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Eigenvalues of fractional Sturm-Liouville problems by successive method

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Abstract In this paper, we consider a fractional Sturm-Liouville equation of the form,

$$-{}^{c}D^{\alpha}_{0+} \circ D^{\alpha}_{0+} y(t) + q(t)y(t) = \lambda y(t), \quad 0 < \alpha < 1, \quad t \in [0,1],$$

with Dirichlet boundary conditions

$$I^{1-\alpha}_{0+} y(t)|_{t=0} = 0, \quad \text{and} \quad I^{1-\alpha}_{0+} y(t)|_{t=1} = 0,$$

where, the sign $\circ$ is composition of two operators and $q \in L^2(0,1)$, is a real-valued potential function. We use a recursive method based on Picard's successive method to find the solution of this problem. We prove the method is convergent and show that the eigenvalues are obtained from the zeros of the Mittag-Leffler function and its derivatives.

Keywords. Fractional Sturm-Liouville, Fractional calculus, Sucssesive methods, Eigenvalues.

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1. Introduction

Over the last century, it has been demonstrated that many linear second order differential equations such as Hermite, Laguerre, Jacobi and others could be transformed into Sturm-Liouville equations. Investigation of fractional counterparts of these equations should lead to some interesting results. It is known that the Legendre polynomials play important roles in numerical analysis. The eigenvalue and eigenfunction properties of Fractional Legendre Fractional Legendre Equation (FLE) as well as the corresponding fractional Rodrigues formula are investigated in paper [8, 19]. It is shown that the Legendre Polynomials resulting, from an (FLE) are the

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same as those obtained from the integer order Legendre Equation. In [28], the authors have considered a regular fractional Sturm-Liouville problem of two kinds RFSLP-I and RFSLP-II of order $\nu \in (0,2)$ with the fractional differential operators both of Riemann-Liouville and Caputo type, of the same fractional-order $\mu = \nu/2 \in (0,1)$. It is proven that the regular boundary-value problems RFSLP-I & -II are indeed asymptotic cases for the singular counterparts SFSLP-I & -II. During the last three decades, fractional calculus has been applied to physics and other natural sciences. The use of differential equations of fractional order appears frequently in several research areas [13, 18, 21, 22, 25]. It has been applied to many fields in science and engineering, such as viscoelasticity, fluid mechanics, control theory, etc. Much effort has focussed on a class of well known fractional Sturm-Liouville problems (FSLPs), for example Mingarelli and Dehghan [11, 12] have investigated the general solution of three or two-term fractional differential equations of mixed Caputo/Riemann Liouville type in the case of Dirichlet boundary conditions. From numerical viewpoint, we also refer the reader for fractional differential equations to [3, 4, 5, 7, 15, 17, 27]. Al-Mdallal [2] applied the adomian decomposition method for solving fractional Sturm-Liouville problems. For more details about these problems and their applications, see [2]. In [6], the aforementioned relation between eigenvalues and zeros of Mittag-leffler function was shown. The Homotopy Analysis method has been used to approximate of the eigenvalues of Sturm-Liouville problems of fractional order [1]. Variational method and Inverse Laplace transform method applied in [10, 20], respectively. In this work, the successive method for solving the following equation

$$-{}^{c}D_{0+}^{\alpha} \circ D_{0+}^{\alpha} y(t) + q(t)y(t) = \lambda y(t), \quad 0<\alpha<1, \quad t\in[0,1], \tag{1.1}$$

is considered.

This paper is organized as follows. In Section 2, we present some preliminaries which we will use in this paper. A description and analysis of the successive method is presented in Section 3. Uniqueness of solution is discussed in Section 4. Two illustrative examples are given in Section 5. The last section includes our conclusions.

2. Preliminaries

We recall some definitions in Fractional Calculus. We refer the reader to [11, 12] for further details.

Definition 2.1. The left-sided and the right-sided Riemann-Liouville fractional integrals I_{a+}^{α} and I_{b-}^{α} of order $\alpha \in \mathbb{R}^+$ are defined by

$$I_{a+}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad t \in (a,b], \tag{2.1}$$

and

$$I_{b-}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s)ds, \quad t \in [a,b), \tag{2.2}$$

respectively. Here $\Gamma(\alpha)$ denotes the Euler's Gamma function. The following property is easily verified.

