Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 9, No. 3, 2021, pp. 774-787
DOI:10.22034/cmde.2020.32950.1532

An accurate method for nonlinear local fractional Wave-Like equations with variable coefficients

Djelloul Ziane
Laboratory of mathematics and its applications (LAMAP),
University of Oran1 Ahmed Ben Bella, Oran, 31000, Algeria.
E-mail: djeloulz@yahoo.com

Mountassir Hamdi Cherif*
Laboratory of mathematics and its applications (LAMAP),
University of Oran1 Ahmed Ben Bella, Oran, 31000, Algeria.
E-mail: mountassir27@yahoo.fr

Kacem Belghaba
Laboratory of mathematics and its applications (LAMAP),
University of Oran1 Ahmed Ben Bella, Oran, 31000, Algeria.
E-mail: belghaba@yahoo.fr

Fethi Bin Muhammad Belgacem
Department of Mathematics, Faculty of Basic Education,
PAAET, Al-Ardhiya, Kuwait.
E-mail: fbmbelgacem@gmail.com

Abstract The basic motivation of the present study is to apply the local fractional Sumudu variational iteration method (LFSVIM) for solving nonlinear wave-like equations with variable coefficients and within local fractional derivatives. The derivatives operators are taken in the local fractional sense. The results show that the LFSVIM is an appropriate method to find non-differentiable solutions for similar problems. The results of the solved examples showed the flexibility of applying this method and its ability to reach accurate results even with these new differential equations.

Keywords. Sumudu variational iteration method, Nonlinear local fractional wave-like equations, Local fractional calculus.

2010 Mathematics Subject Classification. 44A05, 26A33, 44A20, 34K37.

1. Introduction

Many problems of physics and engineering can be expressed by nonlinear partial differential equations, we mention some of them, such as, the Schrödinger, the Burgers, the Bateman-Burgers, the Fokker-Planck, the Korteweg-de Vries, the Klein-Gordon, the Rosenau-Hyman, the Whitham, the Cahn-Hilliard, the Foam Drainage equation and other. Given the importance of these partial differential equations and others as fractionl or local fractional differential equations, many researchers have worked to develop method to solve them, or find approximate solutions [2, 3, 7, 10, 32, 34]. The

Received: 20 April 2020 ; Accepted: 1 June 2020.
* corresponding.

work of these researchers led to the development of several methods [8, 9, 16, 33], including the variational iteration method and in the abbreviation (VIM), which is among the most famous methods developed recently, where it was developed by He [11, 12, 13].

With the advent of fractional differential equations, the use of this method expanded and began to include the solution of this new type of equations [4, 17, 18, 20, 21, 23], and it was also used to solve another fractional equations which include, local fractional differential equations, local fractional partial differential equations and local fractional integro-differential equations [1, 26, 27, 28], or we find them benefit from the combined with some known transforms, such as: Laplace transform and Sumudu transform, in order to facilitate the solution of this type of equations, especially nonlinear ones. Among these works we find local fractional laplace variational iteration method [15, 29, 30] and local fractional Sumudu variational iteration method [31].

The main objective of this work was to apply the method proposed by us in article [31] to solve local fractional differential equations, where we will apply them here to solve nonlinear local fractional wave-like equations with variable coefficients, and these results were compared with the results of other research in the case of $\sigma = 1$.

This work consists of five basic sections, after the introduction comes the second section, which includes basic concepts and some properties related to the local fractional derivative and the local fractional Sumudu transform method. In the third section we presented the basics of this proposed method. In the fourth section we apply the modified method (LFSVIM) to solve nonlinear local fractional wave-like equations. In the last section, we finish our work by presenting the conclusion.

2. Preliminaries

In this section, we will introduce the basics of local fractional calculus, and we will focus on the following concepts: Local fractional derivative, local fractional integral, some important results and local fractional Sumudu transform.

2.1. Local fractional derivative.

Definition 2.1. The local fractional derivative of $\Phi(\chi)$ of order σ at $\chi = \chi_0$ is defined as ([24, 25])

$$\Phi^{(\sigma)}(\chi) = \frac{d^\sigma \Phi}{d\chi^\sigma}\bigg|_{\chi=\chi_0} = \lim_{\chi\to\chi_0} \frac{\Delta^\sigma(\Phi(\chi) - \Phi(\chi_0))}{(\chi-\chi_0)^\sigma}, \tag{2.1}$$

where

$$\Delta^\sigma(\Phi(\chi) - \Phi(\chi_0)) \cong \Gamma(1+\sigma)\left[(\Phi(\chi) - \Phi(\chi_0))\right]. \tag{2.2}$$

For any $\chi \in (\alpha, \beta)$, there exists

$$\Phi^{(\sigma)}(\chi) = D_\chi^\sigma \Phi(\chi),$$

denoted by

$\Phi(\chi) \in D_{\chi}^{\sigma}(\alpha, \beta)$.

Local fractional derivative of high order is

$$\Phi^{(m\sigma)}(\chi) = \overbrace{D_{\chi}^{(\sigma)} \cdots D_{\chi}^{(\sigma)} \Phi(\chi)}^{m \ times}. \tag{2.3}$$

2.2. **Local fractional integral.**

Definition 2.2. The local fractional integral of $\Phi(\chi)$ of order σ in the interval $[\alpha, \beta]$ is defined as ([24, 25])

$$\begin{aligned} {}_{\alpha}I_{\beta}^{(\sigma)}\Phi(\chi) &= \frac{1}{\Gamma(1+\sigma)} \int_{\alpha}^{\beta} \Phi(\tau)(d\tau)^{\sigma} \\ &= \frac{1}{\Gamma(1+\sigma)} \lim_{\Delta\tau \longrightarrow 0} \sum_{j=0}^{N-1} f(\tau_j)(\Delta\tau_j)^{\sigma}, \end{aligned} \tag{2.4}$$

where $\Delta\tau_j = \tau_{j+1} - \tau_j$, $\Delta\tau = \max\{\Delta\tau_0, \Delta\tau_1, \Delta\tau_2, \cdots\}$ and $[\tau_j, \tau_{j+1}]$, $\tau_0 = \alpha$, $\tau_N = \beta$, is a partition of the interval $[\alpha, \beta]$. For any $r \in (\alpha, \beta)$, there exists ${}_{\alpha}I_{\chi}^{(\sigma)}\Phi(\chi)$, denoted by $\Phi(\chi) \in I_{\chi}^{(\sigma)}(\alpha, \beta)$.

