Solving the forward-backward heat equation with a non-overlapping domain decomposition method based on multiquadric RBF mesh-free method

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Abstract   In this paper, we present a numerical technique to deal with the one-dimensional forward-backward heat equations. First, the physical domain is divided into two non-overlapping subdomains resulting in two separate forward and backward subproblems, and then a meshless method based on multiquadric radial basis functions is employed to treat the spatial variables in each subproblem using the Kansa's method. We use a time discretization scheme to approximate the time derivative by the forward and backward finite difference formulas. In order to have adequate boundary conditions for each subproblem, an initial approximate solution is assumed on the interface boundary, and the solution is improved by solving the subproblems in an iterative way. The numerical results show that the proposed method is very useful and computationally efficient in comparison with the previous works.

1. Introduction

In the last two decades, meshfree methods based on radial basis functions (RBFs) have been applied to almost all types of partial differential equations (PDEs) [14, 29]. This method was first introduced by Kansa for the numerical solution of elliptic and parabolic equations [22] and then developed by many other researchers (see for example [10, 15, 16, 23]). The main attractive features of these methods are, firstly, they are truly meshless method meaning that neither in the domain representation nor in the solution procedure, mesh construction is required. Secondly, they can be easily applied to higher dimensional cases due to their radial nature and thirdly, they are

very convenient for the problems with complicated geometry.

The commonly used RBFs are multiquadrics (MQ), Gaussian, Thin plate splines (TPS) and compactly supported RBFs (CS-RBFs) [39]. For a comprehensive discussion about RBFs see [5].

In this work we propose a truly meshless method for the numerical solution of the forward-backward heat equation (FBHE) in a one-dimensional case as follows,

$$
\begin{aligned}
a(x)u_t - u_{xx} &= f(x,t), & (x,t) &\in \Omega = (-1,1) \times (0,1), \\
u(-1,t) &= g_{-1}(t), & t &\in [0,1], \\
u(1,t) &= g_1(t), & t &\in [0,1], \\
u(x,0) &= u_0(x), & x &\in [0,1], \quad \text{(initial condition)} \\
u(x,1) &= u_1(x), & x &\in [-1,0], \quad \text{(terminal condition)}
\end{aligned} \tag{1.1}
$$

where $a(x)$, $g_{-1}(t)$, $g_1(t)$, $u_0(x)$, $u_1(x)$ are given functions and $a(x) > 0$ for $x > 0$, $a(x) < 0$ for $x < 0$ and $a(0) = 0$. For a theoretical and practical consideration of the solution of this equation see for example [8, 35, 38].
The FBHE has appeared in a variety of applications in different areas some of which are briefly mentioned here.

- **Randomly accelerated particle**
  This problem considers the determination of the time that it takes a particle which is restricted to move on a line segment with a specific initial position and an initial velocity undergoing random acceleration to reach either of the boundary points. For full consideration of this problem in details see [18].
- **LaRosas electron beam model**
  This problem considers a beam electron velocity distribution that specifies the measure of how large a population of electrons at location x and time t travel at velocity v. At the leading edge of the beam it is considered appropriate to model the diffusion of electrons by the quasi-linear plasma diffusion equations [33].
- **Prandtl boundary layer equations**
  This problem arises in two dimensional fluid flow near a boundary when separation occurs. Starting with the nondimensionalized Navier-Stokes equations for a viscous incompressible fluid and considering the flow near the boundary leads to a non-linear FBHE [34].
- **Transport during flow reversal**
  This problem arises from modeling of transport by convection dominated flow of temperature or a pollutant or salt in the boundary layer of a fluid undergoing a flow separation or reversal (to see an example in details see [36]).
- **Neutron scattering**
  This is another example of the problems whose modeling lead to a forward-backward heat equation.

The FBHE has been solved by various methods such as finite difference method [7, 20, 38], transformation to a system of first order differential equations [30], least square approach [1] and Galerkin finite element [19].

While the meshless methods are classified into two major categories: meshfree strong form and meshfree weak form, the current work is based on the strong form and collocation method [29]. We apply the MQ to the underlying equation. This RBF was initially suggested by Hardy for applications in cartography to improve the approximation previously obtained by polynomial interpolation [21]. The MQ often gives the most accurate results in two dimensions for different applications [17].

One difficulty with using RBFs such as MQ is the existence of shape parameters whose values seriously affect the quality of the approximation. There have been many works regarding the choice of the shape parameters for various RBFs [24, 28]. Although, there is no mathematical theory for finding optimal values of these parameters. For some applications, suitable values have been suggested based on both theoretical and experimental works [11, 28].

Another disadvantage of using RBFs is that their interpolation matrices are fully populated. This causes the linear system of equations to be ill-conditioned, especially for large scale problems. To avoid this difficulty, some treatments such as domain decomposition method (DDM) [4], the preconditioner [26] and CS-RBFs [39] have been proposed.

This paper is organized as follows: In section 2, some theoretical and practical aspects of RBFs will be discussed. In section 3, the domain partitioning will be carried out for the underlying problem and the RBF meshfree formulation will be applied to the forward and backward subproblems. The solution of the algebraic equations resulted from the local problems will be dealt with via the standard iterative DDM in section 4. Finally, some numerical results will be presented in section 5.