Property 2.2. For a constant C, we have $I^{\alpha}_{a^+}C = \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} \cdot C$.

Definition 2.3. The left-sided and the right-sided Caputo fractional derivatives ${}^cD^{\alpha}_{a^+}$ and ${}^cD^{\alpha}_{b^-}$ are defined by

$$ {}^cD^{\alpha}_{a^+}f(t) := I^{n-\alpha}_{a^+} \circ D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds, \quad t > a, \tag{2.3} $$

and for $t < b$,

$$ {}^cD^{\alpha}_{b^-}f(t) := (-1)^n I^{n-\alpha}_{b^-} \circ D^n f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (s-t)^{\alpha-n+1} f^{(n)}(s)ds, $$

respectively, where f is sufficiently differentiable and $n-1 \leq \alpha < n$.

Definition 2.4. The left-sided and the right-sided Riemann-Liouville fractional derivatives $D^{\alpha}_{a^+}$ and $D^{\alpha}_{b^-}$ are defined by

$$ D^{\alpha}_{a^+}f(t) := D^n \circ I^{n-\alpha}_{a^+}f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s)ds, \quad t > a, $$

and for $t < b$,

$$ D^{\alpha}_{b^-}f(t) := (-1)^n D^n \circ I^{n-\alpha}_{b^-}f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b (s-t)^{n-\alpha-1} f(s)ds, \tag{2.4} $$

respectively, where f is sufficiently differentiable and $n-1 \leq \alpha < n$.

2.1. **The Mittag-Leffler function.** The function $E_{\alpha}(z)$ defined by

$$ E_{\alpha}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} \qquad (z \in \mathbb{C}, \, \Re(\alpha) > 0), $$

was introduced by Mittag-Leffler [14, 24]. In particular, when $\alpha = 1$ and $\alpha = 2$, we have

$$ E_1(z) = e^z, \quad E_2(z) = \cosh(\sqrt{z}). $$

The generalized Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined by

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \tag{2.5} $$

where $z, \beta \in \mathbb{C}$ and $Re(\alpha) > 0$. When $\beta = 1$, $E_{\alpha,\beta}(z)$ coincides with the Mittag-Leffler function (2.4):

$$ E_{\alpha,1}(z) = E_{\alpha}(z). $$

Two other particular cases of (2.5) are as follows:

$$ E_{1,2}(z) = \frac{e^z - 1}{z}, \quad E_{2,2}(z) = \frac{\sinh(\sqrt{z})}{\sqrt{z}}. $$

Further properties of this special function may be found in [16].

2.2. **Laplace transform.**

Definition 2.5 ([23, 26]). The Laplace transform of a function $f(t)$ defined for all real-valued $t \geq 0$, t stands for the time, is the function $F(s)$ which is a unilateral transform defined by

$$F(s) = \mathcal{L}\{f(t)\} := \int_0^{\infty} e^{-st} f(t)dt,$$

where s is the frequency parameter.

Definition 2.6 ([23, 26]). The convolution of $f(t)$ and $g(t)$ supported on only $[0, \infty)$ is defined by

$$(f * g)(t) = \int_0^t f(s)g(t-s)ds, \qquad f, g : [0, \infty) \to \mathbb{R}.$$

Property 2.7 ([26]). The Laplace transform of the convolution of $f(t)$ and $g(t)$ is given by following relation

$$\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\{f(t)\} \times \mathcal{L}\{g(t)\}.$$

Property 2.8 ([18, 26]). The laplace transform of the derivatives of the Mittag-Leffler function is given by

$$\int_0^{\infty} e^{-st} t^{\alpha k+\beta-1} E_{\alpha,\beta}^{(k)}(\pm\lambda t^{\alpha})dt = \frac{k! s^{\alpha-\beta}}{(s^{\alpha} \mp \lambda)^{k+1}}, \qquad (\Re(p) > |a|^{\frac{1}{\alpha}}).$$

Property 2.9 ([16, 26]). The Laplace transform of the Riemann-Liouville fractional derivative is given by

$$\mathcal{L}\{D_{0+}^{\alpha} f(t)\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^k [D_{0+}^{\alpha-k-1} f(t)]_{t=0}, \qquad (n-1 \leq \alpha < n).$$