2.3. **Some important results.**

Definition 2.3. In fractal space, the Mittage Leffler function, Hyperbolic sine and hyperbolic cosine are defined as ([14, 24, 25])

$$E_{\sigma}(\chi^{\sigma}) = \sum_{m=0}^{+\infty} \frac{\chi^{m\sigma}}{\Gamma(1+m\sigma)}, \quad 0 < \sigma \leq 1, \tag{2.5}$$

$$E_{\sigma}(\chi^{\sigma})E_{\sigma}(\upsilon^{\sigma}) = E_{\sigma}(\chi+\upsilon)^{\sigma}, \quad 0 < \sigma \leq 1, \tag{2.6}$$

$$E_{\sigma}(\chi^{\sigma})E_{\sigma}(-\upsilon^{\sigma}) = E_{\sigma}(\chi-\upsilon)^{\sigma}, \quad 0 < \sigma \leq 1, \tag{2.7}$$

$$\sin_{\sigma}(\chi^{\sigma}) = \sum_{m=0}^{+\infty} (-1)^m \frac{\chi^{(2m+1)\sigma}}{\Gamma(1+(2m+1)\sigma)}, \quad 0 < \sigma \leq 1, \tag{2.8}$$

$$\cos_{\sigma}(\chi^{\sigma}) = \sum_{m=0}^{+\infty} (-1)^m \frac{\chi^{2m\sigma}}{\Gamma(1+2m\sigma)}, \quad 0 < \sigma \leq 1. \tag{2.9}$$

The non-differentiable functions within local fractional derivative and integral are given as follows ([24, 25])

$$\frac{d^{\sigma}\chi^{m\sigma}}{d\chi^{\sigma}} = \frac{\Gamma(1+m\sigma)\chi^{(m-1)\sigma}}{\Gamma(1+(m-1)\sigma)}. \tag{2.10}$$

$$\frac{d^{\sigma}}{d\chi^{\sigma}}E_{\sigma}(\chi^{\sigma}) = E_{\sigma}(\chi^{\sigma}). \tag{2.11}$$

$$\frac{d^{\sigma}}{d\chi^{\sigma}}\sin_{\sigma}(\chi^{\sigma}) = \cos_{\sigma}(\chi^{\sigma}). \tag{2.12}$$

$$\frac{d^{\sigma}}{d\chi^{\sigma}}\cos_{\sigma}(\chi^{\sigma}) = -\sin_{\sigma}(\chi^{\sigma}). \tag{2.13}$$

$${}_0I_{\chi}^{(\sigma)}\frac{\chi^{m\sigma}}{\Gamma(1+m\sigma)} = \frac{\chi^{(m+1)\sigma}}{\Gamma(1+(m+1)\sigma)}. \tag{2.14}$$

2.4. **Local fractional Sumudu transform.** We will introduce the definition of a local fractional Sumudu transform method and some its basic properties [19].

If there is a new transform operator ${}^{LF}S_{\sigma} : \Phi(\chi) \longrightarrow F(u)$, namely

$${}^{LF}S_{\sigma}\left\{\sum_{m=0}^{\infty} a_m\chi^{m\sigma}\right\} = \sum_{m=0}^{\infty}\Gamma(1+m\sigma)a_m u^{m\sigma}. \tag{2.15}$$

For examples

$${}^{LF}S_{\sigma}\left\{\frac{\chi^{\sigma}}{\Gamma(1+\sigma)}\right\} = u^{\sigma}. \tag{2.16}$$

Definition 2.4. [19] The local fractional Sumudu transform of $\Phi(\chi)$ of order σ is

$${}^{LF}S_{\sigma}\left\{\Phi(\chi)\right\} = F_{\sigma}(u) = \frac{1}{\Gamma(1+\sigma)}\int_0^{\infty} E_{\sigma}(-u^{-\sigma}\chi^{\sigma})\frac{\Phi(\chi)}{u^{\sigma}}(d\chi)^{\sigma},\ 0<\sigma\leq 1. \tag{2.17}$$

The inverse formula of (2.17) is

$${}^{LF}S_{\sigma}^{-1}\{\ F_{\sigma}(u)\} = \Phi(\chi)\ ,\quad 0<\sigma\leq 1\ . \tag{2.18}$$

Theorem 2.5. *If* ${}^{LF}S_{\sigma}\left\{\Phi(\chi)\right\} = F_{\sigma}(u)$ *and* ${}^{LF}S_{\sigma}\left\{\varphi(\chi)\right\} = \Psi_{\sigma}(u)$, *then one has*

$${}^{LF}S_{\sigma}\left\{\Phi(\chi)+\varphi(\chi)\right\} = F_{\sigma}(u)+\Psi_{\sigma}(u). \tag{2.19}$$

