Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 9, No. 4, 2021, pp. 1028-1041
DOI:10.22034/CMDE.2020.40380.1764

Meshless approach for pricing Islamic Ijarah under stochastic interest rate models

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Abstract Nowadays, the fixed interest rate financing method is commonly used in the capitalist financial system and in a wide range of financial liability instruments, the most important of which is bond. In the Islamic financial system, using these instruments is considered as usury and has been prohibited. In fact, Islamic law, Shariah, forbids Muslims from receiving or paying off the Riba. Therefore, using customary financial instruments such as bond is not acceptable or applicable in countries which have a majority of Muslim citizens.
In this paper, we introduce one financial instrument, Sukuk, as a securities-based asset under stochastic income. These securities can be traded in secondary markets based on the Shariah law.
To this end, this paper will focus on the most common structure of the Islamic bond, the Ijarah and its negotiation mechanism. Then, by presenting the short-term stochastic model, we solve fixed interest rate and model the securities-based asset by the stochastic model.
Finally, we approximate the resulting model by radial basis function method, as well as utilizing the Matlab software

Keywords. Bond, Financing, Finite difference method, Partial differential equations, RBF method, Riba, Stochastic differential equations, Sukuk.
2010 Mathematics Subject Classification. 65L05, 34K06, 34K28.

1. Introduction

The process of financing is one of the most important subjects in financial economics. There are various types of methods and instruments for financing, each of which has its own properties. These instruments and methods are applied according to the needs, affordability, conditions of economic firms, and the diversity of people's behavior in investment and risk-taking.

Securitization is one of the most important methods of the process of financing. During this process, the financial institution assets or founder are first separated from the balance sheet of the institution. Afterwards, investors who have purchased exchange-traded financial instruments finance the capitals. As a matter of fact, this exchanged-traded instrument represents the mentioned liability. In other words, securitization is the process through which financial intermediaries like commercial banks

Received: 18 June 2020 ; Accepted: 14 September 2020.

are eliminated, and the callable bond is sold directly to the investors. Actually, an investment company collects the series of financial assets and sells bond to foreign investors for financing the intended set [1, 3, 4, 5, 9].

In the process of securitization, the company or an institution that needs financing, establishes the special-purpose vehicle (SPV) and sells those assets that have future cash flows to it. This SPV publishes the base asset to provide the necessary payment for buying the intended assets and then offers it to all investors. Later, it pays off the payments which are earned from selling liability bonds to the founder for buying the financial assets.

The investors who have purchased asset-backed liability bond gain efficiency from the cash-flows obtained from financial assets of the SPV. One of the most prominent and efficient financial instruments in the process of securitization is bond.

Financing through bond is how economic firms borrow their own necessary investment directly from the people. Thus, it pays off the coupon rate in certain times and repays the original loan at maturity date.

In the Islamic financial system, using these instruments is usury and has been forbidden by Islamic law. To put it simply, the Shariah, forbids Muslims from receiving or paying off riba. Therefore, using customary financial instruments like bond is not acceptable or applicable in countries with a majority of Muslim citizens. The main conclusion to be drawn is that, governments and every Islamic financial institution that is looking for financing and managing their own liability need to have substitutes according to Islamic principles.

Sukuk, the most widespread type of Islamic bond, can be considered as an appropriate substitute for bond. It is internally consistent and faithful to the Islamic banking principle (free of riba) by involving a physical capital as well as contracts such as rent and Mudaraba. The literal meaning of Sukuk is a written document which can be defined as certificates with an identical face value which, after completion of subscription, represent present value payment contained in them via the buyer to the publisher, and the owner is the holder of a set of assets, interest in assets or interest in a project or a particular investment activity. Although there are various types of Sukuk, only 14 types of them have been recommended by Accounting and Auditing Organization for Islamic Financial Institutions.

The main idea of Sukuk was presented by Ul Haque and Mirakhor (1999) [10], Kahf (1997) [11], Elgari (1997) [8] and Zarka (1997) [19]. Also, Al-Suwailem (2000) [2] and El-Gamal (2000) [7] have precisely studied uncertainty factors in the Islamic financial system. The first idea of the Islamic bond issuance dates back to the 1980s. At that time, many efforts were carried out by Islamic banks to overcome the liquidity problem.