Property 2.10 ([16, 26]). The Laplace transform of the Caputo fractional derivative is given by

$$\mathcal{L}\{{}^c D_t^{\alpha} f(t)\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \qquad (n-1 < \alpha \leq n).$$

Property 2.11 ([18]). If $0 < \Re(\alpha) < 1$, then

$$I_{a^+}^{\alpha} D_{a^+}^{\alpha} f(t) = f(t) - \frac{(I_{a^+}^{1-\alpha} f)(0^+)}{\Gamma(\alpha)} \cdot t^{\alpha-1}.$$

Property 2.12 ([18]). If $\alpha > 0$ and $f(t) \in L_1(\mathbb{R}^+)$, then

$$I_{a^+}^{\alpha} \circ I_{a^+}^{\alpha} f(t) = I_{a^+}^{2\alpha} f(t) = \frac{1}{\Gamma(2\alpha)} \int_a^t (t-s)^{2\alpha-1} f(s)ds.$$

3. Analysis of the iterative method

Theorem 3.1. *Let $q(t)$ be continuous in the interval $[0,1]$ and there exist $M, N > 0$, such that*

$$|\frac{(t-s)^{2\alpha-1}\lambda}{\Gamma(2\alpha)}| \leq N, \quad |\int_0^t (t-s)^{2\alpha-1} q(s)ds| \leq M,$$

then, the following successive method

$$y_n(t) = y_0(t) + \frac{1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1}[\lambda - q(s)]y_{n-1}(s)ds,$$

to the equation,

$$-{}^cD_{0+}^{\alpha} \circ D_{0+}^{\alpha} y(t) + q(t)y(t) = \lambda y(t), \quad 0 < \alpha < 1, \quad t \in [0,1],$$

converges to the solution of the differential equation.

Proof. We consider equation (1.1) we have,

$$I_{0+}^{\alpha}\Big(-{}^cD_{0+}^{\alpha} \circ D_{0+}^{\alpha} y(t)\Big) = I_{0+}^{\alpha}\Big(\big(\lambda - q(t)\big)y(t)\Big),$$

$$I_{0+}^{\alpha} \circ D_{0+}^{\alpha} y(t) = I_{0+}^{\alpha}\Big({}^cD_{0+}^{\alpha} y(t)|_{t=0}\Big) + I_{0+}^{\alpha}\Big(\big(\lambda - q(t)\big)y(t)\Big).$$

Now, by integration in the interval $[0,t]$ yields:

$$\begin{aligned} y(t) = {} & \frac{(t-0)^{\alpha-1}}{\Gamma(\alpha)} \cdot I_{0+}^{1-\alpha} y(t)|_{t=0} + \frac{(t-0)^{\alpha-1}}{\Gamma(\alpha+1)} \cdot {}^cD_{0+}^{\alpha} y(t)|_{t=0} \\ & + \frac{1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1}\big(\lambda - q(s)\big)y(s)ds, \end{aligned} \tag{3.1}$$

we consider a recursive sequence y_n of function $[0,1]$ and correspondingly an infinite series Σu_n where $u_n = y_n - y_{n-1}$. By using M-test weierstrass, we conclude that this series is uniformly convergent to a function u. Since,

$$\sum_{n=1}^{N} u_n = \sum_{n=1}^{N}(y_n - y_{n-1}) = y_N - y_0,$$

therefore y_n tend to $u + y_0$ on $[0,1]$, and from uniform convergence, it follows that $u + y_0$ is a solution of (3.1), and hence solution of (1.1), now first we define y_0 and the y_n on $[0,1]$ by iteration. Let

$$\begin{aligned} y_0(t) &= y(0) + \frac{(t-0)^{\alpha-1}}{\Gamma(\alpha)} . I_{0+}^{1-\alpha} y(t)|_{t=0} + \frac{(t-0)^{\alpha}}{\Gamma(\alpha+1)} . {}^cD_{0+}^{\alpha} y(t)|_{t=0}, \\ y_1(t) &= y_0(t) + \frac{1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1}\big(\lambda - q(s)\big)y_0(s)ds, \end{aligned}$$

So, by induction we obtain:

$$y_n(t) = y_0(t) + \frac{1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1}\big(\lambda - q(s)\big)y_{n-1}(s)ds.$$

