighted residual methods(WRMs). WRMs are included a special class of approximate techniques, in which the residuals (or errors) are minimized in a specific procedure and therewith leading to particular methods containing Galerkin, Petrov-Galerkin, collocation and Tau methods [19].

Assume the general problem:

$$L[y(x)] + N[y(x)] = f(x), \qquad x \in X, \tag{4.1}$$

with the initial conditions

$$y^{(i)}(0) = d_i, \qquad i = 0,1,\dots,m-1, \tag{4.2}$$

where L and N are linear and nonlinear operators, respectively, and $f(x)$ is a function of variable x and $d_i (i = 0, 1, \ldots, m-1)$ are initial values of unknown function $y(x)$. The beginning of the WRM is to approximate the solution $y(x)$ by a finite sum

$$y(x) \simeq y_N(x) = \sum_{i=0}^{N} c_i \phi_i(x), \qquad x \in X, \qquad N \in \mathbb{N}, \tag{4.3}$$

where $\phi_i(x)$ are the basic or trial functions and the expansion coefficients must be specified. Replacing $y(x)$ by $y_N(x)$ in (4.1) leads to the residual function:

$$R_N(x) = L[y_N(x)] + N[y_N(x)] - f(x) \neq 0. \tag{4.4}$$

The idea of the WRM is to compel the residual to zero by needing:

$$\langle R_N, \psi_j \rangle_{w,X} = \int_X R_N(x)\psi_j(x)w(x)dx = 0, \qquad 0 \leq j \leq N, \tag{4.5}$$

where $\psi_j(x)$ are test functions and w is positive weight function.

The choice of trial and test functions is one of the most important properties that recognizes spectral methods from finite-element and finite-difference methods. Spectral methods use globally smooth functions as trial and test functions. The most generally used trial and test functions are trigonometric functions or orthogonal polynomials (typically, the eigenfunctions of singular Sturm-Liouville problems) [19]. Therefore, with selecting MGLPs as the trial functions in (4.3), i.e., $\phi_i(x) = L_i^{(\alpha,\beta)}(x)$ and $\psi_j(x) = \delta(x - x_j)$ as test functions and $w(x) = w^{(\alpha,\beta)}(x)$ in (4.5), the residual function is forced to zero at x_j, i.e., $R_N(x_j) = 0$, which δ, is the Dirac delta function, i.e., $\delta(x) = 1$ for $x = 0$ and $\delta(x) = 0$ for otherwise, and x_j are preassigned collocation points. The name of this method is modified generalized Laguerre collocation method(MGLCM).

Hence, using (4.3) with MGLPs as the trial functions and applying the expressed matrices and approximations, the terms of assumed equations (4.1) are approximated and replaced into (4.1) and the residual function is obtained. With collocating the residual function at $(N - m + 1)$ scaled roots of the $(N - m + 1)-$th MGLP(i.e., Laguerre-Gauss points) on interval $(0, l)$ or other suitable collocation points for each two particular selections of modified generalized Laguerre parameters α and β, the $(N-m+1)$ linear or nonlinear algebraic equations with $(N+1)$ unknowns are obtained. Consequently, this system of algebraic equations alongside m equations are generated by substituting introduced matrices and approximations into initial conditions (4.2) gives an algebraic system of $(N + 1)$ equations with $(N + 1)$ unknowns. This system is solved with the aid of Maple software and the unknown spectral coefficients vector C is determined and approximate solution $y_N(x)$ is calculated.

Notation. All roots of $(N + m - 1)-$th MGLP are positive and non-zero, so the singularity of Lane-Emden type equations does not occur at these collocation points. Therefore the equations obtained at these points, together with the approximate equations of initial conditions, yield the favorable algebraic systems. Furthermore for other suitable collocation points, such as $x_i = \frac{l}{N} i, \;\; i = 0, 1, 2, \ldots, N$, the singularity is eliminate by assuming the first node point $x_0 > 0$, such as $x_0 = 0.0001$. Then, by replacing the approximate equations of initial conditions with the first two equations, the favorable algebraic system is obtained.

4.2. **Existence, uniqueness, convergence and stability of approximation.** Let N be any non-negative integer and $\mathbb{P}_N$ denotes the linear space of polynomials whose degree is at most N on X. Clearly, the dimension of $\mathbb{P}_N$ is $N+1$ and the polynomials $L_n^{(\alpha,\beta)}(x)$, $n = 0, 1, \ldots, N$ are orthogonal with respect to inner product (3.1). Therefore they form a basis for $\mathbb{P}_N$, i.e., $\mathbb{P}_N = Span\{L_0^{(\alpha,\beta)}(x), L_1^{(\alpha,\beta)}(x), \ldots, L_N^{(\alpha,\beta)}(x)\}$.