In July of 1983, the Central Bank of Malaysia, as the pioneer organization in this area, utilized a zero-coupon bond, named the Government Investment Certificate, instead of using government securities or treasury bills, which paid interest. In the 1990s, Islamic financial instruments were used simultaneously with the expansion of various structures of financial transactions by Islamic banks. Afterward, financial researchers and religious scholars studied appropriate strategies.

The first idea of Islamic leasing or Ijarah (Islamic bond) issuance has been shown by Kahaf Monzar. As mentioned before, the focus of this paper is on Ijarah Sukuk. Therefore, this paper is organized as follows: In section 2, Islamic leasing or Ijarah is explained. In section 3, we describe stochastic interest rate models substituted by fixed interest rate models, one of which we later select. In section 4, Ijarah Sukuk is modeled under short-term stochastic interest rate. In section 5, we applied a fairly new numerical solution technique for Ijarah Sukuk called RBF-FD. In the next step, the resulting model is implemented by hypothetical data, using Matlab software, and then the conclusions are explored.

2. Ijarah Sukuk

Ijarah Sukuk is a type of investment that represents ownership of an equal share in the usufruct of an asset that is well-defined, existing and known, and tied to an Ijarah contract as defined by the Shariah. Among Islamic financial products, Ijarah Sukuk is a distinctive product that has gained great popularity among Muslim investors.

On the other hand, publishers (governments and corporations) point that Ijarah Sukuk is an appropriate method for mobilization of resources. The financing method using Ijarah Sukuk is briefly as follows: The company requiring liquidity (founder) selects a set of assets that are suitable to rent. Then, it establishes a SPV (publisher) while is legally independent. The founder sells selected assets as a portfolio to it.

An agreement between them emphasizes that assets will have been offered again to the founder in the form of a contract after purchasing them. The loan term as well as the Ijarah price is determined under an agreement between the parties. One of the most popular types of Ijarah Sukuk used by banks and credit institutions is the one that the founder (bank) sells facilities, which were awarded in the form of lease-purchase contract, to a third party. Generally, the third party first issues Ijarah Sukuk. Later, it purchases assets that the bank transfers to individuals in the form of lease-purchase. The connection between the founder and the third party is interrupted by selling these assets and third-party-owned assets on behalf of investors that bank (founder) are awarded in the form of lease-purchase contracts.

3. The Problem of Bond in Riba-Free Banking

The name Islamic banking law implies the main purpose of an approval and enforcement of this law, namely elimination of usury from banking and financial systems and adaptation to the Shariah. According to this law, certain interests of financial and banking operations are removed and replaced with Islamic contracts because the interests or interest rates of riba are taken into account. Putting in place the condition of increase in lending an object or money is considered as riba. It means that if money or property has been loaned to someone and it is decided that the loaner needs to return an amount more than the original, riba loaning has taken place, which is forbidden according to the Shariah. To tackle this problem, the interest rate is presented by the models which eliminate the fixedness of interest rate, and the interest rate is determined based on market and production computations. In line with this, we consider the following two cases:

3.1. **Interest rate is deterministic and changes accordingly to time.** Rate r_t is a function of t under deterministic and unfixed conditions. In normal conditions, Ijarah Sukuk price depends on interest rate and time. The Ijarah Sukuk price is also altered in terms of t when the interest rate is not an independent variable and known in terms of t.

In this case, it can be assumed that $SI(t)$ and $k(t)$, the Ijarah Sukuk and coupon rate prices are not fixed and they are a function of t. The final condition with the maturity time of T is represented by $SI(T) = F$ in which F is the face value. Taking the derivative on the equation $SI(t)$, at any time $t < T$, leads us to a first order linear differential equation. Let dt be a small time interval from time t, therefore the change in the value of an Ijarah Sukuk during the time dt is acquired by $\frac{dSI}{dt}dt$ and also $k(t)dt$ is the received coupon. In no arbitrage condition, it implies the equation with risk free interest rate $r(t)SI(t)dt$, at any time $t < T$ then:

$$\frac{dSI}{dt}dt + k(t)dt = r(t)SI(t)dt,$$

divided by dt yields:

$$\frac{dSI}{dt} + k(t) = r(t)SI(t), \quad t < T.$$

The resulting equation is the non-homogeneous linear equation as below:

$$\frac{dSI}{dt} - r(t)SI(t) = -k(t), \quad t < T.$$

In order to solve this equation, we multiply both sides, by the integral-making factor of $e^{\int_t^T r(u)du}$, hence:

$$\frac{dSI}{dt}e^{\int_t^T r(u)du} - r(t)SI(t)e^{\int_t^T r(u)du} = -k(t)e^{\int_t^T r(u)du}, \quad t < T,$$

or

$$\frac{d}{dt}\left[SI(t)e^{\int_t^T r(u)du}\right] = -k(t)e^{\int_t^T r(u)du},$$

or

$$\int_t^T \frac{d}{du}\left[SI(u)e^{\int_u^T r(s)ds}\right]du = -\int_t^T k(u)e^{\int_u^T r(s)ds}du.$$