And there exist $M, N > 0$,

$$\Big|\frac{(t-s)^{2\alpha-1}\lambda}{\Gamma(2\alpha)}\Big| \leq N, \quad \Big|\int_0^t (t-s)^{2\alpha-1}q(s)ds\Big| \leq M,$$

thus

$$|u_1(t)| = |y_1(t) - y_0(t)| \leq MNAt,$$

and, by induction, if

$$|u_n(t)| = |y_n(t) - y_{n-1}(t)| \leq \frac{M^n N^n A t^n}{n!},$$

then

$$|u_{n+1}(t)| = |y_{n+1}(t) - y_n(t)| \leq \frac{M^{n+1}N^{n+1}At^{n+1}}{(n+1)!}.$$

We can now define the nonnegative constant E_n as follows:

$$|u_n(t)| = |y_n(t) - y_{n-1}(t)| \leq \frac{M^n N^n A t^n}{n!} \leq \frac{M^n N^n A}{n!} := E_n(t),$$

for $n \geq 1$,

$$\sum_{n=1}^{\infty} E_n = A\sum_{n=1}^{\infty}\frac{(MN)^n}{n!} = A(e^{MN}) - 1.$$

The exponential series for $\exp(t)$ being convergent for all values of its argument t. So, all the hypothesis for the application of the weierstass M- test [9] are satisfied and we can deduce that since,

$$\sum_{n=1}^{\infty}(y_n - y_{n-1}),$$

is uniform convergence on $[0, 1]$, to a function u, then, as we showed above in our general discussion, the sequence y_n converges uniformaly to $y = u + y_0$ on $[0, 1]$. Since every y_n is continus on $[0, 1]$, then y is continus also. So,

$$\lim_{n\to\infty}\int_0^t (t-s)^{2\alpha-1}\big(\lambda - q(s)\big)y_n(s)ds$$

$$= \int_0^t (t-s)^{2\alpha-1}\big(\lambda - q(t)\big)y(t)ds.$$

And from the Lebesque Dominated Convergence Theorem, we arrive

$$\lim_{n\to\infty}\int_0^t (t-s)^{2\alpha-1}\big(\lambda - q(s)\big)y_n(s)ds$$

$$= \int_0^t (t-s)^{2\alpha-1}\big(\lambda - q(t)\big)y(t)ds.$$

Therefore, we conclude that y is the solution of the integral equation (3.1) and the proof is completed. □

4. Uniqueness Of Solution

We suppose $Y = Y(t)$ is another solution of integral equation (1.1). The continuous function $y - Y$ is bounded on $[0, 1]$, suppose that

$$|y(t) - Y(t)| \leq p$$

for all $t \in [0, 1]$. Inductively, we can show that

$$|y(t) - Y(t)| \leq \frac{(MN)^n p}{n!}.$$

Since the right hand side of the inequality tends to zero as $n \to \infty$ then $y(t) = Y(t)$ for all $t \in [0, 1]$.

5. Examples

Example 5.1. We consider the fractional differential equation

$$^cD^{\alpha}_{0+} \circ D^{\alpha}_{0+} y(t) = (B - \lambda)y(t) \tag{5.1}$$

with Dirichlet boundary conditions

$$I^{1-\alpha}_{0+} y(t)|_{t=0} = 0, \quad \text{and} \quad I^{1-\alpha}_{0+} y(t)|_{t=1} = 0.$$

So, from equation (3.1) we have

$$y(t) = y_0(t) + \frac{(t-0)^{\alpha-1}}{\Gamma(\alpha)} \cdot I^{1-\alpha}_{0+} y(t)|_{t=0} + \frac{(t-0)^{\alpha}}{\Gamma(\alpha+1)} \cdot {}^cD^{\alpha}_{0+} y(t)|_{t=0},$$

and without loss of generality we assume $I^{1-\alpha}_{0+} y(t)|_{t=0}$, ${}^cD^{\alpha}_{0+} y(t)|_{t=0}$ are constant to be determined by imposing one or more initial/boundary conditions. Now, we assume $q(t) = B$. Applying, recursive method we obtained $y_0(t) = A$

$$\begin{aligned} y_1(t) &= y_0(t) + \frac{1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1}(B-\lambda)y_0(s)ds \\ &= A + \frac{(B-\lambda)At^{2\alpha}}{\Gamma(2\alpha+1)}, \end{aligned} \tag{5.2}$$