According to Weierstrass theorem in a bounded closed interval and its extended type for unbounded domains, any continuous function can be uniformly approximated by polynomials.[12]

We define the orthogonal projection operator $P_{N,w^{(\alpha,\beta)}} : L^2_{w^{(\alpha,\beta)}}(X) \to \mathbb{P}_N$, for any continuous function y as $P_{N,w^{(\alpha,\beta)}}(y) = y_N := \sum_{n=0}^{N} c_n L_n^{(\alpha,\beta)}(x)$, where Laguerre coefficients c_n are computed according to (3.2). Since $\{L_n^{(\alpha,\beta)}(x)\}_{n=0}^{N}$ are linearly independent, y_N is determined uniquely. It turns out that $P_{N,w^{(\alpha,\beta)}}$ is a linear operator. We call y_N the orthogonal projection of y onto $\mathbb{P}_N$ through the inner product (3.1).

At this point, the best approximation problem is finding one polynomial among all the polynomials of degree less or equal to a fixed integer N, which be best approximates uniformly in X for a given continuous function y. It can be formulated in terms of the norm $\|.\|_{w^{(\alpha,\beta)}}$ and the next proposition fully determines the solution of this problem [12, 15].

Theorem 4.1. *For any $y \in L^2_{w^{(\alpha,\beta)}}(X)$, there exists a unique polynomial $\varphi_{N,w^{(\alpha,\beta)}}(y) \in \mathbb{P}_N$ that*

$$\|y - \varphi_{N,w^{(\alpha,\beta)}}(y)\|_{w^{(\alpha,\beta)}} = inf_{\psi \in \mathbb{P}_N} \|y - \psi\|_{w^{(\alpha,\beta)}}. \tag{4.6}$$

Moreover $\varphi_{N,w^{(\alpha,\beta)}}(y) = y_N$.

To proof of Theorem 4.1 is referred to [12].
In short, we can write

$$\|y - y_N\|_{w^{(\alpha,\beta)}} = inf_{\psi \in \mathbb{P}_N} \|y - \psi\|_{w^{(\alpha,\beta)}}. \tag{4.7}$$

Another interesting characterization is given in the following theorem.

Theorem 4.2. *For any $y \in L^2_{w^{(\alpha,\beta)}}(X)$, we have*

$$\langle y - y_N, \phi\rangle_{w^{(\alpha,\beta)},X} = 0, \qquad \forall \phi \in \mathbb{P}_N. \tag{4.8}$$

To proof of Theorem 4.2 is referred to [12].
By virtue of this theorem, the operator $P_{N,w^{(\alpha,\beta)}}$ takes the name of orthogonal projector, since the error $y - y_N$ is orthogonal to the space $\mathbb{P}_N$. Choosing in particular $\phi = y_N$ in (4.8), application of the Schwartz inequality leads to $\|y_N\|_{w^{(\alpha,\beta)}} \le \|y\|_{w^{(\alpha,\beta)}}$. The convergence of the Laguerre approximation is deduced from the next result[12].

Theorem 4.3. *For any $y(x) \in L^2_{w^{(\alpha,\beta)}}(X)$, the sequence $\{y_N(x)\}_{N\ge 0}$ defined in (4.3) using MGLPs is converges to $y(x)$. i.e. we have*

$$lim_{N\to+\infty} \|y - y_N\|_{w^{(\alpha,\beta)}} = 0. \tag{4.9}$$

Proof. Since $\mathbb{P}_N$ is a complete Hilbert space, so it is sufficient we show the sequence of patrial sums from $\{y_N(x)\}_{N\geq 0}$ is a Cauchy sequence.
Suppose $\{p_k^{(\alpha,\beta)}(x)\}_{k\geq 0}$ be the orthonormal form for MGLPs. Then

$$y_N(x) = \sum_{k=0}^{N} d_k p_k^{(\alpha,\beta)}(x), \qquad d_k = c_k (h_k^{(\alpha,\beta)})^{\frac{1}{2}}, \qquad p_k^{(\alpha,\beta)}(x) = \frac{L_k^{(\alpha,\beta)}(x)}{(h_k^{(\alpha,\beta)})^{\frac{1}{2}}}.$$

Also, we define the sequence of partial sums $\{S_M(x)\}_{M\geq 0}$ from $\{y_N(x)\}_{N\geq 0}$, such that $S_M(x) = \sum_{k=0}^{M} d_k p_k^{(\alpha,\beta)}(x)$.
With $L^2_{w^{(\alpha,\beta)}}$-norm, for arbitrary L and M; $L < M$, we have

$$\begin{aligned}
\|S_M - S_L\|^2 &= \|\sum_{k=L+1}^{M} d_k p_k^{(\alpha,\beta)}(x)\|^2 \\
&= \left\langle \sum_{k=L+1}^{M} d_k p_k^{(\alpha,\beta)}(x), \sum_{j=L+1}^{M} d_j p_j^{(\alpha,\beta)}(x) \right\rangle \\
&= \sum_{k=L+1}^{M} \sum_{j=L+1}^{M} d_k d_j \langle p_k^{(\alpha,\beta)}(x), p_j^{(\alpha,\beta)}(x) \rangle \\
&= \sum_{k=L+1}^{M} |d_k|^2.
\end{aligned}$$

