Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 9, No. 3, 2021, pp. 830-845
DOI:10.22034/cmde.2020.31729.1483

The numerical approximation for the solution of linear and nonlinear integral equations of the second kind by interpolating moving least squares

Morteza Asgari
Faculty of Mathematical Sciences,
Shahrood University of Technology, Shahrood, Iran.
E-mail: morteza.asgari@shahroodut.ac.ir

Ali Mesforush
Faculty of Mathematical Sciences,
Shahrood University of Technology, Shahrood, Iran.
E-mail: ali.mesforush@shahroodut.ac.ir

Alireza Nazemi*
Faculty of Mathematical Sciences,
Shahrood University of Technology, Shahrood, Iran.
E-mail: nazemi20042003@gmail.com

Abstract In this paper, the interpolating moving least-squares (IMLS) method is discussed. The interpolating moving least square methodology is an effective technique for the approximation of an unknown function by using a set of disordered data. Then we apply the IMLS method for numerical solution of Volterra–Fredholm integral equations, and finally some examples are given to show the accuracy and applicability of the method.

Keywords. Moving least-squares method, Volterra-Fredholm integro-differential equations, Error analysis.

2010 Mathematics Subject Classification. 65L05, 34K06, 34K28.

1. Introduction

The theory and application of integral equations is an important subject within applied mathematics. Integral equations of various types appear in many fields of science and engineering. Differential equations with transformed arguments or neural type differential equations can be transformed to VolterraFredholm integral equations [1]. There are several techniques for proximating the solution of such problems, but very few references have been found in technical literature dealing with VolterraFredholm integral equations. Yalcinbas [25] applied Taylor series and Ghasemi et al.[12] used the homopoty perturbation method for solving some of these equations. In [26] I think he meant his variational iteration method is employed to solve nonlinear VolterraFredholm integral equations. In [5] Numerical solution of VolterraFredholm integral equations by moving least square method and Chebyshev polynomials are

Received: 25 January 2019 ; Accepted: 15 May 2020.
∗ corresponding.

used.Triangular functions (TF) method for the solution of nonlinear VolterraFredholm integral equations [8]. Lancaster and Salkauskas [6] presented the interpolating moving least-squares (IMLS) method by using a singular weight function. The shape function of the IMLS method satisfies the property of the Kronecker δ function, and then the meshless method based on the IMLS method can apply the essential boundary condition directly without any additional numerical effort. Kaljevic and Saigal presented an improved formulation of the EFG method based on the IMLS method [3]. Maisuradze et al. applied the IMLS method to fit potential energy surfaces in one dimensional chemical application [7]. Netuzhylov used the meshfree collocation method based on the IMLS method to solve boundary value problems [13]. By revising the formulae of the IMLS method, Ren et al. presented the interpolating boundary element-free (IBEF) method [14, 15] and the interpolating element-free Galerkin (IEFG) method [16, 17] for two-dimensional potential and elasticity problems. Based on the IMLS method and the CVMLS approximation, Ren et al. also presented the complex variable interpolating moving least-squares (CVIMLS) method [18, 19]. To overcome the singularity of the weight function in the IMLS method, Wang et al. presented the improved interpolating moving least-squares (IIMLS) method with the nonsingular weight function [20, 21, 22, 23]. In order to study the mathematical theory of meshless methods based on the IMLS method, it is certainly also important to analyze error estimates of the approximation function of the IMLS method and its derivatives. In the MLS approximation, the weight function is bounded. But in the IMLS method, the weight function is singular. Thus the error estimates of the approximation function of the IMLS method and its derivatives are more difficult than the ones of the MLS approximation. The advantage of the IMLS method is that the meshless method which is constructed based on the IMLS method can apply the essential boundary conditions directly and easily. The error estimate of the approximation function of the IMLS method was presented in a one-dimensional case, and the error estimates of the first and second order derivatives are also given [6].