Proof. Using formula (2.17), we obtain:

$$\begin{aligned}
{}^{LF}S_\sigma\left\{\Phi(\chi)+\varphi(\chi)\right\} =& \frac{1}{\Gamma(1+\sigma)}\int_0^\infty E_\sigma(-u^{-\sigma}\chi^\sigma)\frac{\Phi(\chi)+\varphi(\chi)}{u^\sigma}(d\chi)^\sigma \\
=& \frac{1}{\Gamma(1+\sigma)}\int_0^\infty \left[E_\sigma(-u^{-\sigma}\chi^\sigma)\frac{\Phi(\chi)}{u^\sigma}+E_\sigma(-u^{-\sigma}\chi^\sigma)\frac{\varphi(\chi)}{u^\sigma}\right](d\chi)^\sigma \\
=& \frac{1}{\Gamma(1+\sigma)}\int_0^\infty E_\sigma(-u^{-\sigma}\chi^\sigma)\frac{\Phi(r)}{u^\sigma}(d\chi)^\sigma \\
&+\frac{1}{\Gamma(1+\sigma)}\int_0^\infty E_\sigma(-u^{-\sigma}\chi^\sigma)\frac{\varphi(\chi)}{u^\sigma}(d\chi)^\sigma \\
=& {}^{LF}S_\sigma\left\{\Phi(\chi)\right\}+{}^{LF}S_\sigma\left\{\varphi(\chi)\right\}.
\end{aligned}$$

This ends the proof. □

Theorem 2.6. *The Sumudu transform of local fractional derivative and integral is*

- *If* ${}^{LF}S_\sigma\left\{\Phi(\chi)\right\}=F_\sigma(u)$, *then one has*

$$ {}^{LF}S_\sigma\left\{\frac{d^\sigma\Phi(\chi)}{d\chi^\sigma}\right\}=\frac{F_\sigma(u)-F(0)}{u^\sigma}. \tag{2.20}$$

From (2.20), we have if ${}^{LF}S_\sigma\left\{\Phi(r)\right\}=F_\sigma(u)$, *so*

$$ {}^{LF}S_\sigma\left\{\frac{d^{n\sigma}\Phi(\chi)}{d\chi^{n\sigma}}\right\}=\frac{1}{u^{n\sigma}}\left[F_\sigma(u)-\sum_{k=0}^{n-1}u^{k\sigma}\Phi^{(k\sigma)}(0)\right]. \tag{2.21}$$

When $n=2$, *from (2.21), we get*

$$ {}^{LF}S_\sigma\left\{\frac{d^{2\sigma}\Phi(\chi)}{d\chi^{2\sigma}}\right\}=\frac{1}{u^{2\sigma}}\left[F_\sigma(u)-\Phi(0)-u^\sigma\Phi^{(\sigma)}(0)\right]. \tag{2.22}$$

- *If* ${}^{LF}S_\sigma\left\{\Phi(\chi)\right\}=F_\sigma(u)$, *then we have*

$$ {}^{LF}S_\sigma\left\{{}_0I_\chi^{(\sigma)}\Phi(\chi)\right\}=u^\sigma F_\sigma(u). \tag{2.23}$$

Proof. see [19] □

3. Analytical of Sumudu variational iteration method in the case local fractional derivative

We consider a general nonlinear local fractional differential equation

$$ L_\alpha V(\chi,\tau)+N_\alpha V(\chi,\tau)+R_\alpha V(\chi,\tau)=h_\alpha(\chi,\tau), \tag{3.1}$$

with $L_\alpha=\frac{\partial^{2\alpha}}{\partial\tau^{2\alpha}}$ denotes linear local fractional derivative operator of order 2α, R_α represent linear operator of order less than L_α, N_α denotes nonlinear operator and $h_\alpha(\chi,\tau)$ is the non-differentiable source term.

We apply the local fractional Sumudu transform on both sides of (3.1), we obtain

$$S_\alpha\left[L_\alpha V(\chi,\tau)\right]+S_\alpha\left[N_\alpha V(\chi,\tau)+R_\alpha V(\chi,\tau)\right]=S_\alpha\left[h_\alpha(\chi,\tau)\right]. \tag{3.2}$$

By use the property of the local fractional Sumudu transform, we get

$$\begin{aligned} S_\alpha\left[V(\chi,\tau)\right] &= V(\chi,0)+\frac{\partial^\alpha V(\chi,0)}{\partial\tau^\alpha}u^\alpha+u^{2\alpha}S_\alpha\left[h_\alpha(\chi,\tau)\right] \\ &\quad -u^{2\alpha}S_\alpha\left[N_\alpha V(\chi,\tau)+R_\alpha V(\chi,\tau)\right]. \end{aligned} \tag{3.3}$$

Now, applying the inverse transform on (3.3)

$$V=V(\chi,0)+\frac{\partial^\alpha V(\chi,0)}{\partial t^\alpha}\frac{\tau^\alpha}{\Gamma(1+\alpha)}+S_\alpha^{-1}\left(u^{2\alpha}S_\alpha\left[h_\alpha(\chi,\tau)-N_\alpha V-R_\alpha V\right]\right). \tag{3.4}$$

Applying $\frac{\partial^\alpha}{\partial t^\alpha}$ on both sides of (3.4), we find

$$\frac{\partial^\alpha V}{\partial\tau^\alpha}+\frac{\partial^\alpha}{\partial\tau^\alpha}S_\alpha^{-1}\left(u^{2\alpha}S_\alpha\left[N_\alpha V+R_\alpha V-h_\alpha(\chi,\tau)\right]\right)-\frac{\partial^\alpha V(\chi,0)}{\partial\tau^\alpha}=0. \tag{3.5}$$

The application of the main step of the VIM method (correction functional) [22], gives the following fundamental formula

$$V_{n+1}= V_n-\ {}_0I_\tau^{(\alpha)}\left[\begin{array}{c}\frac{\partial^\alpha V_n}{\partial\tau^\alpha}-\frac{\partial^\alpha V(\chi,0)}{\partial\tau^\alpha}\\ +\frac{\partial^\alpha}{\partial\tau^\alpha}S_\alpha^{-1}\left(u^{2\alpha}S_\alpha\left[N_\alpha V_n+R_\alpha V_n-h_\alpha(\chi,\tau)\right]\right)\end{array}\right]. \tag{3.6}$$

The approximate solution is calculated by the following limit

$$V(\chi,\tau)=\lim_{n\to\infty}V_n(\chi,\tau). \tag{3.7}$$

4. Applications of this method

In this part, we apply local fractional Sumudu variational iteration method [31] for solving some models of nonlinear local fractional wave-like equations with variable coefficients.