By Bessels inequality, since

$$\sum_{k=L+1}^{M} |d_k|^2 \leq \|\sum_{k=L+1}^{M} d_k p_k^{(\alpha,\beta)}(x)\|^2 \leq \|y(x)\|^2,$$

therefore $\sum_{k=0}^{+\infty} |d_k|^2$ is bounded and convergent. Hence $\|S_M - S_L\|^2 \to 0$ as $L, M \to 0$. This implies $\|S_M - S_L\| \to 0$. Then $\{S_M(x)\}_{M\geq 0}$ is a Cauchy sequence and it converges to $S(x) \in \mathbb{P}_N$.
Now, we assert that $S(x) = y(x)$.
By the projection operator, we have

$$\langle S(x) - y(x), p_j^{(\alpha,\beta)}(x) \rangle_{w^{(\alpha,\beta)}} = 0, \quad j = 0, 1, 2, \dots, M.$$

Then $S(x) - y(x) = 0$. Hence $S(x) = y(x)$ and $y_N(x) = \sum_{k=0}^{N} c_k L_k^{(\alpha,\beta)}(x)$ converges to $y(x)$ as $N \to +\infty$. Thus $lim_{N\to+\infty} \|y - y_N\|_{w^{(\alpha,\beta)}} = 0$. □

In order to describe approximation errors precisely and study stability of the method, for any integer $r \geq 0$, we define the non-uniformly weighted Sobolev space $A^r_{w^{(\alpha,\beta)}}(X)$ as follows:

$$A^r_{w^{(\alpha,\beta)}}(X) = \{u \mid u \textit{ is measurable on } X \textit{ and } \|u\|_{A^r_{w^{(\alpha,\beta)}}} < \infty\},$$

equipped with the following norm and semi-norm

$$\|y\|_{A^r_{w^{(\alpha,\beta)}}} = (\sum_{k=0}^{r} |y|^2_{A^r_{w^{(\alpha,\beta)}}})^{1/2}, \qquad |y|_{A^r_{w^{(\alpha,\beta)}}} = \|\partial_x^r y\|_{w^{(\alpha+r,\beta)}}.$$

Clearly, $A^r_{w^{(\alpha,\beta)}}(X) = \{y \mid y \in L^2_{w^{(\alpha,\beta)}}(X),\ \partial_x^k y \in L^2_{w^{(\alpha+r,\beta)}}(X), 0 \leq k \leq r\}$, and $A^0_{w^{(\alpha,\beta)}}(X) = L^2_{w^{(\alpha,\beta)}}(X)$. The following basic result is available on bound of error of the Laguerre approximation [14, 25].

Theorem 4.4. *For any $y \in A^r_{w^{(\alpha,\beta)}}(X)$ and any integers $0 \leq s \leq r$, we have*

$$\|\partial_x^s (y - y_N)\|_{w^{(\alpha+s,\beta)}} \leq c(\beta N)^{\frac{s-r}{2}} \|\partial_x^r y\|_{w^{(\alpha+r,\beta)}}. \tag{4.10}$$

where c is a generic positive constant independent of any function and α, β, N.

To proof of Theorem 4.4 is referred to [14, 15, 19, 25].
We observe that the above result is valid for $y \in A^r_{w^{(\alpha,\beta)}}(X)$ which includes functions that do not decay at infinity, however, the error estimate is given in a weighted space with an exponentially decay rate [19].
By assuming $s = 0$, in (4.10) we have

$$\|y - y_N\|_{w^{(\alpha,\beta)}} \leq c(\beta N)^{\frac{-r}{2}} \|\partial_x^r y\|_{w^{(\alpha+r,\beta)}}, \tag{4.11}$$

and,

$$\frac{\|y - y_N\|_{w^{(\alpha,\beta)}}}{\|y - y_{N-1}\|_{w^{(\alpha,\beta)}}} \leq \left(\frac{N}{N-1}\right)^{\frac{-r}{2}} < 1. \tag{4.12}$$

Therefore, with $L^2_{w^{(\alpha,\beta)}}$-norm, according to (4.11), the order of convergence is $O(N^{-r/2})$ for fixed r. Also, (4.12) concludes that the error of the method decreases at $N-$th step compared to the $(N-1)-$th step. Thus the stability of the method is resulted.