The rest of this paper is organized as follows: the outline of the IMLS method is discussed in section 2. In section 3, the proposed method is employed on nonlinear VolterraFredholm integral equations. An error analysis of the method is demonstrated in section 4. Several test problems are solved and the results are shown in section 5. Section 6 completes this paper with a brief conclusion.

2. Interpolating moving least-squares method

Let $\mathbf{X} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ be a set of all nodes in the bounded domain $\Omega \subset \mathbb{R}^n$ where N is the number of nodes. The parameter ρ_I denotes the radius of the domain of influence of $\mathbf{x}_I$, and $\|\cdot\|$ denotes the Euclidean norm. The domain of influence of $\mathbf{x}_I$ is defined by $\Omega_I = \{\mathbf{x} : \|\mathbf{x} - \mathbf{x}_I\| \leq \rho_I, \mathbf{x} \in \Omega\}$. Let $\rho = \max_{\mathbf{x}_I \in \mathbf{X}} \{\rho_I\}$. For a given point $\mathbf{x} \in \Omega$, define the index set $\tau_{\mathbf{x}} = \{I \,|\, \|\mathbf{x} - \mathbf{x}_I| \leq \rho_I, \mathbf{x}_I \in \mathbf{X}\}$.

Let $u(\mathbf{x})$ be the function of the field variable defined in Ω. The approximation function of $u(\mathbf{x})$ is denoted by $u^h(\mathbf{x})$. In order to let the approximation $u^h(\mathbf{x})$ in the IMLS method satisfy the interpolating property, Lancaster and Salkauskas [6] defined

a singular weight function, i.e.

$$w(\mathbf{x},\mathbf{x}_I)=w(\mathbf{x}-\mathbf{x}_I)=\begin{cases}\|\mathbf{x}-\mathbf{x}_I\|^{-\alpha}, & \|\mathbf{x}-\mathbf{x}_I\|\le\rho_I,\\ 0, & \text{others},\end{cases}\tag{2.1}$$

where the parameter α is an even positive integer.

For the weight function in (2.1), when $\|\mathbf{x}-\mathbf{x}_I\|\le\rho_I$ and $\mathbf{x}\neq\mathbf{x}_I$. We have

$$\lim_{\rho_I\to 0} w(\mathbf{x},\mathbf{x}_I)=+\infty.\tag{2.2}$$

Then if the maximum radius of the domains of influence were very small, the values of the weight function would be very large, which is disadvantageous for a numerical method. However, from the MLS approximation, we know that, in order to guarantee the precision of the numerical solutions, the fine distribution of nodes in the domain is used, which makes the radius of influence small. Then a proper value of the radius of influence must be selected. To overcome the disadvantage of the weight function of (2.2), in this paper, we use the following weight function

$$w(\mathbf{x},\mathbf{x}_I)=w(\mathbf{x}-\mathbf{x}_I)=\begin{cases}\|\frac{\mathbf{x}-\mathbf{x}_I}{\rho_I}\|^{-\alpha}, & \|\mathbf{x}-\mathbf{x}_I\|\le\rho_I,\\ 0, & \text{others}.\end{cases}\tag{2.3}$$

Define the inner product

$$\langle f,g\rangle_{\mathbf{x}}=\sum_{I\in\tau_{\mathbf{x}}} w(\mathbf{x},\mathbf{x}_I)f(\mathbf{x}_I)g(\mathbf{x}_I),\quad \forall f,g\in C^0(\Omega),\tag{2.4}$$

where the subscript $\mathbf{x}$ denotes a point in Ω. Then the corresponding norm at $\mathbf{x}$ is

$$\|f\|_{\mathbf{x}}=\left[\sum_{I\in\tau_{\mathbf{x}}} w(\mathbf{x},\mathbf{x}_I)f^2(\mathbf{x}_I)\right]^{\frac{1}{2}}.\tag{2.5}$$