As a result

$$SI(T)e^0 - SI(t)e^{\int_t^T} r(s)ds = -\int_t^T k(u)e^{\int_u^T r(s)ds}du.$$

Considering the final condition $SI(T) = F$, the Ijarah Sukuk price function can be written as follows:

$$SI(t) = e^{-\int_t^T r(s)ds}\left[F + \int_t^T k(u)e^{\int_u^T r(s)ds}du\right].$$

Financial interpretation related to this formula encompasses an appropriate financial explanations.

The amount of coupon rate $k(u)du$ in the maturity time of T will be enhanced to $k(u)e^{\int_u^T r(s)ds}du$ in the $[u, u+du]$ time span.

The future value in T is acquired by $\int_t^T k(u)e^{\int_u^T r(s)ds}du$ in all received coupons at any time during the life of the Ijarah Sukuk.

The present value and face value of coupon is obtained by discount factor $e^{-\int_t^T r(s)ds}$ with deduction of the received amount in T. The result equals the price function and reveals the future value of Ijarah Sukuk in the time of t which is in accordance with the price function.

Regarding to the amount of $k(t)$ and $r(t)SI(t)$, the Ijarah Sukuk price function could be regarded as an increasing or decreasing function at the time of t.

3.2. **Stochastic interest rate.** Consider that the interest rate r_t follows a Ito's process and can be written as the following stochastic differential equation:

$$dr = \mu(r,t)dt + \rho(r,t)dZ$$

where $\mu(r,t)$ and $\rho(r,t)$ denote instantaneous drift and instantaneous variance respectively, and dZ is the Standard Brownian Motion process. Although, it's a very general model for the interest rate, for now, this model would be enough for us. in the remainder of the paper, we introduce another models which are more famous.
We draw a conclusion for the Ijarah Sukuk price differential equation via the concept of the absence of arbitrage. The differential equation contains market price of risk r since the interest rate is not a trade securities. The price of Ijarah Sukuk has been shown by different pay off times to observe stability of relations and guarantee no arbitrage opportunities. If $SI(r,t)$ is Ijarah Sukuk price, then by applying Itos lemma, we get the Ijarah Sukuk price dynamic as the following:

$$\begin{aligned} dSI &= \tfrac{\partial SI}{\partial t}dt + \tfrac{\partial SI}{\partial r}dr + \tfrac{1}{2}\rho^2\tfrac{\partial^2 SI}{\partial r^2}dt \\ &= \tfrac{\partial SI}{\partial t}dt + \tfrac{\partial SI}{\partial r}(\mu dt + \rho dZ_t) + \tfrac{1}{2}\rho^2\tfrac{\partial^2 SI}{\partial r^2}dt \\ &= \left[\tfrac{\partial SI}{\partial t} + \mu\tfrac{\partial SI}{\partial r} + \tfrac{1}{2}\rho^2\tfrac{\partial^2 SI}{\partial r^2}\right]dt + \rho\tfrac{\partial SI}{\partial r}dZ. \end{aligned}$$

Moreover, suppose that stochastic dynamic SI follows Brownian Motion process as a short-term rate dynamic:

$$\frac{dSI}{SI} = \mu_{SI}dt + \sigma_{SI}dZ.$$

By unifying dSI from recent relations, we get:

$$\begin{aligned} &\left[\tfrac{\partial SI}{\partial t} + \mu\tfrac{\partial SI}{\partial r} + \tfrac{1}{2}\rho^2\tfrac{\partial^2 SI}{\partial r^2}\right]dt + \rho\tfrac{\partial SI}{\partial r}dZ = \\ &\qquad (\mu_{SI}(r,t)dt + \sigma_{SI}(r,t)dZ)\,SI. \end{aligned}$$

The above relation holds under the assumption of equality of coefficients dt and dZ on both sides. Therefore, the instantaneous drift rate $\mu_{SI}(r,t)$ and volatility $\sigma_{SI}(r,t)$ from bond price process are attained:

$$\begin{aligned} \mu_{SI}(r,t) &= \tfrac{1}{SI}\left[\tfrac{\partial SI}{\partial t} + \mu\tfrac{\partial SI}{\partial r} + \tfrac{1}{2}\rho^2\tfrac{\partial^2 SI}{\partial r^2}\right], \\ \sigma_{SI}(r,t) &= \tfrac{\rho}{SI}\tfrac{\partial SI}{\partial r}. \end{aligned} \tag{3.1}$$

As the short-term interest rate is not an equated bond, it can not be used together with the trading security, and it does not play the role of an underlying asset in stock options.