5. Numerical examples

To demonstrate the applicability and accuracy of the MGLCM, we apply the method for special cases of Lane-Emden type equations. In some cases, the solutions obtained by the proposed method and exact solutions are equal, which we provide the value of unknown vector C and solution $y(x)$. In some other cases, the numerical solutions of the presented method are in excellent agreement with the exact solutions, which we provide a table of results including the numerical solutions and absolute errors of the presented method and other methods at selected points. All of the numerical computations are done on a computer using written codes in Maple software.

Example 1. The standard Lane-Emden equation
For $f(x) = 1$, $g(y) = y^r$, $h(x) = 0$, $k = 2$, $a = 1$, $b = 0$, (1.1) is the standard Lane-Emden equation of index r, in the following general form,

$$y''(x) + \frac{2}{x} y'(x) + y(x)^r = 0, \tag{5.1}$$

with initial conditions $y(0) = 1$, $y'(0) = 0$.
The Lane-Emden equation of index r is a fundamental equation in the theory of stellar

structure. It is a beneficial equation in astrophysics for calculating the structure of interiors of the polytropic stars. This equation explains the temperature variation of a spherical gas cloud under the mutual attraction of its molecules and subject to the classical laws of thermodynamics [9]. The solutions of (5.1) for a given index r are known as polytropic of index r, where the parameter r has physical significance in the range $0 \leq r \leq 5$. Unfortunately, exact solutions to (5.1) in closed form are possible only for values of the polytropic index $r = 0,\ 1$ and 5. For other values of r between 0 and 5 only numerical solutions are available in the literature.

case 1. For $r = 0$, (5.1) turns into the following linear nonhomogeneous form

$$y''(x) + \frac{2}{x}y'(x) + 1 = 0, \tag{5.2}$$

with initial conditions $y(0) = 1$, $y'(0) = 0$ and has exact solution $y(x) = 1 - \frac{x^2}{6}$. Applying the present method for $N = 2$, we have

$$\begin{aligned} &y(x) \simeq c_0 L_0^{(\alpha,\beta)}(x) + c_1 L_1^{(\alpha,\beta)}(x) + c_2 L_2^{(\alpha,\beta)}(x) = C^T\Phi(x),\\ &y'(x) \simeq C^T D^1 \Phi(x), \qquad y''(x) \simeq C^T D^2 \Phi(x), \end{aligned}$$

where the operational matrices are obtained from Theorem 3.2,

$$D^1 = -\beta \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \qquad D^2 = \beta^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Firstly, substituting above approximations into (5.2) leads to the residual function

$$R_N(x) \simeq C^T(D^2 + \frac{2}{x}D)\Phi(x) + 1 = 0,$$

and collocating $R_N(x)$ in the root of $L_1^{(\alpha,\beta)}(x)$, yields

$$-2\beta c_1 + \beta(\alpha - 1)c_2 + \frac{1+\alpha}{\beta} = 0. \tag{5.3}$$

Secondly, substituting above approximations into initial conditions yields

$$\begin{aligned} &y(0) \simeq C^T\Phi(0) = c_0 + (1+\alpha)c_1 + (\frac{1}{2}\alpha^2 + \frac{3}{2}\alpha + 1)c_2 = 1,\\ &y'(0) \simeq C^T D^2 \Phi(0) = -\beta c_1 + \beta(-2-\alpha)c_2 = 0. \end{aligned} \tag{5.4}$$

Finally, by solving the linear system of Eqs (5.3)-(5.4) the three unknown coefficients are determined as:

$$c_0 = -\frac{\alpha^2 - 6\beta^2 + 3\alpha + 2}{6\beta^2}, \qquad c_1 = \frac{2+\alpha}{3\beta^2}, \qquad c_2 = -\frac{1}{3\beta^2}.$$

Consequentially, the numerical solution for arbitrary choices of $\alpha > -1$ and $\beta > 0$ is obtained as:

$$y(x) = (\frac{-\alpha^2 + 6\beta^2 - 3\alpha - 2}{6\beta^2}, \frac{2+\alpha}{3\beta^2}, \frac{-1}{3\beta^2}) \begin{pmatrix} 1 \\ -\beta x + \alpha + 1 \\ \frac{\beta^2 x^2}{2} - \beta(\alpha+2)x + \frac{\alpha^2}{2} + \frac{3\alpha}{2} + 1 \end{pmatrix}$$

$$= 1 - \frac{x^2}{6},$$

which is same the exact solution.
case 2. For $r = 1$, (5.1) has the following linear homogeneous type equation

$$y''(x) + \frac{2}{x}y'(x) + y(x) = 0, \tag{5.5}$$

with initial conditions $y(0) = 1$, $y'(0) = 0$ and exact solution $y(x) = \frac{sin(x)}{x}$.
Applying the present method for an assumed N and constructing the residual function and collocating in the scaled roots of $L_{N-1}^{(\alpha,\beta)}(x)$ on interval $(0,1)$ along with approximating the initial conditions equations, we will obtain a linear algebraic system with $(N+1)$ equations and $(N+1)$ unknowns. After solving this algebraic system, the maximum absolute errors for various choices of N, α and β are shown in Table 1. It shows that for various values of α and β by increasing the value of N the maximum absolute errors are decreased, which demonstrates the suggested method is of high accuracy.