Let $p_0(\mathbf{x})\equiv 1, p_1(\mathbf{x}),\dots,p_{\bar{m}}(\mathbf{x})$ be given basis functions, where $\bar{m}+1$ denotes the number of the basis functions. We will generate a new set of basis functions from these given basis. First normalizing $p_0(\mathbf{x})$ and let

$$\tilde{p}_0(\mathbf{x};\bar{\mathbf{x}})=\frac{1}{\left[\sum_{I\in\tau_{\mathbf{x}}} w(\mathbf{x},\mathbf{x}_I)\right]^{\frac{1}{2}}}.\tag{2.6}$$

Then we can generate new basis functions orthogonal to $\tilde{p}_0(\mathbf{x};\bar{\mathbf{x}})$ as

$$\tilde{p}_i(\mathbf{x},\bar{\mathbf{x}})=p_i(\bar{\mathbf{x}})-\mathcal{S}p_i(\mathbf{x}),\quad i=1,2,\dots,\bar{m},\tag{2.7}$$

where $\mathcal{S}p_i$ is a linear operator defined as

$$\mathcal{S}p_i(\mathbf{x})=\sum_{I\in\tau_{\mathbf{x}}} v(\mathbf{x},\mathbf{x}_I)p_i(\mathbf{x}_I),\tag{2.8}$$

and

$$v(\mathbf{x},\mathbf{x}_I)=\frac{w(\mathbf{x},\mathbf{x}_I)}{\sum_{J\in\tau_{\mathbf{x}}} w(\mathbf{x},\mathbf{x}_J)}.\tag{2.9}$$

The function $v(\mathbf{x},\mathbf{x}_I)$ has the following properties [6].

Lemma 2.1. *If the weight function of* (2.3) *is used, then* $v(\mathbf{x},\mathbf{x}_I)\in C^\infty(\bar{\Omega})$, *and*

(a) $v(\mathbf{x}_I, \mathbf{x}_J) = \delta_{IJ}$, $\forall I, J \in \tau_{\mathbf{x}}$,
(b) $\sum_{I\in\tau_{\mathbf{x}}} v(\mathbf{x}, \mathbf{x}_I) = 1$, $\forall \mathbf{x} \in \tau_{\mathbf{x}}$,
(c) $0 \le v(\mathbf{x}, \mathbf{x}_I) \le 1$, $\forall \mathbf{x} \in \Omega$, and $v(\mathbf{x}, \mathbf{x}_I) = 0$ if and only if $\mathbf{x} = \mathbf{x}_J$, $J \neq I$,
(d) $\frac{\partial v(\mathbf{x}_I, \mathbf{x}_J)}{\partial \mathbf{x}} = 0$, $\forall I, J \in \tau_{\mathbf{x}}$.

To obtain the expression of the approximation function $u^h(\mathbf{x})$ which satisfies the interpolating property, Lancaster and Salkauskas [6] defined a local approximation, i.e.,

$$u^h(\mathbf{x}, \bar{\mathbf{x}}) = \tilde{p}_0(\mathbf{x}; \bar{\mathbf{x}}) a_0(\mathbf{x}) + \sum_{i=1}^{\bar{m}} \tilde{p}_i(\mathbf{x}; \bar{\mathbf{x}}) a_i(\mathbf{x}), \tag{2.10}$$

where $\bar{\mathbf{x}}$ is the point in the domain of influence of $\mathbf{x}$, and $a_i(\mathbf{x})$, $i = 1, 2, \ldots, \bar{m}$, are the unknown coefficients of the basis functions.

For a given $\mathbf{x}$, the difference between the local approximation function $u^h(\mathbf{x}, \bar{\mathbf{x}})$ and the function $u(\bar{\mathbf{x}})$, will be minimized by a weighted least-squares method. Then define the weighted discrete L_2 norm as

$$J(\mathbf{x}) = \sum_{I\in\tau_{\mathbf{x}}} w(\mathbf{x}, \mathbf{x}_I) \left[u^h(\mathbf{x}, \mathbf{x}_I) - u_I\right]^2, \tag{2.11}$$

where $w(\mathbf{x}, \mathbf{x}_I)$, as shown in (2.3), is a weight function with compact support, $\mathbf{x}_I$ for $I \in \tau_{\mathbf{x}}$ are the nodes with domains of influence that cover the point $\mathbf{x}$, and $u_I = u(\mathbf{x}_I)$.