Actually, our purpose is to maintain the Ijarah Sukuk at different maturity dates. This is possible because the instantaneous rate of Ijareh Sukuk at different times is related to the stochastic short rate which exists in Ijareh Sukuk prices. Let us construct a portfolio Π.

We purchase the Ijarah Sukuk with the value of V_1 at the maturity time of T_1 and sell the other ones with the value of V_2 at the maturity time of T_2. The value of this portfolio is $\Pi = V_1 - V_2$. As mentioned before, the change in the value of portfolio during the time of dt is given by:

$$\begin{aligned} d\Pi = &\left[V_1 \mu_{SI}(r,t;T_1) - V_2 \mu_{SI}(r,t;T_2)\right] dt \\ &+ \left[V_1 \sigma_{SI}(r,t;T_1) - V_2 \sigma_{SI}(r,t;T_2)\right] dZ. \end{aligned}$$

Assume that the portfolio is formed in a way that we can purchase and sell any amount of Ijareh Sukuk. So V_1 and V_2 can be chosen as follows:

$$\begin{aligned} V_1 &= \frac{\sigma_{SI}(r,t;T_2)}{\sigma_{SI}(r,t;T_2) - \sigma_{SI}(r,t;T_1)} \Pi, \\ V_2 &= \frac{\sigma_{SI}(r,t;T_1)}{\sigma_{SI}(r,t;T_2) - \sigma_{SI}(r,t;T_1)} \Pi. \end{aligned}$$

Therefore, the random term in $d\Pi$ is dropped to zero and we have:

$$\frac{d\Pi}{\Pi} = \frac{\mu_{SI}(r,t;T_1)\sigma_{SI}(r,t;T_2) - \mu_{SI}(r,t;T_2)\sigma_{SI}(r,t;T_1)}{\sigma_{SI}(r,t;T_2) - \sigma_{SI}(r,t;T_1)} dt. \tag{3.2}$$

Since this portfolio has no risk, risk-free short-term rate must be acquired to avoid arbitrage opportunities.

$$d\Pi = r\Pi dt. \tag{3.3}$$

Now, putting (3.3) in (3.2) we simply obtain:

$$\frac{r\Pi dt}{\Pi} = \frac{\mu_{SI}(r,t;T_1)\sigma_{SI}(r,t;T_2) - \mu_{SI}(r,t;T_2)\sigma_{B}(r,t;T_1)}{\sigma_{SI}(r,t;T_2) - \sigma_{B}(r,t;T_1)} dt,$$

which leads us to:

$$r = \frac{\mu_{SI}(r,t;T_1)\sigma_{SI}(r,t;T_2) - \mu_{SI}(r,t;T_2)\sigma_{SI}(r,t;T_1)}{\sigma_{SI}(r,t;T_2) - \sigma_{SI}(r,t;T_1)}. \tag{3.4}$$

Let rewrite (3.4) and obtain the following:

$$\frac{\mu_{SI}(r,t;T_1) - r}{\sigma_{SI}(r,t;T_1)} = \frac{\mu_{SI}(r,t;T_2) - r}{\sigma_{SI}(r,t;T_2)}.$$

It goes without saying that, the above equation is written for two arbitrary maturity time, namely, T_1 and T_2, hence, we can conclude that, those fractions are independent from maturity time. In other terms, we have:

$$\frac{\mu_{SI}(r,t;T) - r}{\sigma_{SI}(r,t;T)} = \Lambda.$$

Which is called the market price of risk. After putting, μ_{SI} and σ_{SI} from (3.1), into above equation, we would have the following:

$$\frac{\partial SI}{\partial t} + \frac{1}{2}\rho^2 \frac{\partial^2 SI}{\partial r^2} + (\mu - \Lambda\rho)\frac{\partial SI}{\partial r} - rSI = 0, \tag{3.5}$$

which is an equation for pricing the Sukuk contracts.
However, it's more common to use some well-known model, such as Vasichek or Cox-Ingerson Ross, for the dynamic of interest rate. Now, we do the same to achieve a better model. In this paper, we assume the interest rate model follows Vasichek model. So, we would have:

$$dr = (b - ar)dt + \sigma dZ.$$

Then

$$\begin{aligned} dr &= \mu(r,t)dt + \rho(r,t)dZ \\ &= (b - ar)dt + \sigma dZ. \end{aligned}$$