2. RBF INTERPOLATION

This section discusses the theoretical aspects concerning interpolation (or collocation) via RBFs. These meshfree schemes are flexible methods that effectively work in scattered data points; therefore, it can be easily applied to problems with irregular domains. Some of the most popular RBFs are listed in Table 1.

As seen in Table 1, some RBFs contain a parameter ϵ known as a shape parameter which affects the flatness of the RBF profile. Many of these RBFs have been proven to be the smoothest interpolant and for solving PDEs, an RBF must be chosen with adequate order of smoothness up to the order of the underlying PDE.

A given function $u(x)$ in scattered data interpolation can be approximated by an interpolant such as $U(x)$ which is a linear combination of RBFs at N distinct nodes or centers $X = \{x_1, ..., x_N\}$ augmented by certain polynomials as follows,

$$U(x) = \sum_{i=1}^{N} \alpha_i \phi_i(x) + \sum_{j=1}^{M} \beta_j p_j(x) = \Phi^T(x)\alpha + P^T(x)\beta, \quad x \in \mathbb{R}^d, \qquad (2.1)$$

TABLE 1. Example of some popular radial basis functions

Name	$\phi(r)$	Restrictions	Smoothness
Generalized Multiquadric	$(r^2 + \epsilon^2)^{\frac{s}{2}}$	$s \neq 2n, n \in \mathbb{N}, s > 0$	C^∞
Generalized Inverse Multiquadric	$(r^2 + \epsilon^2)^{-s}$	$s > 0$	C^∞
Gaussian	$e^{-(\frac{r}{\epsilon})^2}$		C^∞
Matern (M2)	$e^{\frac{r}{\epsilon}}(\frac{r}{\epsilon} + 1)$		C^2
Wendland (W2)	$(1 - \frac{r}{\epsilon})_+^4 (4\frac{r}{\epsilon} + 1)$		C^2
Polyharmonic (odd s)	r^s	$s \neq 2n, n \in \mathbb{N}, s > 0$	$C^{\lceil s \rceil - 1}$
Polyharmonic	$r^{2s} \log(r)$	$s \in \mathbb{N}, s > 0$	C^{2s-1}

where $\phi_i : \mathbb{R}^d \longrightarrow \mathbb{R}$ is a radial basis function $\Phi^T(x) = [\phi_1(x), \phi_2(x), ..., \phi_N(x)]$, and $P^T(x)$ is a polynomial basis function that has the monomial terms as $P^T(x) = [p_1(x), p_2(x), ..., p_M(x)]$, where, $p_1(x), p_2(x), ..., p_M(x)$ form a base for the M-dimensional space $\pi_{m-1}(\mathbb{R}^d)$ of polynomials of total degree $\leq m - 1$ in d variables. Also $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)^T$ and $\beta = (\beta_1, \beta_2, ..., \beta_M)^T$.

Similarly for any differential operator $\mathbf{L}$, $\mathbf{L}u$ may be approximated by

$$\mathbf{L}U(x) = \sum_{i=1}^{N} \alpha_i \mathbf{L}\phi_i(x) + \sum_{j=1}^{M} \beta_j \mathbf{L}p_j(x) = \mathbf{L}\Phi^T(x)\alpha + \mathbf{L}P^T(x)\beta, \quad x \in \mathbb{R}^d \quad (2.2)$$

where $\phi(x) = \varphi(r_i) : \mathbb{R}^+ \longrightarrow \mathbb{R}$ in which r_i is a distace between an interpolating point x and the node x_i. The function ϕ_i depends only on the distance between x and a fixed point x_i. This distance in the Euclidean d-dimensional space for $x = (x^1, ..., x^d)$ and $x_i = (y_i^1, ..., y_i^d)$ is given by

$$r_i = ||x - x_i|| = ((x^1 - y_i^1)^2 + ... + (x^d - y_i^d)^2)^{\frac{1}{2}},$$

The vectors α and β are determined by enforcing the interpolation pass through all N scattered nodal points as follows,

$$U_k = \sum_{i=1}^{N} \alpha_i \phi_i(x_k) + \sum_{j=1}^{M} \beta_j p_j(x_k), \quad k = 1, 2, ..., N. \tag{2.3}$$

The above equations form an undetermined system as the number of unknown coefficients is $N + M$ whereas there are only N equations. In order to make a square system, some additional equations are requaired. Based on the theory of the conditionally positive definite functions, the following equations are added to the linear system (2.3) (see [15]).