$$\begin{aligned} y_2(t) &= A + \frac{(B-\lambda)A}{\Gamma(2\alpha)} \int_0^t \Big[(t-s)^{2\alpha-1}\Big(1 + \frac{(B-\lambda)s^{2\alpha}}{\Gamma(2\alpha+1)}\Big)\Big]ds \\ &= A + \frac{(B-\lambda)At^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{(B-\lambda)^2At^{4\alpha}}{\Gamma(4\alpha+1)}, \end{aligned} \tag{5.3}$$

and

$$y_3(t) = A + \frac{(B-\lambda)A}{\Gamma(2\alpha)} \int_0^t \Big[(t-s)^{2\alpha-1} \tag{5.4}$$

$$\Big(1 + \frac{(B-\lambda)s^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{(B-\lambda)^2 s^{4\alpha}}{\Gamma(4\alpha+1)}\Big)\Big]ds$$

$$= A + \frac{(B-\lambda)At^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{(B-\lambda)^2At^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{(B-\lambda)^3At^{6\alpha}}{\Gamma(6\alpha+1)}, \tag{5.5}$$

and, by induction

$$y_n(t) = A\Big(1 + \sum_{k=1}^{n} \frac{(B-\lambda)^k t^{2\alpha k}}{\Gamma(2\alpha k + 1)}\Big),$$

so,

$$y(t) = \lim_{n\to\infty} y_n(t) = A\Big(1 + E_{2\alpha}(B-\lambda)t^{2\alpha}\Big). \tag{5.6}$$

Now, in order to obtain eigenvalues, by choosing terms from equation (5.6) for $\alpha \to 1$, $B = 1$ and with the following boundary condition

$$I_{0^+}^{1-\alpha} y(t)|_{t=1} = 0,$$

we get

$$A\Big(1 + E_2\big(-(\sqrt{\lambda - 1})^2\big)\Big) = A\Big(1 + \cos(\sqrt{\lambda - 1})\Big) = 0,$$

or

$$\lambda_n = 1 + (n\pi)^2.$$

Table 1 gives numerical results for different values of α and the curves of eigenfunctions.

TABLE 1. The eigenvalues λ_n of the $FSLP$ of Example 5.1

N	α	0.88	0.92	0.96	0.98	0.99	1
5	λ_n	11.39062114	11.23807884	11.08067271	11.00014195	10.95942025	10.53008205
		33.93455591	33.51171736	33.10109493	32.90039591	32.80120245	11.30670785
							20.23021501
							32.70278010
15	λ_n	92.38190053	91.54902644	90.69821292	90.26623055	90.04861476	10.86960440
		176.2339295	175.0023995	173.7938292	173.1980121	172.9022529	89.63610735
							90.02372131
							172.6080703
30	λ_n	500.3802320	486.6805498	486.1842897	488.7341969	248.1558112	10.86960440
						474.4613399	247.1677423
						499.4931910	248.3250404
							472.0298801
							499.9540715

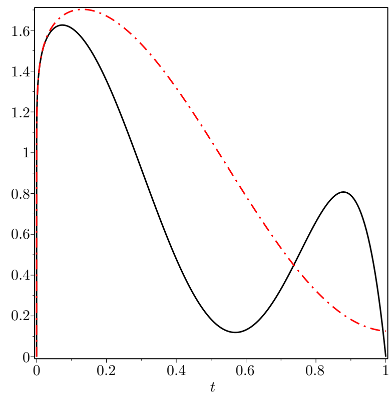

FIGURE 1. The curves of eigenfunction, $N = 5$ for $n = 1$ (solid line), $n = 2$ (dash dot line), where $\alpha = 0.92$, $\lambda_1 = 11.23807884$ and $\lambda_2 = 33.51171736$ for Example 5.1