After putting the corresponding value of μ and ρ into (3.5), we simply obtain:

$$\frac{\partial SI}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 SI}{\partial r^2} + (b - ar - \Lambda\sigma)\frac{\partial SI}{\partial r} - rSI = 0. \tag{3.6}$$

Whereas, $\Lambda(r,t)$ is the market price of risk under interest rate r. Since it states the enhancement in expected rate of Ijarah Sukuk should take the market price of risk, the hedging Ijarah Sukuk should take the market price of risk in no-arbitrage market, and it is not related to maturity. It's easy now to verify that the Ijarah Sukuk obtained under Vasichek interest rate satisfies an initial and boundary value problem:

$$\begin{aligned} &SI(r,T) = F, && 0 < r < \infty, \\ &SI(0,t) = p(t), && 0 < t < T, \\ &SI(\infty,t) = g(t), && 0 < t < T. \end{aligned} \tag{3.7}$$

Concerning the already mentioned initial and boundary conditions, the reader should note that, there are multiple kind of Sukuk contracts, each of which has its own circumstances which implies different type of initial and boundary condition. Therefore, here we just consider them in a general way and then in the following section, we look for an appropriate conditions, specific for Ijarah Sukuk. (For better understanding, the reader may find [12] useful, while it is about bonds, there are some similarity with our discussion.)

4. Numerical solution

In this section, we attempt to suggest numerical solution base on the combination of Finite Difference method and Radial Basis Function (RBF) approximation. For the readers who lacks information about these methods, please check the references [6, 14, 17]. The main idea of RBF method is to approximate the desired function, here SI with the sum of basic functions, where, these basic functions can be selected as Gaussian, multi-quadratic, splines or other common functions such as Wedlands functions[13, 15, 16, 18]. Each of them, has its advantages and disadvantages, however in this paper we consider Gaussian functions as our desired basis functions.

Lets assume that $t = T - \tau$, $0 < \tau < T$, $\tau^0 = 0$, $\tau^{n+1} = \tau^n + \delta\tau$, $\delta\tau N = T$, $n = 0, 1, \ldots, N$. By applaying the finite difference method, the derivatives is approximated by the following differential formulas:

$$\frac{\partial SI(r,\tau^n)}{\partial \tau} \simeq \frac{SI(r,\tau^{n+1}) - SI(r,\tau^n)}{\delta\tau}. \tag{4.1}$$

By replacing (4.1) into (3.6), we discretize the problem in time direction. Then we obtain:

$$SI^{n+1} = \frac{1}{2}\sigma^2\delta\tau\frac{\partial^2 SI^n}{\partial r^2} + (b - ar - \Lambda\sigma)\delta\tau\frac{\partial SI^n}{\partial r} - (r\delta\tau - 1)SI^n. \tag{4.2}$$

where $SI^{n+1} = SI(r,\tau^{n+1})$, $n = 0, 1, \ldots, N$.

Now, we discretize the equation (4.2) in space direction by the RBF approximation method. Therefore, the obtained recursive equation can be rewritten at $n = 0$ and $n \geq 1$, respectively as follows:

$$SI^1 = \frac{1}{2}\sigma^2\delta\tau\frac{\partial^2 SI^0}{\partial r^2} + (b - ar - \Lambda\sigma)\delta\tau\frac{\partial SI^0}{\partial r} - (r\delta\tau - 1)SI^0, \tag{4.3}$$

and

$$SI^{n+1} = \frac{1}{2}\sigma^2\delta\tau\frac{\partial^2 SI^n}{\partial r^2} + (b - ar - \Lambda\sigma)\delta\tau\frac{\partial SI^n}{\partial r} - (r\delta\tau - 1)SI^n. \tag{4.4}$$

Then, we approximate the $SI^n(r)$ by RBF as follows:

$$SI^n(r) = \sum_{j=1}^{N} \lambda_j^n \varphi(r_j), \tag{4.5}$$

where λ_j, $j = 1, \ldots, N$, are unknowns parameters. So, we consider N collocation points to obtain the values of coefficients λ_j, $j = 1, \ldots, N$, in the interpolate of $SI^n(r)$ as:

$$SI^{n+1} = SI^{n+1}(r_i) \simeq \sum_{j=1}^{N} \lambda_j^{n+1}\varphi(r_{ij}), \quad i = 1, \ldots, N \tag{4.6}$$

where $r_{ij} =\parallel r_i - r_j \parallel$ and $\parallel . \parallel$ is the Eucliden norm, φ is a radial basis function.