$$\sum_{i=1}^{N} \alpha_i p_j(x_i) = 0, \quad j = 1, 2, ..., M. \tag{2.4}$$

The above equations can be reduced to the matrix form

$$\hat{U} = \left[\begin{array}{c} U \\ \mathbf{0} \end{array} \right] = \left[\begin{array}{cc} \mathbf{\Phi} & \mathbf{P} \\ \mathbf{P}^T & \mathbf{O} \end{array} \right] \left[\begin{array}{c} \alpha \\ \beta \end{array} \right], \tag{2.5}$$

where the matrix $\mathbf{\Phi} \in \mathbb{R}^{N \times N}$ is given by $\Phi_{ij} = \varphi(\|x_i - x_j\|), i, j = 1, ..., N,$ $\mathbf{P} \in \mathbb{R}^{N \times M}$ has entries $\mathbf{P}_{jk} = p_k(x_j), j = 1, ..., N, k = 1, ..., M, U^T = (U_1, ..., U_n), \mathbf{0}$ is a zero vector of length M, and $\mathbf{O}$ is a $M \times M$ zero matrix. The system (2.5) is solvable if the coefficient matrix on the left-hand side is invertible. The invertibility of the coefficient matrix in (2.5) for a class of RBFs have been considered in [15].
In eq. (2.1), if ϕ is conditionally positive definite i.e. ϕ has a polynomial growth towards infinity such as Multiquadric and Thin Plate Spline, then we need to add polynomial $\sum_{j=1}^{M} \beta_j p_j(x)$ in order to have a unique solution for the system (2.5).
Unlike these functions, RBFs such as Gaussian, Inverse Multiquadric functions are strictly positive definite RBFs. Adding a polynomial to these functions is not necessary since the solvability of the resulting interpolation system, in this case, is guaranteed. If ϕ is a positive definite function then by omitting the polynomial $\sum_{j=1}^{M} \beta_j p_j(x)$ in (2.1) we may have

$$U(x) = \sum_{i=1}^{N} \alpha_i \phi_i(x) = \Phi^T(x)\alpha, \quad x \in \mathbb{R}^d, \tag{2.6}$$

by enforcing the interpolation pass through all N scattered nodal points, the following equations are obtained

$$U_k = \sum_{i=1}^{N} \alpha_i \phi_i(x_k) = \mathbf{\Phi}\alpha, \quad k = 1, 2, ..., N, \tag{2.7}$$

for which the associated matrix Φ is invertible and the RBF interpolation problem is well-posed, hence a unique solution exists .
Regardless of the property of the positive definiteness in these two general type of RBFs, the interpolation matrix of these functions is dense and it may be very ill-conditioned by choosing some particular RBF shape parameters [15]. That would happen because of the globally supported of RBFs.

To overcome the conditioning dilemma in the system of collocation equations, several recipes such as compactly supported RBFs [39], preconditioning [3], domain decomposition [26] and local methods have been applied [37].
A few works have been accomplished to consider the accuracy and convergence of the RBF collocation method. Madych and Nelson in [31] have theoretically shown that the error converges at the exponential rate $\mathbf{O}(\lambda^{\frac{1}{h}})$ for a class of RBFs covering Multiquadric and the Gaussian, where $0 < \lambda < 1$ and h is the maximum mesh size. Along the same works for the Multiquadrics, Madych in [32] presented $\mathbf{O}(e^{ac}\lambda^{\frac{c}{h}})$, where c is the shape parameter and $a > 0$ is a constant. For PDEs, Cheng et al in [6]

obtained the error estimate $\mathbf{O}(\lambda^{\frac{\sqrt{c}}{h}})$ to the convergence of Gaussian and Multiquadrics for elliptic problems.
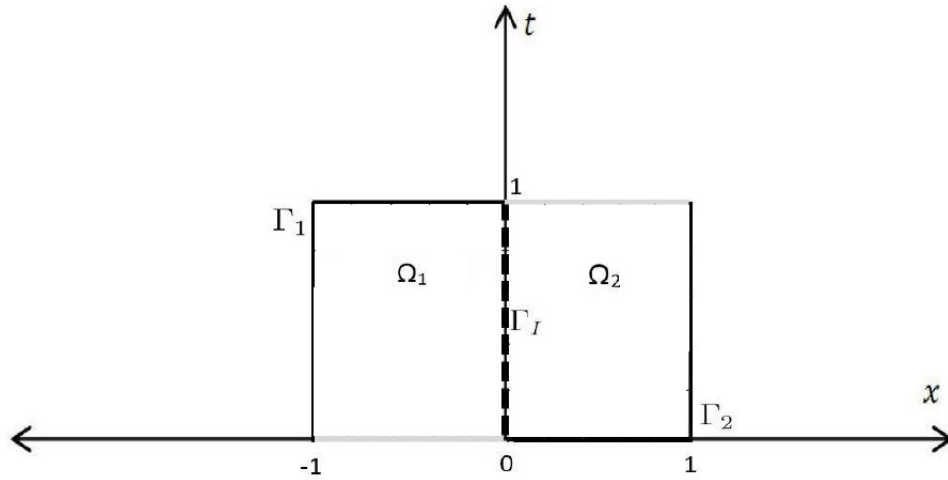
3. DOMAIN PARTITIONING

FIGURE 1. The domain partitioning scheme

Let the domain Ω be divided into two subdomains, $\Omega_1 = (-1, 0) \times (0, 1)$ and $\Omega_2 = (0, 1) \times (0, 1)$ with the real boundaries Γ_1 and Γ_2 and an artificial boundary $\Gamma_I = \{0\} \times (0, 1)$ in between (Fig. 1). This partitioning divides problem (1.1) into two subproblems as follows,