Example 4.1. We consider the following nonlinear local fractional wave-like equation, which takes the following form

$$V_{tt}^{(2\alpha)}=x^{2\alpha}\frac{\partial}{\partial x^\alpha}\left(V_x^{(\alpha)}V_{xx}^{(2\alpha)}\right)-x^{2\alpha}\left(V_{xx}^{(2\alpha)}\right)^2-V,\ 0<\alpha\le 1, \tag{4.1}$$

subject to the initial conditions

$$V(x,0)=0,\ V_t^{(\alpha)}(x,y,0)=x^{2\alpha}. \tag{4.2}$$

According to (3.6) and (4.1), we obtain the formula of successive approximations which given by

$$V_{n+1} = V_n - {}_0I_t^{(\alpha)}\left[\frac{\partial^\alpha V_n}{\partial\tau^\alpha} - \frac{\partial^\alpha V(x,0)}{\partial\tau^\alpha} + \frac{\partial^\alpha}{\partial\tau^\alpha}S_\alpha^{-1}\left(u^{2\alpha}S_\alpha\left[- x^{2\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(V_{nx}^{(\alpha)}V_{nxx}^{(2\alpha)}\right) + x^{2\alpha}\left(V_{nxx}^{(2\alpha)}\right)^2 + V_n\right]\right)\right]. \tag{4.3}$$

Using the formula (4.3), we obtain the following successive approximations:

$$V_0(x,t) = x^{2\alpha}\frac{t^\alpha}{\Gamma(1+\alpha)},$$

$$V_1 = V_0 - {}_0I_t^{(\alpha)}\left[\frac{\partial^\alpha V_0}{\partial\tau^\alpha} - \frac{\partial^\alpha V(x,0)}{\partial\tau^\alpha} + \frac{\partial^\alpha}{\partial\tau^\alpha}S_\alpha^{-1}\left(u^{2\alpha}S_\alpha\left[- x^{2\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(V_{0x}^{(\alpha)}V_{0xx}^{(2\alpha)}\right) + x^{2\alpha}\left(V_{0xx}^{(2\alpha)}\right)^2 + V_0\right]\right)\right]$$

$$V_2 = V_1 - {}_0I_t^{(\alpha)}\left[\frac{\partial^\alpha V_1}{\partial\tau^\alpha} - \frac{\partial^\alpha V(x,0)}{\partial\tau^\alpha} + \frac{\partial^\alpha}{\partial\tau^\alpha}S_\alpha^{-1}\left(u^{2\alpha}S_\alpha\left[- x^{2\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(V_{1x}^{(\alpha)}V_{1xx}^{(2\alpha)}\right) + x^{2\alpha}\left(V_{1xx}^{(2\alpha)}\right)^2 + V_1\right]\right)\right],$$

$$V_3 = V_2 - {}_0I_t^{(\alpha)}\left[\frac{\partial^\alpha V_2}{\partial\tau^\alpha} - \frac{\partial^\alpha V(x,0)}{\partial\tau^\alpha} + \frac{\partial^\alpha}{\partial\tau^\alpha}S_\alpha^{-1}\left(u^{2\alpha}S_\alpha\left[- x^{2\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\left(V_{2x}^{(\alpha)}V_{2xx}^{(2\alpha)}\right) + x^{2\alpha}\left(V_{2xx}^{(2\alpha)}\right)^2 + V_2\right]\right)\right], \tag{4.4}$$

$$\vdots$$

and so on.

Depending on the relations (4.5), we get the first terms of this method that take the following form

$$
\begin{aligned}
V_0(x,t) &= x^{2\alpha}\frac{t^{\alpha}}{\Gamma(1+\alpha)}, \\
V_1(x,t) &= x^{2\alpha}\left(\frac{t^{\alpha}}{\Gamma(1+\alpha)}-\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right), \\
V_2(x,t) &= x^{2\alpha}\left(\frac{t^{\alpha}}{\Gamma(1+\alpha)}-\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{t^{5\alpha}}{\Gamma(1+5\alpha)}\right), \\
V_3(x,t) &= x^{2\alpha}\left(\frac{t^{\alpha}}{\Gamma(1+\alpha)}-\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{t^{5\alpha}}{\Gamma(1+5\alpha)}-\frac{t^{7\alpha}}{\Gamma(1+7\alpha)}\right), \\
&\vdots \\
V_n(x,t) &= x^{2\alpha}\left(\frac{t^{\alpha}}{\Gamma(1+\alpha)}-\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{t^{5\alpha}}{\Gamma(1+5\alpha)}-\frac{t^{7\alpha}}{\Gamma(1+7\alpha)}+\cdots\right. \\
&\quad \left.+(-1)^n\frac{t^{(2n+1)\alpha}}{\Gamma(1+(2n+1)\alpha)}\right).
\end{aligned} \tag{4.5}
$$

Then the non-differentiable solution of (4.1), is calculated by

$$
\begin{aligned}
V(x,t) &= \lim_{n\to\infty} V_n(x,t) \\
&= x^{2\alpha}\lim_{n\to\infty}\left(\begin{array}{c}\frac{t^{\alpha}}{\Gamma(1+\alpha)}-\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{t^{5\alpha}}{\Gamma(1+5\alpha)}-\frac{t^{7\alpha}}{\Gamma(1+7\alpha)} \\ +\cdots+(-1)^n\frac{t^{(2n+1)\alpha}}{\Gamma(1+(2n+1)\alpha)}\end{array}\right).
\end{aligned} \tag{4.6}
$$

This leads to the following result

$$
V(x,t) = x^{2\alpha}\sin_{\alpha}(t^{\alpha}). \tag{4.7}
$$

Note that, our solution (4.7) satisfies the initial conditions (4.2), and in the case $\sigma = 1$, we obtain the same solution obtained in [5] by Adomian Decomposition Method and in [6] by homotopy perturbation method.