TABLE 1. Maximum absolute error for different values of α, β and N for case 2.

N	α	β	Max. Abs. Error	N	α	β	Max. Abs. Error
4	0	1	4.16E-06	4	5	5	7.61E-06
8			2.39E-11	8			9.76E-12
12			3.76E-15	12			2.10E-17
4	5	1	1.11E-05	4	3	5	3.18E-07
8			9.76E-12	8			1.43E-11
12			9.98E-16	12			1.50E-18

case 3. For $r = 5$, (5.1) has the nonlinear homogeneous form

$$y''(x) + \frac{2}{x}y'(x) + y(x)^5 = 0, \tag{5.6}$$

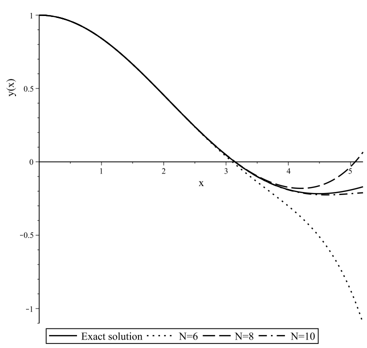
with initial conditions $y(0) = 1$, $y'(0) = 0$ and the exact solution $y(x) = (1 + \frac{x^2}{3})^{-\frac{1}{2}}$.
This solution is called the Talenti-Aubin solution for this critical Lane-Emden type equation.
Applying the MGLCM for arbitrary value of N by replacing $y(x) \simeq C^T\Phi(x)$, $y'(x) \simeq C^T D\Phi(x)$, $y''(x) \simeq C^T D^2\Phi(x)$ and $y^5(x) \simeq C^T\tilde{C}^4\Phi(x)$ in (5.6) where D is the operational matrix of derivative and $\tilde{C}$ is the operational matrix of product for the vector C, constructing the residual function and collocating in the scaled roots of $L_{N-1}^{(\alpha,\beta)}(x)$ on interval $(0,1)$ along with inserting the above approximations into the initial conditions equations, are formed a nonlinear algebraic system with $(N+1)$ equations and $(N+1)$ unknowns. By solving this algebraic system, comparison of the maximum absolute errors obtained by the present method for $\alpha = 0$, $\beta = 7$ and various choices of N and those obtained by the second kind Chebyshev operational matrix(S2CTM) [11] are shown in Table 2. It observes that the order of errors for the MGLCM is better than the order of errors for the S2CTM.

As shown in Figures 1a and 1b, the present method has an appropriate convergence rate by increasing the value of N. Also, that is working well with only few MGLPs

Table 2. Comparison of the maximum absolute error of the present method and the S2CTM for case 3.

N	Max. Abs. Error[11]	Max. Abs. Error(MGLCM)
4	8.97E-03	1.19E-02
5	2.43E-03	3.24E-04
6	5.55E-03	1.65E-04
7	3.81E-03	2.74E-05
8	-	3.89E-06
10	-	2.71E-07

and providing the better order of errors and the better solutions in comparison with the recently developed methods.

(a) 

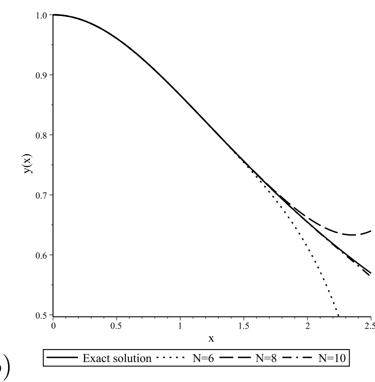
(b) 

Figure 1. (a): Graphs of the exact and approximate solutions for $\alpha = 3, \beta = 5$ in case 2, (b): Graphs of the exact and approximate solutions for $\alpha = 0, \beta = 7$ in case 3.