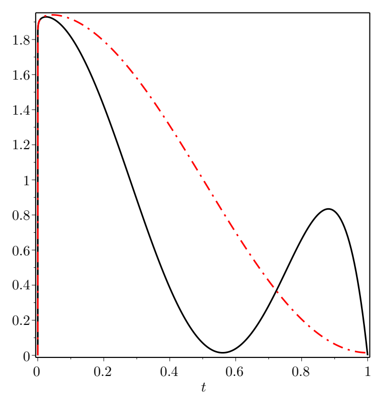

FIGURE 2. The curves of eigenfunction, $N = 5$ for $n = 1$ (solid line), $n = 2$ (dash dot line), where $\alpha = 0.99$, $\lambda_1 = 10.95942025$ and $\lambda_2 = 32.80120245$ for Example 5.1

Example 5.2. We consider the fractional differential equation

$$ {}^cD^{\alpha}_{0+} \circ D^{\alpha}_{0+} y(t) + (\lambda - t^{\beta}) y(t) = 0 \tag{5.7} $$

with Dirichlet boundary conditions

$$ I^{1-\alpha}_{0+} y(t)|_{t=0} = 0 \quad , \quad I^{1-\alpha}_{0+} y(t)|_{t=1} = 0. \tag{5.8} $$

So, from equation (3.1) we have

$$ y(t) = y(0) + \frac{(t-0)^{\alpha-1}}{\Gamma(\alpha)} \cdot I^{1-\alpha}_{0+} y(t)|_{t=0} + \frac{(t-0)^{\alpha}}{\Gamma(\alpha+1)} \cdot {}^cD^{\alpha}_{0+} y(t)|_{t=0}, \tag{5.9} $$

and without loss of generality we assume $I_{0+}^{1-\alpha}y(t)|_{t=0}=0$, ${}^{c}D_{0+}^{\alpha}y(t)|_{t=0}\neq 0$. Now, we assume $q(s)=s^{\beta}$. Applying, recursive method we obtain $y_0(t)=A$

$$y_1(t)=y_0(t)+\frac{1}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}(\lambda-s^{\beta})y_0(s)ds,$$

$$=A+\frac{A\lambda t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{At^{2\alpha+\beta}\Gamma(\beta+1)}{\Gamma(2\alpha+\beta+1)}. \tag{5.10}$$

And

$$y_2(t)=y_0(t)+\frac{1}{\Gamma(2\alpha)}\int_0^t \Big((t-s)^{2\alpha-1}(\lambda-s^{\beta}) \tag{5.11}$$

$$\Big[A+\frac{A\lambda s^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{As^{2\alpha+\beta}\Gamma(\beta+1)}{\Gamma(2\alpha+\beta+1)}\Big]\Big)ds,$$

$$=y_1(t)+\frac{A\lambda^2 t^{4\alpha}}{\Gamma(4\alpha+1)}-\frac{A\lambda\Gamma(2\alpha+\beta+1)t^{4\alpha+\beta}}{\Gamma(2\alpha+1)\Gamma(4\alpha+\beta+1)}$$

$$-\frac{A\lambda\Gamma(\beta+1)t^{4\alpha+\beta}}{\Gamma(4\alpha+\beta+1)}+\frac{A\Gamma(\beta+1)\Gamma(2\alpha+2\beta+1)t^{4\alpha+2\beta}}{\Gamma(2\alpha+\beta+1)\Gamma(4\alpha+2\beta+1)}. \tag{5.12}$$

Finally,

$$y_3(t)=y_0(t)+\frac{1}{\Gamma(2\alpha)}\int_0^t \Big((t-s)^{2\alpha-1}(\lambda-s^{\beta})$$

$$\Big[y_1(s)+\frac{A\lambda s^{4\alpha}}{\Gamma(4\alpha+1)}-\frac{A\lambda\Gamma(2\alpha+\beta+1)s^{4\alpha+\beta}}{\Gamma(2\alpha+1)\Gamma(4\alpha+\beta+1)}$$

$$-\frac{A\lambda\Gamma(\beta+1)s^{4\alpha+\beta}}{\Gamma(4\alpha+\beta+1)}+\frac{A\Gamma(\beta+1)\Gamma(2\alpha+2\beta+1)s^{4\alpha+2\beta}}{\Gamma(2\alpha+\beta+1)\Gamma(4\alpha+2\beta+1)}\Big]\Big)ds,$$

$$=y_2(t)+\frac{A\lambda^3 t^{6\alpha}}{\Gamma(6\alpha+1)}-\frac{A\lambda^2\Gamma(4\alpha+\beta+1)t^{6\alpha+\beta}}{\Gamma(4\alpha+1)\Gamma(6\alpha+\beta+1)} \tag{5.13}$$