Example 4.2. Now, we consider a model of two dimensional nonlinear local fractional wave-like equation which defined by

$$
V_{tt}^{(2\alpha)} = \frac{\partial^{2\alpha}}{\partial x^{\alpha}y^{\alpha}}\left(V_{xx}^{(2\alpha)}V_{yy}^{(2\alpha)}\right) - \frac{\partial^{2\alpha}}{\partial x^{\alpha}y^{\alpha}}\left(x^{\alpha}y^{\alpha}V_x^{(\alpha)}V_y^{(\alpha)}\right) - V,\ 0<\alpha\le 1, \tag{4.8}
$$

or otherwise

$$
V_{tt}^{(2\alpha)} = \left(V_{xx}^{(2\alpha)}V_{yy}^{(2\alpha)} - x^{\alpha}y^{\alpha}V_x^{(\alpha)}V_y^{(\alpha)}\right)_{xy}^{(2\alpha)} - V,\ 0<\alpha\le 1, \tag{4.9}
$$

subject to the initial conditions

$$V(x,y,0)=E_\alpha(x^\alpha y^\alpha),\ V_t^{(\alpha)}(x,y,0)=E_\alpha(x^\alpha y^\alpha). \tag{4.10}$$

By using (3.6) and (4.8), we obtain the successive approximations

$$\begin{aligned}V_{n+1}= \ & V_n-\ {}_0I_t^{(\alpha)}\Bigg[\frac{\partial^\alpha V_n}{\partial\tau^\alpha}-\frac{\partial^\alpha V(x,y,0)}{\partial\tau^\alpha}+\frac{\partial^\alpha}{\partial\tau^\alpha}S_\alpha^{-1}\Bigg(u^{2\alpha}S_\alpha\Bigg[\\ &\Big(-V_{nxx}^{(2\alpha)}V_{nyy}^{(2\alpha)}+x^\alpha y^\alpha V_{nx}^{(\alpha)}V_{ny}^{(\alpha)}\Big)_{xy}^{(2\alpha)}+V_n\Bigg]\Bigg)\Bigg]\end{aligned} \tag{4.11}$$

According to (4.10) and (4.11), we find

$$V_0(x,y,t)=E_\alpha(x^\alpha y^\alpha)+E_\alpha(x^\alpha y^\alpha)\frac{t^\alpha}{\Gamma(1+\alpha)},$$

$$\begin{aligned}V_1&=V_0-\ {}_0I_t^{(\alpha)}\Bigg[\frac{\partial^\alpha V_0}{\partial\tau^\alpha}-\frac{\partial^\alpha V(x,y,0)}{\partial\tau^\alpha}+\frac{\partial^\alpha}{\partial\tau^\alpha}S_\alpha^{-1}\Bigg(u^{2\alpha}S_\alpha\Bigg[\\ &\Big(-V_{0xx}^{(2\alpha)}V_{0yy}^{(2\alpha)}+x^\alpha y^\alpha V_{0x}^{(\alpha)}V_{0y}^{(\alpha)}\Big)_{xy}^{(2\alpha)}+V_0\Bigg]\Bigg)\Bigg],\\ V_2&=V_1-\ {}_0I_t^{(\alpha)}\Bigg[\frac{\partial^\alpha V_1}{\partial\tau^\alpha}-\frac{\partial^\alpha V(x,y,0)}{\partial\tau^\alpha}+\frac{\partial^\alpha}{\partial\tau^\alpha}S_\alpha^{-1}\Bigg(u^{2\alpha}S_\alpha\Bigg[\\ &\Big(-V_{1xx}^{(2\alpha)}V_{1yy}^{(2\alpha)}+x^\alpha y^\alpha V_{1x}^{(\alpha)}V_{1y}^{(\alpha)}\Big)_{xy}^{(2\alpha)}+V_1\Bigg]\Bigg)\Bigg],\\ V_3&=V_2-\ {}_0I_t^{(\alpha)}\Bigg[\frac{\partial^\alpha V_2}{\partial\tau^\alpha}-\frac{\partial^\alpha V(x,y,0)}{\partial\tau^\alpha}+\frac{\partial^\alpha}{\partial\tau^\alpha}S_\alpha^{-1}\Bigg(u^{2\alpha}S_\alpha\Bigg[\\ &\Big(-V_{2xx}^{(2\alpha)}V_{2yy}^{(2\alpha)}+x^\alpha y^\alpha V_{2x}^{(\alpha)}V_{2y}^{(\alpha)}\Big)_{xy}^{(2\alpha)}+V_2\Bigg]\Bigg)\Bigg],\\ &\vdots\end{aligned} \tag{4.12}$$

So, the first terms of solution is given by

$$\begin{aligned}
V_0(x,y,t) &= E_\alpha(x^\alpha y^\alpha) + E_\alpha(x^\alpha y^\alpha)\tfrac{t^\alpha}{\Gamma(1+\alpha)},\\
V_1(x,y,t) &= E_\alpha(x^\alpha y^\alpha)\left(1+\tfrac{t^\alpha}{\Gamma(1+\alpha)}-\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)}-\tfrac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right),\\
V_2(x,y,t) &= E_\alpha(x^\alpha y^\alpha)\left(1+\tfrac{t^\alpha}{\Gamma(1+\alpha)}-\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)}-\tfrac{t^{3\alpha}}{\Gamma(1+3\alpha)}+\tfrac{t^{4\alpha}}{\Gamma(1+4\alpha)}\right.\\
&\qquad\left.+\tfrac{t^{5\alpha}}{\Gamma(1+5\alpha)}\right),\\
V_3(x,y,t) &= E_\alpha(x^\alpha y^\alpha)\left(1+\tfrac{t^\alpha}{\Gamma(1+\alpha)}-\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)}-\tfrac{t^{3\alpha}}{\Gamma(1+3\alpha)}+\tfrac{t^{4\alpha}}{\Gamma(1+4\alpha)}\right.\\
&\qquad\left.+\tfrac{t^{5\alpha}}{\Gamma(1+5\alpha)}-\tfrac{t^{6\alpha}}{\Gamma(1+6\alpha)}-\tfrac{t^{7\alpha}}{\Gamma(1+7\alpha)}\right),\\
&\ \vdots\\
V_n(x,y,t) &= E_\alpha(x^\alpha y^\alpha)\left[\tfrac{t^\alpha}{\Gamma(1+\alpha)}-\tfrac{t^{3\alpha}}{\Gamma(1+3\alpha)}+\tfrac{t^{5\alpha}}{\Gamma(1+5\alpha)}-\cdots+\tfrac{(-1)^n t^{(2n+1)\alpha}}{\Gamma(1+(2n+1)\alpha)}\right]\\
&\quad+E_\alpha(x^\alpha y^\alpha)\left[1-\tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)}+\tfrac{t^{4\alpha}}{\Gamma(1+4\alpha)}-\tfrac{t^{6\alpha}}{\Gamma(1+6\alpha)}+\cdots+\tfrac{(-1)^n t^{2n\alpha}}{\Gamma(1+2n\alpha)}\right].
\end{aligned} \tag{4.13}$$