Subproblem 1:

$$
\begin{aligned}
a(x)u_t - u_{xx} &= f(x, t), & (x, t) \in \Omega_1 = (-1, 0) \times (0, 1), \\
u(-1, t) &= g_{-1}(t), & t \in [0, 1], \\
u(x, 1) &= u_1(x), & x \in [-1, 0],
\end{aligned}
\tag{3.1}
$$

Subproblem 2:

$$
\begin{aligned}
a(x)u_t - u_{xx} &= f(x, t), & (x, t) \in \Omega_2 = (0, 1) \times (0, 1), \\
u(1, t) &= g_1(t), & t \in [0, 1], \\
u(x, 0) &= u_0(x), & x \in [0, 1].
\end{aligned}
\tag{3.2}
$$

To solve the subproblems (3.1) and (3.2), we first deal with the spatial variable by the RBF method.

Let $\{x_i\}_{i=1}^{N}$ be a set of nodes in $[-1, 0]$ where $\{x_i\}_{i=2}^{N-1}$ and $\{x_1 = -1, x_N = 0\}$ represent the interior and boundary points respectively.

An approximation of the function $u(x,t)$ in the backward equations (3.1), can be written as

$$u(x,t) \approx \sum_{j=1}^{N} \lambda_j(t)\varphi(||x - x_j||), \quad x \in \mathbb{R}^d, \tag{3.3}$$

where $\{\varphi(||x - x_j||)\}_{j=1}^{N}$ is a set of the Multiquadric functions, centered at x_j as

$$\varphi(||x - x_j||) = \sqrt{||x - x_j||^2 + c^2},$$

and the constant c is a shape parameter. Also $\{\lambda_j(t)\}_{j=1}^{N}$ are time-dependent unknown coefficients to be determined.

Substituting (3.3) into (3.1), we obtain

$$a(x) \sum_{j=1}^{N} \frac{d\lambda_j(t)}{dt}\varphi(||x - x_j||) - \lambda_j(t))\varphi_{xx}(||x - x_j|| = f(x,t), \tag{3.4}$$

$$u(-1,t) = \sum_{j=1}^{N} \lambda_j(t)\varphi(|| - 1 - x_j||) = g_{-1}(t), \tag{3.5}$$

$$\sum_{j=1}^{N} \lambda_j(1)\varphi(||x - x_j||) = u_1(x), \tag{3.6}$$

where $\varphi_{xx}(||x - x_j||$ represents the second derivative of $\varphi(||x - x_j||$ with respect to x. It should be noted that the last equation is associated with the terminal condition, that is, the solution function at $t = 1$.
Similarly, using N nodes $\{x_i\}_{i=N}^{2N-1}$ in $[0,1]$, the solution function $u(x,t)$ for subproblem 2 can be approximated as

$$u(x,t) \approx \sum_{j=N}^{2N-1} \gamma_j(t)\varphi(||x - x_j||, \tag{3.7}$$

where $\{x_N = 0, x_{2N-1} = 1\}$ and $\{x_j\}_{j=N+1}^{2N-2}$ are the boundary and interior nodes, respectively, for this subproblem and γ_j's are the corresponding unknown coefficients. Substituting (3.7) into (3.2) yields

$$a(x) \sum_{j=N}^{2N-1} \frac{d\gamma_j(t)}{dt} \varphi(||x - x_j||) - \gamma_j(t)\varphi_{xx}(||x - x_j||) = f(x, t), \tag{3.8}$$

$$u(1, t) = \sum_{j=N}^{2N-1} \gamma_j(t)\varphi(||1 - x_j||) = g_1(t), \tag{3.9}$$

$$\sum_{j=N}^{2N-1} \gamma_j(0)\varphi(||x - x_j||) = u_0(x). \tag{3.10}$$

We notice that this subproblem represents a forward type and the last equation is associated with the initial condition.

Employing the collocation method at the interior points gives

$$a(x) \sum_{j=1}^{N} \frac{d\lambda_j(t)}{dt} \varphi(||x_i - x_j||) - \lambda_j(t))\varphi_{xx}(||x_i - x_j|| = f(x_i, t), \ \ i = 2, \ldots, N-1. \tag{3.11}$$

We now discretize the time by the first order forward and backward schemes, respectively, for subproblems 1 and 2.

Let $\delta t = 1/M$ and $t_j = j\delta t$ for $j = 0, ..., M$. By a first order forward difference scheme in Ω_1, we approximate the time derivative as

$$\frac{d\lambda_j(t)}{dt} \approx \frac{1}{\delta t}(\lambda_j(t + \delta t) - \lambda_j(t)). \tag{3.12}$$