$$-\frac{A\lambda^2\Gamma(2\alpha+\beta+1)t^{6\alpha+\beta}}{\Gamma(2\alpha+1)\Gamma(6\alpha+\beta+1)}-\frac{A\lambda^2\Gamma(\beta+1)t^{6\alpha+\beta}}{\Gamma(6\alpha+\beta+1)}$$

$$+\frac{A\lambda\Gamma(2\alpha+\beta+1)\Gamma(4\alpha+2\beta+1)t^{6\alpha+2\beta}}{\Gamma(2\alpha+1)\Gamma(4\alpha+\beta+1)\Gamma(6\alpha+2\beta+1)}$$

$$+\frac{A\lambda\Gamma(\beta+1)\Gamma(4\alpha+2\beta+1)t^{6\alpha+2\beta}}{\Gamma(4\alpha+\beta+1)\Gamma(6\alpha+2\beta+1)} \tag{5.14}$$

$$+\frac{A\lambda\Gamma(\beta+1)\Gamma(2\alpha+2\beta+1)t^{6\alpha+2\beta}}{\Gamma(2\alpha+\beta+1)\Gamma(6\alpha+2\beta+1)}$$

$$-\frac{A\Gamma(\beta+1)\Gamma(2\alpha+2\beta+1)\Gamma(4\alpha+3\beta+1)t^{6\alpha+3\beta}}{\Gamma(2\alpha+\beta+1)\Gamma(4\alpha+2\beta+1)\Gamma(6\alpha+3\beta+1)}. \tag{5.15}$$

Now, in order to obtain eigenvalues, by choosing terms from equation (5.15) and with following boundary conditions, we have,

$$I_{0+}^{1-\alpha}y_3(t)|_{t=1}=\mathcal{L}^{-1}\Big(\mathcal{L}\Big\{\frac{A}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\Big[y_2(\tau)+\frac{A\lambda^3\tau^{6\alpha}}{\Gamma(6\alpha+1)}$$

$$
\begin{aligned}
&- \frac{A\lambda^2\Gamma(4\alpha+\beta+1)\tau^{6\alpha+\beta}}{\Gamma(4\alpha+1)\Gamma(6\alpha+\beta+1)} - \frac{A\lambda^2\Gamma(2\alpha+\beta+1)\tau^{6\alpha+\beta}}{\Gamma(2\alpha+1)\Gamma(6\alpha+\beta+1)} \\
&- \frac{A\lambda^2\Gamma(\beta+1)\tau^{6\alpha+\beta}}{\Gamma(6\alpha+\beta+1)} + \frac{A\lambda\Gamma(2\alpha+\beta+1)\Gamma(4\alpha+2\beta+1)\tau^{6\alpha+2\beta}}{\Gamma(2\alpha+1)\Gamma(4\alpha+\beta+1)\Gamma(6\alpha+2\beta+1)} \\
&+ \frac{A\lambda\Gamma(\beta+1)\Gamma(4\alpha+2\beta+1)\tau^{6\alpha+2\beta}}{\Gamma(4\alpha+\beta+1)\Gamma(6\alpha+2\beta+1)} \\
&+ \frac{A\lambda\Gamma(\beta+1)\Gamma(2\alpha+2\beta+1)\tau^{6\alpha+2\beta}}{\Gamma(2\alpha+\beta+1)\Gamma(6\alpha+2\beta+1)} \\
&- \left.\frac{A\Gamma(\beta+1)\Gamma(2\alpha+2\beta+1)\Gamma(4\alpha+3\beta+1)\tau^{6\alpha+3\beta}}{\Gamma(2\alpha+\beta+1)\Gamma(4\alpha+2\beta+1)\Gamma(6\alpha+3\beta+1)}\right]_{t=1}\Bigg\}\Bigg) d\tau, \\
&= \frac{A}{\Gamma(2-\alpha)} + \frac{A\lambda}{\Gamma(\alpha+2)} - \frac{A\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} + \frac{A\lambda^2}{\Gamma(3\alpha+2)} \\
&- \frac{A\lambda\Gamma(2\alpha+\beta+1)}{\Gamma(2\alpha+1)\Gamma(3\alpha+\beta+2)} - \frac{A\lambda\Gamma(\beta+1)}{\Gamma(3\alpha+\beta+2)} \\
&+ \frac{A\Gamma(\beta+1)\Gamma(2\alpha+2\beta+1)}{\Gamma(2\alpha+\beta+1)\Gamma(3\alpha+2\beta+2)} + \frac{A\lambda^3}{\Gamma(5\alpha+2)} \\
&- \frac{A\lambda^2\Gamma(4\alpha+\beta+1)}{\Gamma(4\alpha+1)\Gamma(5\alpha+\beta+2)} - \frac{A\lambda^2\Gamma(2\alpha+\beta+1)}{\Gamma(2\alpha+1)\Gamma(5\alpha+\beta+2)} \\
&- \frac{A\lambda^2\Gamma(\beta+1)}{\Gamma(5\alpha+\beta+2)} + \frac{A\lambda\Gamma(2\alpha+\beta+1)\Gamma(4\alpha+2\beta+1)}{\Gamma(2\alpha+1)\Gamma(4\alpha+\beta+1)\Gamma(5\alpha+2\beta+2)} \\
&- \frac{A\lambda\Gamma(\beta+1)\Gamma(4\alpha+2\beta+1)}{\Gamma(4\alpha+\beta+1)\Gamma(5\alpha+2\beta+2)} + \frac{A\lambda\Gamma(\beta+1)\Gamma(2\alpha+2\beta+1)}{\Gamma(2\alpha+\beta+1)\Gamma(5\alpha+2\beta+2)} \\
&- \frac{A\Gamma(\beta+1)\Gamma(2\alpha+2\beta+1)\Gamma(4\alpha+3\beta+1)}{\Gamma(2\alpha+\beta+1)\Gamma(4\alpha+2\beta+1)\Gamma(5\alpha+3\beta+2)} = 0.
\end{aligned}
$$