By considering equation (4.6) in a matrix form, we have:

$$[SI]^{n+1} = A[\lambda]^{n+1} \tag{4.7}$$

where $[\lambda]^{n+1} = [\lambda_1^{n+1}, \ldots, \lambda_N^{n+1}]^T$, $[SI]^{n+1} = [SI_1^{n+1}, \ldots, SI_N^{n+1}]^T$ and A is an $N \times N$ matrix:

$$A = \begin{bmatrix} \varphi_{11} & \cdots & \varphi_{1j} & \cdots & \varphi_{1N} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \varphi_{i1} & \cdots & \varphi_{ij} & \cdots & \varphi_{iN} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varphi_{N1} & \cdots & \varphi_{Nj} & \cdots & \varphi_{NN} \end{bmatrix}.$$

By reconstruction of equation (4.3) and equation (4.4) in the matrix form, we obtain:

$$[D]^1 = B[\lambda]^1,$$

whereas

$$B = \begin{bmatrix} G(\varphi_{11}) & \dots & G(\varphi_{1j}) & \dots & G(\varphi_{1N}) \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ G(\varphi_{i1}) & \dots & G(\varphi_{ij}) & \dots & G(\varphi_{iN}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G(\varphi_{N1}) & \dots & G(\varphi_{Nj}) & \dots & G(\varphi_{NN}) \end{bmatrix},$$

such that G is an operator defined by equation (4.3), convert matrix elements A to matrix elements B to be a suitable coefficient for the above expression.

$[D]^1 = [D_1^1, \dots, D_N^1]^T$, whereas

$$D_i^1 = \begin{cases} p_i^1, & i = 1, \\ \left(\frac{1}{2}\sigma^2\delta\tau\frac{\partial^2}{\partial r^2} + (b - ar - \Lambda\sigma)\delta\tau\frac{\partial}{\partial r} - (r\delta\tau - 1)\right) SI^0, & 1 < i < N, \\ g_i^1, & i = N \end{cases}$$

and for $n \geq 1$ of equation (4.4), we obtain:

$$[D]^{n+1} = B[\lambda]^{n+1},$$

and $[D]^{n+1} = [D_1^{n+1}, \dots, D_N^{n+1}]^T$, where

$$D_i^{n+1} = \begin{cases} p_i^{n+1}, & i = 1, \\ \left(\frac{1}{2}\sigma^2\delta\tau\frac{\partial^2}{\partial r^2} + (b - ar - \Lambda\sigma)\delta\tau\frac{\partial}{\partial r} - (r\delta\tau - 1)\right) SI^n, & 1 < i < N, \\ g_i^{n+1}, & i = N. \end{cases}$$

Therefore, for solving the desired pricing equation, as the first step, time variable t was changed to τ. This is because, we prefer an equation to be forward in numerical scheme. Then based on finite difference and RBF approximation method, we proposed a numerical solution for solving the pricing equation.

5. The stability and convergence analysis

This section is allocated to analyze the convergence and stability of the suggested numerical method in the previous section. For the beginning, let R_i^n be the local truncation error of our scheme at time t^n, and let $\tilde{SI}^n = (SI(r_1, \tau^n), \dots, SI(r_N, \tau^n))$ and $SI^n = (SI_1^n, \dots, SI_N^n)$ be the exact and numerical solutions of equation (3.6) at time τ^n. R_i^n has the order $O(\delta\tau + h^2)$ because

$$\begin{aligned} R_i^n = &-\tfrac{SI(r_i,\tau^n+\delta\tau)-SI(r_i,\tau^n)}{\delta\tau} \\ &+\tfrac{1}{2}\sigma^2\tfrac{\partial^2 SI(r_i,\tau^n)}{\partial r^2} + (b - ar - \Lambda\sigma)\tfrac{\partial SI(r_i,\tau^n)}{\partial r} - rSI(r_i, \tau^n). \end{aligned}$$