Example 2. Isothermal gas spheres equation
For $f(x) = 1$, $g(y) = e^{y(x)}$, $h(x) = 0$, $k = 2$, $a = 0$, $b = 0$, (1.1) is the isothermal gas spheres equation,

$$y''(x) + \frac{2}{x}y'(x) + e^{y(x)} = 0, \tag{5.7}$$

with initial conditions $y(0) = y'(0) = 0$ which are modeled by Davis [9]. Wazwaz has obtained the following approximate solution for (5.7) by using the ADM [26]

$$y(x) = -\frac{1}{6}x^2 + \frac{1}{120}x^4 - \frac{1}{1890}x^6 + \frac{61}{1632960}x^8 - \frac{4087}{1796256000}x^{10},$$

and (5.7) is solved by other methods such as, the HFCM [16], Legendre operational matrix(LOM) [17], wavelet series collocation method [22] and other methods. We apply the MGLCM to solve the isothermal gas spheres equation.

Expanding $g(y) = e^{y(x)}$ by Taylor series, we get

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \frac{y^5}{5!} + \dots,$$

and considering only first five terms we can write

$$e^y = 1 + y + \frac{(C^T\Phi(x))^2}{2!} + \frac{(C^T\Phi(x))^3}{3!} + \frac{(C^T\Phi(x))^4}{4!},$$

and from Theorems 3.2, 3.4 and 3.5 we construct the residual function,

$$R_N(x) \simeq C^T\{D^2 + \frac{k}{x}D^1 + I + \frac{1}{2}\tilde{C} + \frac{1}{6}\tilde{C}^2 + \frac{1}{24}\tilde{C}^3\}\Phi(x) + 1 = 0,$$

where D is the operational matrix of derivative, $\tilde{C}$ is the operational matrix of the product and I is the identify matrix. Collocating $R_N(x)$ in the scaled roots of $L_{N-1}^{(\alpha,\beta)}(x)$ on interval $(0, 1)$ along with the approximating the initial conditions $y(0) \simeq C^T\Phi(0) = 0$, $y'(0) \simeq C^TD^2\Phi(0) = 0$ are formed a nonlinear algebraic system with $(N + 1)$ equations and $(N + 1)$ unknowns. Solving the system resulted the unknown vector C is determined and numerical solution $y(x)$ is calculated.

Table 3 shows the comparison of the absolute errors obtained by the present method in the selected points for various choices of α and β and $N = 10$. It shows that by changing the values of α and β the numerical solutions obtained by the MGLCM have an appropriate convergence rate. Also, the classical Laguerre polynomial ($\alpha = 0$, $\beta = 1$) is not the best one for approximating the solution of differential equations.

Table 4 shows the comparison of numerical solution $y(x)$ and the absolute errors obtained by the MGLCM for $\alpha = 1$, $\beta = 10$ and those obtained by Wazwaz [26], the HFCM [16], the LOM [17] and wavelet series collocation method [22]. It indicates that a few terms of MGLPs are sufficient in order to achieve a better approximation and the method has an appropriate convergence rate. The order of errors for the MGLCM is better than the order of errors obtained at [16, 17, 22].

Table 3. Absolute error using the present method at $N = 10$ for Example 2.

x_i	$\alpha = 1,\ \beta = 5$	$\alpha = 3,\ \beta = 7$	$\alpha = 4,\ \beta = 9$
0.0	0.00E-00	0.00E-00	0.00E-00
0.1	4.48E-07	4.78E-09	5.51E-10
0.2	1.60E-07	7.10E-09	6.57E-10
0.3	9.72E-07	5.78E-09	6.32E-10
0.4	5.01E-07	5.29E-09	6.72E-10
0.5	1.19E-06	7.18E-09	2.64E-10
0.6	2.66E-06	1.10E-08	2.12E-09
0.7	2.28E-06	1.96E-08	1.15E-08
0.8	6.66E-07	4.72E-08	4.19E-08
0.9	5.26E-06	1.29E-07	1.26E-07
1.0	9.09E-06	3.31E-07	3.27E-07

TABLE 4. Comparison of the numerical solution and absolute error obtained by MGLCM with other methods in Example 2.

x_i	**Wazwaz**[26]	**Present method**	**Error (N=10)**	**Error**[17] **(N=10)**	**Error**[16] **(N=30)**	**Error**[22] **(M=10)**
0.0	0.0000000000	0.0000000000	0.00E-00	9.24E-18	0.00E-00	-
0.1	-0.0016658339	-0.0016658339	2.11E-14	5.28E-10	5.85E-07	1.09E-12
0.2	-0.0066533671	-0.0066533671	1.35E-14	3.37E-08	6.04E-07	1.86E-12
0.3	-0.0149328833	-0.0149328833	4.15E-12	-	-	1.48E-12
0.4	-0.0264554763	-0.0264554763	5.01E-11	-	-	5.16E-11
0.5	-0.0411539568	-0.0411539573	4.59E-10	8.12E-06	5.58E-07	4.63E-10
0.6	-0.0589440720	-0.0589440748	2.74E-09	-	-	2.72E-09
0.7	-0.0797259923	-0.0797260044	1.21E-08	-	-	1.20E-08
0.8	-0.1033860110	-0.1033860536	4.27E-08	-	-	4.22E-08
0.9	-0.1297983988	-0.1297985256	1.27E-07	-	-	1.26E-07
1.0	-0.1588273537	-0.1588276816	3.28E-07	4.93E-04	8.20E-07	3.24E-07