As the non-differentiable solution is calculate by

$$V(x,y,t) = \lim_{n\to\infty} V_n(x,y,t), \tag{4.14}$$

which

$$\begin{aligned}
V(x,y,t) = \lim_{n\longrightarrow\infty}&\left(E_\alpha(x^\alpha y^\alpha)\left[\frac{t^\alpha}{\Gamma(1+\alpha)}-\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{t^{5\alpha}}{\Gamma(1+5\alpha)}\right.\right.\\
&\left.-\cdots+\frac{(-1)^n t^{(2n+1)\alpha}}{\Gamma(1+(2n+1)\alpha)}\right]+\left[1-\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}\right.\\
&\left.\left.-\frac{t^{6\alpha}}{\Gamma(1+6\alpha)}+\cdots+\frac{(-1)^n t^{2n\alpha}}{\Gamma(1+2n\alpha)}\right]\right).
\end{aligned} \tag{4.15}$$

This leads to the following result

$$V(x,y,t) = E_\alpha(x^\alpha y^\alpha)\left(\sin_\alpha(t^\alpha)+\cos_\alpha(t^\alpha)\right). \tag{4.16}$$

Note that, in the case $\sigma = 1$, we obtain the same solution obtained in [5] by Adomian Decomposition Method and in [6] by homotopy perturbation method.

Example 4.3. Finally, we consider this model of nonlinear local fractional wave-like equation

$$V_{tt}^{(2\alpha)} = V^2\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\left(V_x^{(\alpha)}V_{xx}^{(2\alpha)}V_{xxx}^{(3\alpha)}\right)+\left(V_x^{(\alpha)}\right)^2\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\left(V_x^{(\alpha)}\right)^3-18V^5+V,\ \ 0<\alpha\leq 1, \tag{4.17}$$

with the initial conditions

$$V(x,0)=E_\alpha(x^\alpha),\ V_t^{(\alpha)}(x,0)=E_\alpha(x^\alpha). \tag{4.18}$$

By using (3.6) and (4.17), we get the following formula

$$\begin{aligned} V_{n+1} = \ & V_n - {}_0I_t^{(\alpha)}\left[\frac{\partial^\alpha V_n}{\partial\tau^\alpha} - \frac{\partial^\alpha V(x,0)}{\partial\tau^\alpha}\right. \\ & - \frac{\partial^\alpha}{\partial\tau^\alpha}S_\alpha^{-1}\left(u^{2\alpha}S_\alpha\left[V_n^2\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\left(V_{nx}^{(\alpha)}V_{nxx}^{(2\alpha)}V_{nxxx}^{(3\alpha)}\right)\right]\right) \\ & \left. - \frac{\partial^\alpha}{\partial\tau^\alpha}S_\alpha^{-1}\left(u^{2\alpha}S_\alpha\left[\left(V_{nx}^{(\alpha)}\right)^2\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\left(V_{nx}^{(\alpha)}\right)^3 - 18V_n^5 + V_n\right]\right)\right] \end{aligned} \tag{4.19}$$

And according to the formula (4.19), we obtain the following first terms

$$V_0(x,t) = E_\alpha(x^\alpha) + E_\alpha(x^\alpha)\frac{t^\alpha}{\Gamma(1+\alpha)},$$

$$\begin{aligned} V_1 = V_0 - {}_0I_t^{(\alpha)}&\left[\frac{\partial^\alpha V_0}{\partial\tau^\alpha} - \frac{\partial^\alpha V(x,0)}{\partial\tau^\alpha}\right. \\ & - \frac{\partial^\alpha}{\partial\tau^\alpha}S_\alpha^{-1}\left(u^{2\alpha}S_\alpha\left[V_0^2\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\left(V_{0x}^{(\alpha)}V_{0xx}^{(2\alpha)}V_{0xxx}^{(3\alpha)}\right)\right]\right) \\ & \left. - \frac{\partial^\alpha}{\partial\tau^\alpha}S_\alpha^{-1}\left(u^{2\alpha}S_\alpha\left[\left(V_{0x}^{(\alpha)}\right)^2\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\left(V_{0x}^{(\alpha)}\right)^3 - 18V_0^5 + V_0\right]\right)\right], \end{aligned}$$

$$\begin{aligned} V_2 = V_1 - {}_0I_t^{(\alpha)}&\left[\frac{\partial^\alpha V_1}{\partial\tau^\alpha} - \frac{\partial^\alpha V(x,0)}{\partial\tau^\alpha}\right. \\ & - \frac{\partial^\alpha}{\partial\tau^\alpha}S_\alpha^{-1}\left(u^{2\alpha}S_\alpha\left[V_1^2\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\left(V_{1x}^{(\alpha)}V_{1xx}^{(2\alpha)}V_{1xxx}^{(3\alpha)}\right)\right]\right) \\ & \left. - \frac{\partial^\alpha}{\partial\tau^\alpha}S_\alpha^{-1}\left(u^{2\alpha}S_\alpha\left[\left(V_{1x}^{(\alpha)}\right)^2\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\left(V_{1x}^{(\alpha)}\right)^3 - 18V_1^5 + V_1\right]\right)\right] \end{aligned}$$