Substituting the difference formula (3.12) into (3.11) at $t = t_n$ leads to the equations

$$a(x) \sum_{j=1}^{N} \frac{(\lambda_j(t_{n+1}) - \lambda_j(t_n))}{\delta t} \varphi(||x_i - x_j||) - \lambda_j(t))\varphi_{xx}(||x_i - x_j||) = f(x_i, t_n),$$
$$i = 2, \ldots, N-1,$$

which can be reduced to a matrix form as follows,

$$(D^{(1)}C^{(1)} + \delta t C_{xx}^{(1)})\lambda^{\mathbf{n}} = D^{(1)}C^{(1)}\lambda^{\mathbf{n+1}} - \delta t \mathbf{f^n}, \ \ n = M - 1, ..., 0, \tag{3.13}$$

where $C^{(1)}$ and $C_{xx}^{(1)}$ are $(N-2) \times N$ matrices with

$$C_{ij}^{(1)} = \varphi(||x_i - x_j||), \ \ (C_{xx}^{(1)})_{ij} = \varphi_{xx}(||x_i - x_j||); \ i = 2, ..., N-1, \ j = 1, ...N$$

, $D^{(1)}$ is a diagonal matrix of the same size with $D_{ii}^{(1)} = a(x_i); \ i = 2, ..., N - 1$, $f_i^n = f(x_i, n\delta t)$,

$$\lambda^{\mathbf{n}} = [\lambda_1(t_n), \ldots, \lambda_N(t_n)]^T, \ \ \mathbf{f^n} = \left[f_2^n, \ldots, f_{N-1}^n\right],$$

and the superscripts n and (1), respectively, denote the time steps and the association of the matrices with the subdomain Ω_1. Note that this system includes $N-2$ equations

associated with the interior nodes. Equations formed by the real boundary $x = -1$ and the virtual boundary $x = 0$ are given by

$$
\begin{aligned}
B_1 \lambda^{\mathbf{n}} &= g_{-1}(t_n), \\
B_N^{(1)} \lambda^{\mathbf{n}} &= \psi_1^n,
\end{aligned}
\tag{3.14}
$$

where B_1 and $B_N^{(1)}$ are $1 \times N$ matrices as follows,

$$
B_1 = [\varphi(||x_1 - x_1||), \ldots, \varphi(||x_1 - x_N||)],
$$

$$
B_N^{(1)} = [\varphi(||x_N - x_1||), \ldots, \varphi(||x_N - x_N||)],
$$

and ψ_1^n is a fictitious function acted at the time step n on the virtual boundary $x = 0$ for subproblem 1.

Using a backward scheme for the time derivative in the subproblem 2, and repeating a similar process, the final linear system of equations takes the form

$$
\begin{aligned}
D^{(2)} C^{(2)} - \delta t C_{xx}^{(2)}) \gamma^{\mathbf{n+1}} &= D^{(2)} C^{(2)} \gamma^{\mathbf{n}} + \delta t \mathbf{f^{n+1}}, \quad n = 0, ..., M - 1, \\
B_N^{(2)} \gamma^{\mathbf{n+1}} &= \psi_2^n, \\
B_{2N-1} \gamma^{\mathbf{n+1}} &= g_1(t_n),
\end{aligned}
\tag{3.15}
$$

where

$$
\gamma^{\mathbf{n}} = [\gamma_N(t_n), \ldots, \gamma_{2N-1}(t_n)]^T,
$$

and all the matrices with superscript (2) have the same duty as those for subproblem 1 and B_{2N-1} is associated with the boundary point $x_{2N-1} = 1$.

As seen above, for the subproblem associated with Ω_1, which is a backward problem, a forward finite difference is used, whereas for the subproblem associated with Ω_2 which is a forward problem, a backward formula is utilized.

4. Applying iterative DDM

The linear system of equations (3.13) and (3.14) can not be solved as the boundary conditions on the virtual boundary point $x = 0$ is not available, that is, the values of ψ_1^n, are not known. The same problem occurs for subproblem 2. To fix this issue, we use some approximate values on the interface boundary Γ_I.

To proceed, we need some approximations on the virtual boundary $x = 0$. Let $\{u_{0j}^{(0)} \approx u(0, t_j) \quad 1 \leq j \leq M - 1\}$ be the initial approximate values of the solution on this boundary where the superscript (.) indicates the number of performed iterations. In this case, the above linear systems can be dealt with separately. To do so, we first need to find $\lambda^{\mathbf{M}}$ and $\gamma^{\mathbf{0}}$, respectively, for the backward and the forward problems. To this end, we use the functions $u_1(x)$ and $u_0(x)$ which specify the terminal and initial conditions for the subproblems 1 and 2, respectively. For instance, applying the terminal condition for subproblem 1 and using (3.3), we derive the following equations:

$$
A^{(1)} \lambda^{\mathbf{M}} = \mathbf{u_1},
\tag{4.1}
$$

where

$$A^{(1)} = \begin{bmatrix} B_1 \\ C^{(1)} \\ B_N^{(1)} \end{bmatrix},$$

$\mathbf{u_1}$ is a vector of the nodal values of function $u_1(x)$ and the matrices are those introduced in (3.13) and (3.14).

Similarly, the equation associated with initial conditions for subproblem 2 can be written as

$$A^{(2)}\gamma^{\mathbf{0}} = \mathbf{u_0}, \tag{4.2}$$

where $\mathbf{u_0}$ is a vector of the nodal values of function $u_0(x)$, and

$$A^{(2)} = \begin{bmatrix} B_N^{(2)} \\ C^{(2)} \\ B_{2N-1} \end{bmatrix}.$$

Now the linear equations (4.1) and (4.2) can be solved to obtain $\lambda^{\mathbf{M}}$ and $\gamma^{\mathbf{0}}$ followed by finding $\lambda^{\mathbf{n}}$, $n = M - 1, ..., 0$ and $\gamma^{\mathbf{n}}$, $n = 1, ..., M$ via solving equations (3.13), (3.14) and (3.15) at each time step.