Also, Figure 2a shows the resulting graphs from numerical solutions of the present method for $N = 10$ and various choices of α and β, and Figure 2b shows the resulting graphs from numerical solutions of the present method for $\alpha = 1$, $\beta = 5$ and various choices of N in comparison with the solution obtained by Wazwaz [26]. Comparison the graphs show that the numerical solutions of the MGLCM are convergence to the Wazwaz solution with changing choices of α and β or increasing the value of N.

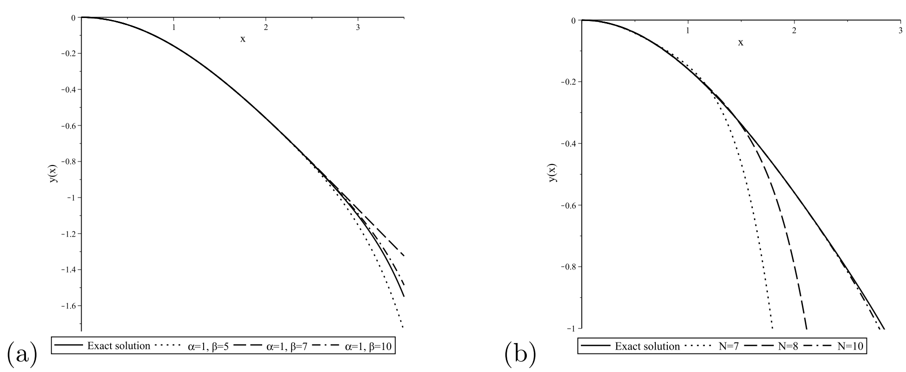


FIGURE 2. (a): Graphs of the exact and approximate solutions for $N = 10$ and various choices of α and β, (b): Graphs of the exact and approximate solutions for $\alpha = 1, \beta = 5$ and various choices of N in Example 2.

Example 3. linear and nonhomogeneous Lane-Emden type equation
For $f(x) = x$, $g(y) = y$, $h(x) = x^5 - x^4 + 44x^2 - 30x$, $k = 8$, $a = 0$, $b = 0$, (1.1) will

be one of the linear and nonhomogeneous Lane-Emden type equations, that is:

$$y''(x) + \frac{8}{x}y'(x) + xy(x) = x^5 - x^4 + 44x^2 - 30x, \tag{5.8}$$

with initial conditions $y(0) = y'(0) = 0$ which has the exact solution $y(x) = x^4 - x^3$. By applying the present method for $N = 4$ with using the approximations

$$y(x) \simeq c_0 L_0^{(\alpha,\beta)}(x) + c_1 L_1^{(\alpha,\beta)}(x) + \ldots + c_4 L_4^{(\alpha,\beta)}(x) = C^T \Phi(x),$$
$$y'(x) \simeq C^T D\Phi(x), \qquad y''(x) \simeq C^T D^2 \Phi(x),$$

and constructing the residual function and collocating in the roots of $L_3^{(\alpha,\beta)}(x)$ along with approximating the initial conditions equations, we will obtain a linear algebraic system with five equations and five unknowns. By solving this system the unknown vector C is determined

$$c_0 = \frac{\alpha^4 - \alpha^3\beta + 10\alpha^3 - 6\alpha^2\beta + 35\alpha^2 - 11\alpha\beta + 50\alpha - 6\beta + 24}{\beta^4},$$
$$c_1 = \frac{-(4\alpha^3 - 3\alpha^2\beta + 36\alpha^2 - 15\alpha\beta + 104\alpha - 18\beta + 96)}{\beta^4},$$
$$c_2 = \frac{6(2\alpha^2 - \alpha\beta + 14\alpha - 3\beta + 24)}{\beta^4}, \qquad c_3 = \frac{-(6(4\alpha - \beta + 16)}{\beta^4}, \qquad c_4 = \frac{24}{\beta^4}.$$

Consequentially, the numerical solution is obtained $y(x) = x^4 - x^3$ for arbitrary choices of $\alpha > -1$ and $\beta > 0$ which is same the exact solution.