By minimizing the weighted discrete L_2 norm of (2.11) we have

$$\langle u(\cdot) - u^h(\mathbf{x}, \cdot), \tilde{p}_0 \rangle_{\mathbf{x}} = 0, \tag{2.12}$$

$$\langle u(\cdot) - u^h(\mathbf{x}, \cdot), \tilde{p}_i \rangle_{\mathbf{x}} = 0, \quad i = 1, 2, \ldots, \bar{m}. \tag{2.13}$$

In terms of the orthogonality, (2.12) and (2.13) can be rewritten as

$$a_0(\mathbf{x}) = \langle u, \tilde{p}_0 \rangle_{\mathbf{x}}, \tag{2.14}$$

$$a_0(\mathbf{x}) \langle \tilde{p}_0, \tilde{p}_j \rangle_{\mathbf{x}} + \sum_{i=1}^{\bar{m}} a_i(\mathbf{x}) \langle \tilde{p}_i, \tilde{p}_j \rangle_{\mathbf{x}} = \langle u, \tilde{p}_j \rangle_{\mathbf{x}}, \quad j = 1, 2, \ldots, \bar{m}. \tag{2.15}$$

According to (2.6) and the definition of inner product, we have

$$\tilde{p}_0(\mathbf{x}, \bar{\mathbf{x}}) a_0(\mathbf{x}) = \frac{1}{\left[\sum_{I\in\tau_{\mathbf{x}}} w(\mathbf{x}, \mathbf{x}_I)\right]^{\frac{1}{2}}} \langle u, \tilde{p}_0 \rangle_{\mathbf{x}} = \sum_{I\in\tau_{\mathbf{x}}} v(\mathbf{x}, \mathbf{x}_I) u_I = \mathcal{S}u. \tag{2.16}$$

Then (2.15) reduces to

$$\sum_{i=1}^{\bar{m}} a_i(\mathbf{x}) \langle \tilde{p}_i, \tilde{p}_j \rangle_{\mathbf{x}} = \langle u - \mathcal{S}u, \tilde{p}_j \rangle_{\mathbf{x}}, \quad j = 1, 2, \ldots, \bar{m}. \tag{2.17}$$

In [6], the unknown parameters $a_i(\mathbf{x})$ $(i = 1, 2, \ldots, \bar{m})$ are solved from (2.17). In fact, by the following lemma, (2.17) can be simplified.

Lemma 2.2. [24]. *If the weight function of* (2.3) *is used, for all* $\mathbf{x} \in \Omega$,

$$\langle \mathcal{S}u, \tilde{p}_i \rangle_{\mathbf{x}} = 0, \quad i = 1, 2, \ldots, \bar{m}. \tag{2.18}$$

According to lemma 2.2, (2.17) can be simplified as

$$\sum_{i=1}^{\bar{m}} a_i(\mathbf{x})\langle \tilde{p}_i, \tilde{p}_j\rangle_{\mathbf{x}} = \langle u, \tilde{p}_j\rangle_{\mathbf{x}}, \quad j = 1, 2, \ldots, \bar{m}. \tag{2.19}$$

(2.19) is simpler than the corresponding expression in [6], and can be rewritten as

$$\mathbf{A}(\mathbf{x})\mathbf{a}(\mathbf{x}) = \mathbf{F}_w(\mathbf{x})\mathbf{u}, \tag{2.20}$$

where

$$\mathbf{a}^T(\mathbf{x}) = (a_1(\mathbf{x}), a_2(\mathbf{x}) \ldots, a_{\bar{m}}(\mathbf{x})), \tag{2.21}$$

$$\mathbf{u}^T = (u_1, u_2, \ldots, u_N), \tag{2.22}$$

$$\mathbf{A}(\mathbf{x}) = \mathbf{F}_w(\mathbf{x})\mathbf{F}^T(\mathbf{x}), \tag{2.23}$$