Having solved the subproblems for the approximate values $u_{0j}^{(0)}$, we need to update the solution on Γ_I.

We can use a number of nodes in the neighborhood of $x = 0$ to interpolate the solution function on the interface at each time step as follows,

Suppose that

$$u(x, t_j) \approx \sum_{l=N-L}^{N+L} \eta_k(t_j)\varphi(||x - x_l||), \quad j = 0, \ldots, M. \tag{4.3}$$

where φ can be the MQ or any other RBFs, η_k's are unknown coefficients and L is chosen such that $L < N$. To solve the interpolation problem, we use the interior approximate solutions of the subproblems to find the unknown coefficients η_j, that is,

$$u(x_i, t_j) = \sum_{l=N-L}^{N+L} \eta_l(t_j)\varphi(||x_i - x_l||), \quad i = N - L, \ldots, N + L, \ i \neq N. \tag{4.4}$$

We also know that in the governing equation of problem (1.1), $a(x) = 0$ at $x = 0$, Hence, $x_N = 0$, as an interior point of the main equation, satisfies

$$-u_{xx} = f(x, t) \quad (x, t) \in \Gamma_I = \{0\} \times (0, 1), \tag{4.5}$$

Thus, by substituting (4.3) into (4.5) and applying the collocation method at $x = x_N$, we obtain

$$\sum_{l=N-L}^{N+L} \eta_l \varphi_{xx}(||x_N - x_l||) = -f(0, t_j), \quad j = 1, \ldots, M-1. \tag{4.6}$$

Solving equations (4.4) and (4.6) will give the values of η_k followed by evaluating $u(x_N, t_j)$ by the use of (4.3) to update the interface boundary solution. This process is performed iteratively to produce the interface boundary solution $u_{0j}^{(m)}$ (mth iteration) until a desired accuracy is achieved.

5. NUMERICAL RESULTS

In order to demonstrate the performance of the proposed method, we present some numerical results by considering a forward-backward heat equation using MQ with the shape parameter ϵ.
To measure the accuracy of the numerical solution, the maximum error (Max error) and the root-mean-square error (RMSE) are used as follows:

$$Max\ error = \max_{j=1}^{N} |\hat{u}_j - u_j|, \quad RMSE = \sqrt{\frac{1}{N} \sum_{j=1}^{N} (\hat{u}_j - u_j)^2},$$

where N is the number of nodes, u_j and $\hat{u}_j$ denote the exact and approximate solutions at the jth node.

Example 5.1. We consider equation (1.1) with $g_{-1}(t) = 0$, $g_1(t) = 0$, $u_0(x) = 0$, $u_1(x) = 0$ and

$$f(x,t) = \begin{cases} 2x(x^2-1)t[(t-1)^2 - 4x^2 + t(t-1)] - 2t^2[(t-1)^2 - 24x^2 + 4], \\ x \geq 0, \quad t \in [0,1], \\ \\ 2x(x^2-1)(t-1)(2t^2 - t - 4x^2) - 2(t-1)^2(t^2 - 24x^2 + 4), \\ x \leq 0, \quad t \in [0,1]. \end{cases}$$

$$\tag{5.1}$$

The exact solution of the above problem is given by:

$$u(x,t) = \begin{cases} (x^2-1)t^2[(t-1)^2 - 4x^2], & x \geq 0 \quad t \in [0,1], \\ \\ (x^2-1)(t^2 - t - 4x^2)(t-1)^2, & x \leq 0 \quad t \in [0,1], \end{cases} \tag{5.2}$$

This problem has been solved by the proposed method and the numerical results are presented for various numbers of time steps, M, the total number of nodes in each subproblem, N and two values of the shape parameter ϵ. Also, in each case, the number of iterations, represented by k is given. The numerical errors in the cases of (I) $\epsilon = 0.5$, $N = 10$, and (II) $\epsilon = 1.4$, $N = 20$, for various number of time steps are given, respectively, in Tables 2 and 3. As seen, in both cases, the results demonstrate the convergence of the proposed method as by increasing the number of nodes, the

errors are decreased.

It should be noted that the number of nodes used, for the interpolation part of the up-dating stage in (4.3), is taken as twenty percent of the total number, that is, $L = N/5$. Of course by increasing the number of interpolation nodes, L, the number of iterations required for a desired accuracy is reduced.

In addition, the exact solution and the approximate solutions obtained by the new method, are compared in Figures 2 and 3 for various number of nodes and the time steps and different initial approximate solution ψ^n on the virtual boundary. As observed in the right sides of the Figures, the approximate solution is in a good agreement with the exact solution. Also the plots of the error values, in the left sides, demonstrate the accuracy of the numerical results.