Using Taylors series expansion, we have:

$$|R_i^n| \leq c(\delta\tau + h^2), \tag{5.1}$$

where c is a constant parameter. By considering the inner product of equation (4.2) in ν, we obtain:

$$\begin{aligned} (SI^{n+1}, \nu) = &\tfrac{1}{2}\sigma^2\delta\tau(\tfrac{\partial SI^n}{\partial r}, \tfrac{\partial \nu}{\partial r}) + (b - ar - \Lambda\sigma)\delta\tau(SI^n, \nu) \\ &-(r\delta\tau - 1)(SI^n, \nu), \end{aligned} \tag{5.2}$$

where $(u,\nu)=\int_\Omega u.\nu dx$ and $\|\nu\|=(\nu,\nu)^{\frac{1}{2}}$.

Theorem 5.1. *The semidiscrete (5.2) is an unconditional stable for $\delta\tau>0$ and*

$$\| SI^{n+1} \|\leq\| SI^0 \|,$$

Proof. Let we have $n=0$ in (5.2):

$$\begin{array}{l}(SI^1,\nu)=\frac{1}{2}\sigma^2\delta\tau(\frac{\partial SI^0}{\partial r},\frac{\partial\nu}{\partial r})+(b-ar-\Lambda\sigma)\delta\tau(SI^0,\nu)\\ \qquad -(r\delta\tau-1)(SI^0,\nu),\end{array}\tag{5.3}$$

By substituting $\nu=SI^1$ in (5.3) and by using the Schwartz inequality, we have:

$$\|SI^1\|^2\leq(\|SI^0\|\|SI^1\|)\Rightarrow\|SI^1\|\leq\|SI^0\|.\tag{5.4}$$

Now, we assume:

$$\|SI^j\|\leq\|SI^0\|,\qquad j=1,\ldots,n.\tag{5.5}$$

Then we prove $\|SI^{n+1}\|\leq\|SI^0\|$.

By replacing $\nu=SI^{n+1}$ in the (5.2), we obtain:

$$\begin{array}{l}(SI^{n+1},SI^{n+1})=\frac{1}{2}\sigma^2\delta\tau(\frac{\partial SI^n}{\partial r},\frac{\partial SI^{n+1}}{\partial r})+(b-ar-\Lambda\sigma)\delta\tau(SI^n,SI^{n+1})\\ \qquad -(r\delta\tau-1)(SI^n,SI^{n+1}),\end{array}\tag{5.6}$$

By using the Schwartz inequality, we have:

$$\|SI^{n+1}\|^2\leq(\|SI^n\|\|SI^{n+1}\|),$$

$$\Rightarrow\|SI^{n+1}\|\leq(\|SI^n\|.$$

Accordind to the (5.5)

$$\|SI^{n+1}\|\leq(\|SI^0\|.$$

□

Theorem 5.2. *Let $(SI^n)_{n=0}^N$ be discreted time solution of (4.2) and $(SI(r,\tau^n))_{n=0}^N$ be a exact solution of (4.2) with $SI(r,0)=F$, then we have the following error estimates:*

$$\| SI(r,\tau^n)-SI^n \|\leq c(\tau+h^2)$$

.

Proof. According to the equation (5.1), and the inner product of (4.2) in ν, we have:

$$\begin{array}{l}(SI(r,\tau^{n+1}),\nu)=\frac{\sigma^2}{2}\delta\tau(\frac{\partial SI(r,\tau^n)}{\partial r},\frac{\partial\nu}{\partial r})+(b-ar-\Lambda\sigma)\delta\tau(SI(r,\tau^n),\nu)\\ \qquad -(r\delta\tau-1)(SI(r,\tau^n),\nu)+(R^n,\nu).\end{array}$$

We define $e^{n+1}=SI(r,\tau^{n+1})-SI^n$, then we obtain:

$$\begin{array}{l}(e^{n+1},\nu)=\frac{\sigma^2}{2}\delta\tau(\frac{\partial e^n}{\partial r},\frac{\partial\nu}{\partial r})+(b-ar-\Lambda\sigma)\delta\tau(e^n,\nu)\\ \qquad -(r\delta\tau-1)(e^n,\nu)+(R^n,\nu).\end{array}\tag{5.7}$$