Example 4. nonlinear and homogeneous Lane-Emden type equation
For $f(x) = 1$, $g(y) = -6y - 4ylny$, $h(x) = 0$, $k = 2$, $a = 1$, $b = 0$, (1.1) will be one of the nonlinear and homogeneous Lane-Emden type equations, that is:

$$y''(x) + \frac{2}{x}y'(x) - 6y(x) = 4y(x)ln(y(x)), \tag{5.9}$$

with initial conditions $y(0) = 1, y'(0) = 0$ which has the analytical solution $y(x) = e^{x^2}$. In this model we have $y(x)ln(y(x))$ term that increases the order of calculation; therefore, we can use the transform $y(x) = e^{z(x)}$ in which $z(x)$ is unknown; where upon transformed form of the model will become the nonlinear and nonhomogeneous equation as follows:

$$z''(x) + \frac{2}{x}z'(x) - 4z(x) + z'(x)^2 = 6, \tag{5.10}$$

with initial conditions $z(0) = z'(0) = 0$.
Now, we approximate $z(x)$ for $N = 2$ by the MGLPs

$$z(x) \simeq c_0 L_0^{(\alpha,\beta)}(x) + c_1 L_1^{(\alpha,\beta)}(x) + c_2 L_2^{(\alpha,\beta)}(x) = C^T \Phi(x),$$

and by using the Theorems 3.2, 3.4 and 3.5 we have

$$z'(x) \simeq C^T D\Phi(x), \qquad z''(x) \simeq C^T D^2 \Phi(x),$$
$$z'(x)^2 \simeq C^T D\Phi(x)\Phi(x)^T D^T C = C^T D\tilde{S}\Phi(x),$$

where $S = D^T C$ and $\tilde{S}$ is the operational matrix of product for the vector S. Substituting above approximations into (5.10) and constructing the residual function and Collocating in the root of $L_1^{(\alpha,\beta)}(x)$ yields

$$\begin{aligned} &-4(\alpha+1)c_0 - 2\beta^2 c_1 + (\alpha\beta^2 + 2\alpha^2 - \beta^2 + 4\alpha + 2)c_2 + \beta^2(\alpha+1){c_2}^2 \\ &+ 2\beta^2(\alpha+1)c_1 c_2 + \beta^2(\alpha+1){c_2}^2 = 6(1+\alpha), \end{aligned} \tag{5.11}$$

and substituting above approximations into initial conditions yields

$$\begin{aligned} &y(0) \simeq C^T\Phi(0) = c_0 + (1+\alpha)c_1 + (\frac{1}{2}\alpha^2 + \frac{3}{2}\alpha + 1)c_2 = 1, \\ &y'(0) \simeq C^T D^2 \Phi(0) = -\beta c_1 + \beta(-2-\alpha)c_2 = 0. \end{aligned} \tag{5.12}$$

Finally, solving the linear system of Eqs (5.11)-(5.12) the three unknown coefficients are determined as:

$$c_0 = \frac{\alpha^2 + 3\alpha + 2}{\beta^2}, \qquad c_1 = \frac{-2(\alpha+2)}{\beta^2}, \qquad c_2 = \frac{2}{\beta^2}.$$

Consequentially, the numerical solution for arbitrary choices of $\alpha > -1$ and $\beta > 0$ is obtained as:

$$\begin{aligned} z(x) = (\frac{\alpha^2+3\alpha+2}{\beta^2}, \frac{-2(\alpha+2)}{\beta^2}, \frac{2}{\beta^2}) \begin{pmatrix} 1 \\ -\beta x + \alpha + 1 \\ \frac{1}{2}\beta^2 x^2 - \beta(\alpha+2)x + \frac{1}{2}\alpha^2 + \frac{3}{2}\alpha + 1 \end{pmatrix} \\ = x^2. \end{aligned}$$

Therefore, $y(x) = e^{z(x)} = e^{x^2}$ which is same the exact solution.

Corollary 5.1. *From Examples* 3−4*, if the exact solution to equation be a polynomial, then the present method will obtain in the real solution.*

6. Conclusion

All things considered, we introduced a collocation method based on MGLPs for solving Lane-Emden type equations with initial conditions. In this method, we used the properties and operational matrices of derivatives and product from two MGLPs to reduce Lane-Emden type equations and their initial conditions to solve a linear or nonlinear algebraic system. From illustrated examples, we can conclude that this method can obtain more accurate and strong convergence results and also a few terms of MGLPs are sufficient to achieve a better approximation. In Examples $1-2$, we observed that the order of error in the presented method is better than the order of error in other existing methods [11, 16, 17, 22] and by changing the values of α and β the numerical solutions obtained by the MGLCM have an appropriate convergence rate. Also, in Examples $3-4$, we clearly see that in the equations which have an exact solution of the polynomial functions, the numerical solutions of the MGLCM and exact solutions are equal. Hence, the numerical results demonstrate high accuracy, excellent efficiency and rapid convergency rate.

Acknowledgment

The authors are very grateful to the reviewers for carefully reading the paper and for their comments and suggestions which have improved the paper.

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