To consider the convergence behavior of the current method, we applied it to the above problem while using different values of the initial approximate values ψ^n. To do this, we chose this values to be a multiple of the exact solution, that is, $\psi^n = cu(x, y)$, when $c = 0, 1.5, 2$. The results are displayed in each case and the number of iterations required for the accuracy achieved are mentioned in the Figures. One can see that the number of iterations required, in each case, is proportional to the closeness of the ψ^n to the exact solution.

TABLE 2. Error values in the case of $\epsilon = 0.5$. for Example 5.1.

N	M	Max error	RMSE	k
10	10	3.2E-2	1.3E-2	18
10	20	1.9E-2	6.9E-3	15
10	30	1.3E-2	4.8E-3	13
10	40	1.1E-2	3.9E-3	12

TABLE 3. Error values in the case of $\epsilon = 1.4$ for Example 5.1.

N	M	$Maxerror$	$RMSE$	k
20	10	2.9E-2	1.1E-2	14
20	20	1.5E-2	4.1E-3	14
20	30	8.3E-3	2.1E-3	13
20	40	7.0E-3	2.1E-3	13

The numerical results achieved by the new method and the results presented in [20] are compared in Table 4. Both methods are based on an iterative non-overlapping DDM, but the previous work used the FDM for the spatial dimension. As a result, being a meshfree method, the current work is more computationally efficient than the other one which is a mesh-dependent method, taking into account, nearly the same accuracy is gained in the two methods. Moreover, in the previous work, an appropriate iteration parameter is required in order to make the method convergent. Finding a suitable parameter is a difficult task and needs time and care.
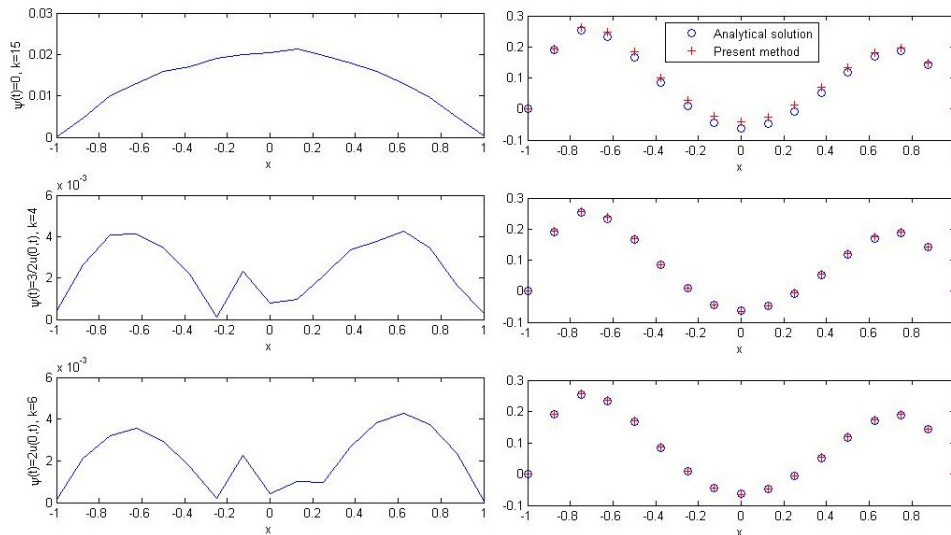
FIGURE 2. Configuration of the the error values in the left side and comparison the exact and approximate solutions in the right side with N=8, M=16 and $\epsilon = 0.14$ (Example 1).

TABLE 4. Comparison of the Max error for Example 5.1 with the previous numerical results.

N	M	Method of [20]	k	New method	ϵ	k
4	4	1.36E-2	24	4.37E-2	.033	5
8	16	3.68E-2	38	2.19E-2	.14	15
16	64	9.22E-3	63	5.4E-3	.84	20

Example 5.2. Consider eqs. (1.1) with

$$f(x,t) = -x\frac{x^2+1}{(1+t)^2} - \frac{2}{1+t}, \tag{5.3}$$

with the initial and terminal conditions

$$u(x,0) = x^2 + 1, \quad u(x,1) = \frac{x^2+1}{2},$$

and the dirichlet condition

$$u(x,t) = \frac{x^2+1}{1+t}, \quad x \in \partial\Omega, \ 0 < t < 1.$$

The exact solution, in this case, is given by:
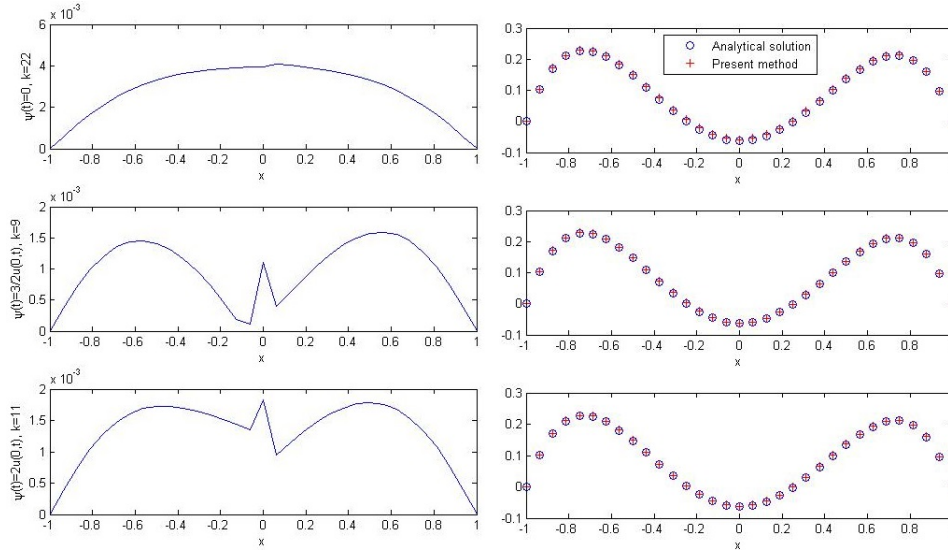
$$u(x,y,t) = \frac{x^2+1}{1+t}. \tag{5.4}$$