$$
\begin{aligned}
V_3 = V_2 - {}_0I_t^{(\alpha)} \Bigg[& \frac{\partial^\alpha V_2}{\partial \tau^\alpha} - \frac{\partial^\alpha V(x,0)}{\partial \tau^\alpha} \\
& - \frac{\partial^\alpha}{\partial \tau^\alpha} S_\alpha^{-1} \left(u^{2\alpha} S_\alpha \left[V_2^2 \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \left(V_{2x}^{(\alpha)} V_{2xx}^{(2\alpha)} V_{2xxx}^{(3\alpha)} \right) \right] \right) \\
& - \frac{\partial^\alpha}{\partial \tau^\alpha} S_\alpha^{-1} \left(u^{2\alpha} S_\alpha \left[\left(V_{2x}^{(\alpha)} \right)^2 \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \left(V_{2x}^{(\alpha)} \right)^3 - 18 V_2^5 + V_2 \right] \right) \Bigg] \\
& \vdots
\end{aligned}
\tag{4.20}
$$

Using the previous formulas (4.20), we get the first terms of this method as follows

$$
\begin{gathered}
V_0(x,t) = E_\alpha(x^\alpha) \left(1 + \tfrac{t^\alpha}{\Gamma(1+\alpha)} \right), \\
V_1(x,t) = E_\alpha(x^\alpha) \left(1 + \tfrac{t^\alpha}{\Gamma(1+\alpha)} + \tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \tfrac{t^{3\alpha}}{\Gamma(1+3\alpha)} \right), \\
V_2(x,t) = E_\alpha(x^\alpha) \left(1 + \tfrac{t^\alpha}{\Gamma(1+\alpha)} + \tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \tfrac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \tfrac{t^{4\alpha}}{\Gamma(1+4\alpha)} + \tfrac{t^{5\alpha}}{\Gamma(1+5\alpha)} \right), \\
\vdots \\
\vdots \\
V_n(x,t) = E_\alpha(x^\alpha) \left(1 + \tfrac{t^\alpha}{\Gamma(1+\alpha)} + \tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \tfrac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \tfrac{t^{4\alpha}}{\Gamma(1+4\alpha)} + \tfrac{t^{5\alpha}}{\Gamma(1+5\alpha)} \right. \\
\left. + \cdots + \tfrac{t^{n\alpha}}{\Gamma(1+n\alpha)} \right).
\end{gathered}
\tag{4.21}
$$

And therefore, then the non-differentiable solution of (4.17) is calculated by

$$
V(x,t) = E_\alpha(x^\alpha) \lim_{n \longrightarrow \infty} \sum_{k=0}^{n} \frac{t^{k\alpha}}{\Gamma(1+k\alpha)}, \tag{4.22}
$$

we know that this limit is

$$
V(x,t) = E_\alpha(x^\alpha) E_\alpha(t^\alpha). \tag{4.23}
$$

Which represent the exact solution of equation (4.17).

Depending on the results presented in [14], we can write the solution (4.23) as follows

$$
V(x,t) = E_\alpha(x^\alpha + t^\alpha). \tag{4.24}
$$

Note that, in the case $\sigma = 1$, we obtain the same solution obtained in [5] by Adomian Decomposition Method and in [6] by homotopy perturbation method.

5. Conclusion

In this work, we applied the local fractional Sumudu variational iteration method (LFSVIM) to solve some nonlinear wave-like equations with variable coefficients and within local fractional derivative. Through this work done, we can say that this algorithm is easy to use during the solution and provides us with a solution in the form of a series that converges rapidly towards the exact solution if it exists, as shown from examples that we have solved it though it looks very complicated. From the results obtained, it can be concluded that this algorithm is powerful and effective in applying to this type of equations, and thus can be applied to other nonlinear local fractional partial differential equations with fixed or variable coefficients.

References

[1] D. Baleanu, J. A. T. Machado, C. Cattani, M. C. Baleanu, and X. J. Yang, *Local Fractional Variational Iteration and Decomposition Methods for Wave Equation on Cantor Sets within Local Fractional Operators,* Abst. Appl. Anal., A. ID: 535048, *2014* (2014), 6 pp.

[2] A. Bouhassoun and M. Hamdi Cherif, *Homotopy Perturbation Method For Solving The Fractional Cahn-Hilliard Equation*, J. Interdiscip. Math., *18*(5) (2015), 513-524.

[3] H. Bulut, H. M. Baskonus, and F. B. Belgacem, *The Analytical Solutions of Some Fractional Ordinary Differential Equations By Sumudu Transform Method*, Abs. Appl. Anal., A. ID: 203875, *2013* (2013), 6pp.

[4] C. Chun and S. Abbasbandy, *New Application of Variational Iteration Method for Analytic Treatment of Nonlinear Partial Differential Equations*, World Appl. Sci. J., *16*(12) (2012), 1677-1681.

[5] M. Ghoreishi, *Adomian Decomposition Method (ADM) for Nonlinear Wave-like Equations with Variable Coefficient,* Appl. Math. Sci., *4*(49) (2010), 2431-2444.

[6] S. Gupta, D. Kumar, and J. Singh, *Application of He's homotopy perturbation method for solving nonlinear wave-like equations with variable coefficients*, Int. J. Adv. Appl. Math. Mech., *1*(2) (2013), 65-79.

[7] M. Hamdi Cherif, K. Belghaba, and D. Ziane, *Homotopy Perturbation Method for Solving the Fractional Fisher's Equation*, Int. J. Anal. appl., *10*(1) (2016), 9-16.