Now, by substituting $\nu = e^{n+1}$ in the (5.7) we obtain:

$$\begin{aligned}(e^{n+1}, e^{n+1}) = &\tfrac{\sigma^2}{2}\delta\tau(\tfrac{\partial e^n}{\partial r}, \tfrac{\partial e^{n+1}}{\partial r}) + (b - ar - \Lambda\sigma)\delta\tau(e^n, e^{n+1}) \\ &-(r\delta\tau - 1)(e^n, e^{n+1}) + (R^n, e^{n+1}).\end{aligned} \tag{5.8}$$

and for $n = 0$ in (5.8), we have:

$$\begin{aligned}(e^1, e^1) = &\tfrac{\sigma^2}{2}\delta\tau(\tfrac{\partial e^0}{\partial r}, \tfrac{\partial e^1}{\partial r}) + (b - ar - \Lambda\sigma)\delta\tau(e^0, e^1) \\ &-(r\delta\tau - 1)(e^0, e^1), +(R^0, e^1).\end{aligned}$$

The Schwartz inequality gives:

$$\| e^1 \|^2 \leq \| e^0 \| \| e^1 \| + \| R^0 \| \| e^1 \|,$$

$$\| e^1 \| \leq \| e^0 \| + \| R^0 \| \leq c(\delta\tau + h^2).$$

Then

$$\| e^{n+1} \| \leq c(\delta\tau + h^2).$$

□

In the following section, we would try to represent numerical result using simulated parameter.

6. Numerical results

Before going through the numerical results, as an example, consider, a petrochemical company entered into a partnership with SPV, for selling its polyethylene production line. Now, the SPV, in line with the financial value of this line of production in the market, issues bonds and collects the financial resources of investors. At this point, the SPV, using the collected capital, purchases the production line, and then assigns it again to the company itself, in the form of Sukuk Contarct. Eventually, Petrochemical Company pays the rent to investors through a trustworthy financial company (for example, a construction and mining bank) and at the end of the contract period it becomes the owner of the production line again.

Now, if, in the initial and boundary condition, we let F, $p(t)$ and $g(t)$, 1, $1 + W$ and 0 respectively, using the following simulated data, the numerical results can be achieved. Among these parameters N is the number of collocation points while C

Table 1. Estimated parameters of the Model

Parametre	**Estimated value**
σ	0.1
C	0.5
W	0.01
a	0.1
b	0.1
Λ	0.2
N	16
T	1

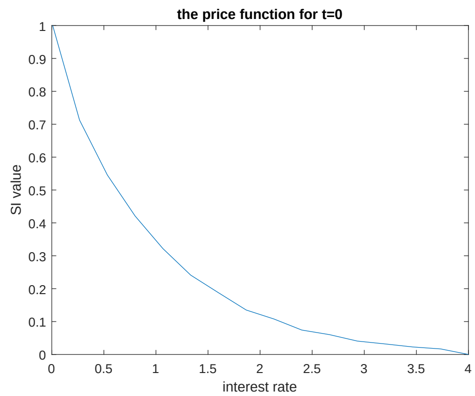

Figure 1. RBF-FD solution using 16 collocation points

serves as the shape parameter that plays a mjor role in the approximation procedure. It's because the large number of collocation points might lead us to high computational cost as well as potential ill-conditioning of the coefficient matrix. On the other hand, the small number of collocation points will provide us with inaccurate function approximation. On the whole, the number and the location of the centers are extremely crucial. Moreover, the shape parameter, or width parameter in RBF Neural Networks standpoint, is able to control the stability and the accuracy of the method simultaneously. Also, as it can be seen we assume the maturity date to be one year. Now, using MATLAB software, we implement the numerical scheme, suggested in section (4).

The figure shows the price of Ijarah-Sukuk, regarding to the simulated paramters and restricted interval for interest rate. However, the importatn thing which should be taken into consideration is that, it can simply change using different parameters.

7. Conclusion and remarks

In this paper, securities-based asset and the deterministic interest rate has been studied. Afterwards, the Vasicek model has been utilized for interest rate. Since interest rates can be altered based on policy changes, these changes may affect the model. We can import these changes to the model by applying the Switching Model and finding new appropriate numerical methods to solve them. Therefore, sensitivity analysis is studied and Ijarah Sukuk can be evaluated. As a further research, one can extend the idea of this paper to study other Islamic securities. It's worth noting that, due to the urgent need for bonds to transfer risk from the insurance market to the capital market, these bonds and this type of modeling can be used to finance and transfer risk for countries that have Sharia problems with bonds.

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