FIGURE 3. Configuration of the error values in the left side and comparison the exact and approximate solutions in the right side with N=16, M=64 and $\epsilon = 0.84$ (Example 1).

The numerical errors are presented in Tables 5 and 6. Also the exact and approximate solutions together with the error function are plotted in Figure 4. Again we observe reasonable accuracy for this example while having the advantages mentioned for the previous example.

TABLE 5. Error values of the new method for Example 5.2.

N	M	ϵ	k	$Max\ error$	MSRE
			12	6.30E-2	1.36E-2
8	8	0.05	14	2.23E-2	6.60E-3
			16	1.41E-2	4.30E-3
			12	7.60E-3	2.40E-3
10	10	0.14	14	3.00E-3	9.49E-4
			16	4.20E-3	7.68E-4

TABLE 6. Error values of the new method for Example 5.2.

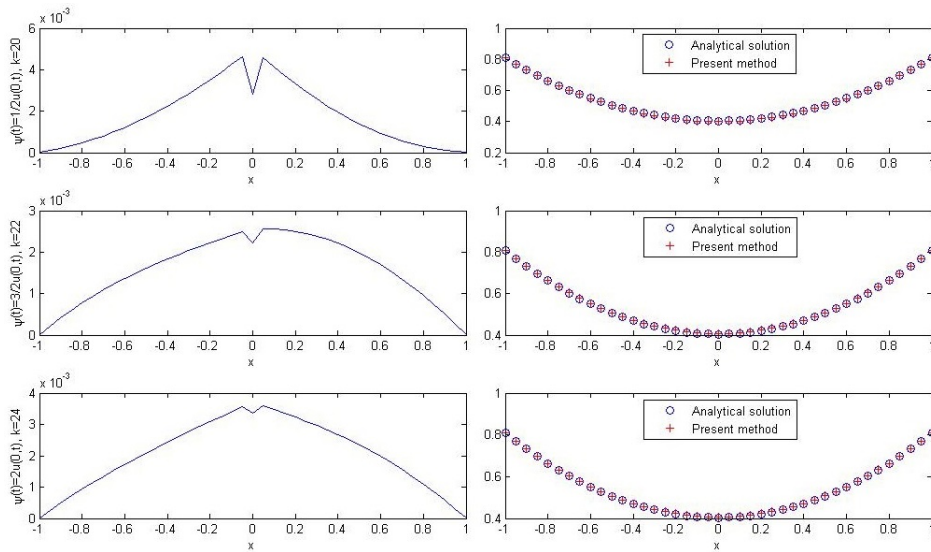| N	M	ϵ	k	$Max\ error$	MSRE
			24	1.90E-3	4.97E-4
20	20	1.54	26	1.30E-3	2.45E-4
			28	6.29E-4	2.40E-4

FIGURE 4. Configuration of the error values in the left side and comparison the exact and approximate solutions in the right side with N=20, M=20 and $\epsilon = 1.54$ (Example 2).

It should be emphasized that due to the nature of the forward-backward problem, the physical domain is essentially divided into a couple of subdomains. However, applying domain decomposition with more than two subdomains could improve both the computational efficiency and the accuracy, since on the one hand, it leads to solving a number of small systems of equations rather than a single large system and on the other hand, the method improve the conditioning of the matrices which results in improving the accuracy of the solution. However, because the current work concerns about a 1D forward-backward problem and, in this case, we are not dealing with large scale matrices, the DDM with many subdomains is not motivated. In a primary attempt, we have considered a DDM with many subdomains for 2D problems confirming the improvement of the results. These achievements together with consideration of 2D problems are not presented here and left for a future work.

6. CONCLUSION

An iterative non-overlapping domain decomposition method based on Multiquadric RBF meshfree method was developed for the numerical solution of the forward-backward heat equation. The proposed method was established by considering two separate backward and forward subproblems via splitting the domain.

The effectiveness of the new method was demonstrated by considering two examples and comparing the results with the numerical solutions obtained by the finite difference method in a previous work.

The method was presented for the case of one dimensional spatial variable. Extending the method to the cases of 2D and 3D are also possible and left for a future work. The method can offer much more computational efficiency in the higher dimensional cases, taking into account that mesh-dependent methods have more difficulty in these cases and their efficiency considerably decreases, especially when an adaptive analysis is required.

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