[8] M. Hamdi Cherif and D. Ziane, *A new technique for solving systems of nonlinear fractional partial differential equations*, Int. J. Anal. appl., *15*(2) (2017), 188-197.

[9] M. Hamdi Cherif and D. Ziane, *Variational iteration method combined with new transform to solve fractional partial differential equations*, Univ. J. Math. Appl., *1*(2) (2018), 113-120.

[10] S. H. M. Hamed, E. A. Yousif, and A. I. Arbab, *Analytic and Approximate Solutions of the Space-Time Fractional Schrdinger Equations by Homotopy Perturbation Sumudu Transform Method*, Abs. Appl. Anal, A. ID: 863015, *2014* (2014), 13pp.

[11] J. H. He, *A new approach to nonlinear partial differential equations*, Comm. Nonlinear Sci. Numer. Simul., *2* (1997), 230-235.

[12] J. H. He, *A variational iteration approach to nonlinear problems and its applications*, Mech. Appl., *20*(1) (1998), 30-31.

[13] J. H. He, *Variational iteration method for autonomous ordinary differential systems*, Appl. Math. Comput., *114*(2/3) (2000), 115-123.

[14] M. S. Hu, R. P. Agarwal, and X. J. Yang, *Local Fractional Fourier Series with Application to Wave Equation in Fractal Vibrating String*, Abs. Appl. Anal, A. ID: 567401, *2012* (2012), 15pp.

[15] H. K. Jassim, *Local fractional variational iteration transform method to solve partial differential equations arising in mathematical physics*, Int. J. Adv. Appl. Math. Mech., *3*(1) (2015), 71-76.

[16] N. A. Khan, O. Abdul Razzaq, and M. Ayyaz, *On the solution of fuzzy differential equations by Fuzzy Sumudu Transform*, Nonlinear Engineering., *4*(1) (2015), 49–60.

[17] R. Y. Molliq and M. S. M. Noorani, *Solving the Fractional Rosenau-Hyman Equation via Variational Iteration Method and Homotopy Perturbation Method*, Int. J. Diff. Equ., 2012, A. ID *472030* (2012), 14 pp.

[18] A. S. V. Ravi Kanth and K. Aruna, *He's variational iteration method for treating nonlinear singular boundary value problems*, Comp. Math. Appl., *60* (2010), 821-829.

[19] H. M. Srivastava, A.K. Golmankhaneh, D. Baleanu, and X.J. Yang, *Local Fractional Sumudu Transform with Application to IVPs on Cantor Sets*, Abs. Appl. Anal., A. ID: 176395, *2014* (2014), 7pp.

[20] V. Turut and N. Güze, *On solving Partial Differential Equations of Fractional Order by Using the Variational Iteration Method and Multivariate Padé Approximations*, Europ. J. Pure. Appl. Math., *6*(2) (2013), 147-171.

[21] A. Wazwaz, *The variational iteration method for rational solutions for KdV, K(2,2), Burgers, and cubic Boussinesq equations,* J. Comp. Appl. Math., *207* (2007), 18-23.

[22] G. C. Wu, *Variational iteration method for solving the time-fractional diffusion equations in porous medium*, Chin. Phys B., *21*(12) (2012), 1-5.

[23] G. C. Wu and E. W. M. Lee, *Fractional variational iteration method and its application*, Physics Letters A., *374* (2010), 2506-2509.

[24] X. J. Yang, *Fractional Functional Analysis and Its Applications*, Asian Academic, Hong Kong, 2011.

[25] X. J. Yang, *Local Fractional Calculus and Its Applications*, World Science Publisher, New York, USA, 2012.

[26] X. J. Yang and D. Baleanu, *Fractal Heat Conduction Problem Solved by Local Fractional Variational Iteration Method*, Thermal Science., *17*(2) (2013), 625-628.

[27] X. J. Yang, D. Baleanu, Y. Khan, and S. T. Mohyud-din, *Local Fractional Variational Iteration Method for Diffusion and Wave Equation on Cantor Sets,* Rom. J. Phys., *59*(1-2) (2014), 36-48.

[28] Y. J. Yang, D. Baleanu, and X. J. Yang, *A Local Fractional Variational Iteration Method for Laplace Equation within Local Fractional Operators,* Abs. Appl. Anal., A. ID: 202650, *2013* (2013), 6 pp.

[29] Y. J. Yang and L. Q. Hua, *Variational Iteration Transform Method for Fractional Differential Equations with Local Fractional Derivative,* Abs. Appl. Anal., A. ID: 760957, *2012* (2014), 9 pp.

[30] A. M. Yang, J. Li, H. M. Srivastava, G. N. Xie, and X. J. Yang, *Local Fractional Laplace Variational Iteration Method for Solving Linear Partial Differential Equations with Local Fractional Derivative*, Dis. Dyn. Nat. Soc., A. ID:365981, *2014* (2014), 8 pp.

[31] D. Ziane, *Local Fractional Sumudu Variational Iteration Method for Solving Partial Differential Equations with Local Fractional Derivative*, Int. J. Open Prob. Compt. Math., *10*(3) (2017), 29-42.

[32] D. Ziane, K. Belghaba, and M. Hamdi Cherif, *Fractional homotopy perturbation transform method for solving the time-fractional KdV, K(2,2) and Burgers equations*, Int. J. Open Prob. Compt. Math., *8*(2) (2015), 63-75.

[33] D. Ziane, M. Hamdi Cherif, D. Baleanu, K. Belghaba, and M. Al Qurashi, *An Efficient Algorithm for Solving Nonlinear Systems of Partial Differential Equations with Local Fractional Operators*, Punjab Univ. J. Math., *51*(9) (2019), 85-99.

[34] D. Ziane, T. Mohyeldin Elzaki, and M. Hamdi Cherif, *Elzaki transform combined with variational iteration method for partial differential equations of fractional order*, Fund. J. Math. Appl., *1*(1) (2018), 102-108.