There is not explicit relation for $y_n(t)$, but there is only recursive relation. Now, in order to obtain eigenvalues, by choosing terms from equation (5.9) and with the boundary conditions (5.8) we have the following tabel and some curves of the following eigenfunctions.

TABLE 2. The eigenvalues λ_n of the $FSLP$ of Example 5.2

α	0.88	0.92	0.96	0.98	0.99	1
λ_n	2.405039224	2.305068561	2.229222390	2.1978936370	2.183555975	2.170000602
	17.97371524	18.89035781	19.95455855	20.800803451	21.26965088	21.77123076
		25.23795748	34.10701183	38.990140030	41.62707551	44.36889646
		26.08584957	34.75041823	40.119474191	43.09634431	46.30580379

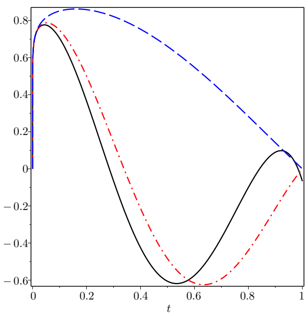

FIGURE 3. The curves of eigenfunctions, for $n = 1$ (solid line), $n = 2$ (dash dot line), $n = 3$ (dash line), where $\alpha = 0.92$, $\lambda_1 = 2.305068561$, $\lambda_2 = 18.89035781$ and $\lambda_3 = 26.08584957$ for Example 5.2

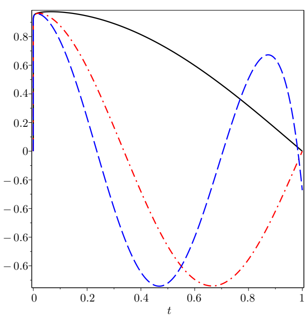

FIGURE 4. The curves of eigenfunctions, for $n = 1$ (solid line), $n = 2$ (dash dot line), $n = 3$ (dash line), where $\alpha = 0.99$, $\lambda_1 = 2.183555975$, $\lambda_2 = 21.26965088$ and $\lambda_3 = 43.09634431$ for Example 5.2

6. Conclusion

In this paper, we use a successive method to find the solution of a typical fractional Sturm-Liouville problem. We also find the approximate of the eigenvalues by the zeros of Mittag-Leffler function and its derivatives. The eigenvalues coincide with the asymptotic behavior given by [11, 12], when α tends to 1.

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