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} \tilde{p}_1(\mathbf{x};\mathbf{x}_1) & \tilde{p}_1(\mathbf{x};\mathbf{x}_2) & \ldots & \tilde{p}_1(\mathbf{x};\mathbf{x}_N) \\ \tilde{p}_2(\mathbf{x};\mathbf{x}_1) & \tilde{p}_2(\mathbf{x};\mathbf{x}_2) & \ldots & \tilde{p}_2(\mathbf{x};\mathbf{x}_N) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{p}_{\bar{m}}(\mathbf{x};\mathbf{x}_1) & \tilde{p}_{\bar{m}}(\mathbf{x};\mathbf{x}_2) & \ldots & \tilde{p}_{\bar{m}}(\mathbf{x};\mathbf{x}_N) \end{bmatrix}, \tag{2.24}$$

and $\mathbf{F}_w(\mathbf{x}) = \bar{\omega}_{kJ}(\mathbf{x})_{\bar{m}\times N}$ is a $\bar{m} \times N$ matrix, and

$$\bar{\omega}_{kJ}(\mathbf{x}) = \begin{cases} w(\mathbf{x}, \mathbf{x}_J)\tilde{p}_k(\mathbf{x};\mathbf{x}_J), & \mathbf{x} \neq \mathbf{x}_J, \\ \sum\limits_{I\in\tau_{\mathbf{x}}, I\neq J} w(\mathbf{x}_J, \mathbf{x}_I)\left[p_k(\mathbf{x}_J) - p_k(\mathbf{x}_I)\right], & \mathbf{x} = \mathbf{x}_J. \end{cases} \tag{2.25}$$

Then we can obtain

$$\mathbf{a}(\mathbf{x}) = \mathbf{A}^{-1}(\mathbf{x})\mathbf{F}_w(\mathbf{x})\mathbf{u}. \tag{2.26}$$

Then the local approximation function can be obtained as

$$u^h(\mathbf{x}, \bar{\mathbf{x}}) = \mathcal{S}u + \sum_{i=1}^{\bar{m}} a_i(\mathbf{x})\tilde{p}_i(\mathbf{x};\bar{\mathbf{x}}). \tag{2.27}$$

Thus the global interpolating approximation function of $u(\mathbf{x})$ can be obtained as

$$u^h(\mathbf{x}) = \mathcal{S}u + \sum_{i=1}^{\bar{m}} a_i(\mathbf{x})g_i(\mathbf{x}) \equiv \Phi(\mathbf{x})\mathbf{u} = \sum_{I=1}^{N} \phi_I(\mathbf{x})u(\mathbf{x}_I), \tag{2.28}$$

where $\Phi(\mathbf{x})$ is a matrix of shape functions,

$$\Phi(\mathbf{x}) = (\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \ldots, \phi_N(\mathbf{x})) = \mathbf{v}^T + \mathbf{p}^T(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{F}_w(\mathbf{x}), \tag{2.29}$$

where

$$\mathbf{v}^T = (v(\mathbf{x}, \mathbf{x}_1), v(\mathbf{x}, \mathbf{x}_2), \ldots, v(\mathbf{x}, \mathbf{x}_N)), \tag{2.30}$$

$$\mathbf{p}^T(\mathbf{x}) = (g_1(\mathbf{x}), g_2(\mathbf{x}), \ldots, g_{\bar{m}}(\mathbf{x})), \tag{2.31}$$

$$g_i(\mathbf{x}) = p_i(\mathbf{x}) - \mathcal{S}p_i(\mathbf{x}). \tag{2.32}$$

Then the first partial derivatives of the shape functions of the IMLS method can be obtained as

$$\phi_{,i}(\mathbf{x}) = \mathbf{v}_{,i}^T + \mathbf{p}_{,i}^T(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{F}_w(\mathbf{x}) + \mathbf{p}^T(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{F}_{w,i}(\mathbf{x}) + \mathbf{p}^T(\mathbf{x})\mathbf{A}_{,i}^{-1}(\mathbf{x})\mathbf{F}_w(\mathbf{x}), \tag{2.33}$$

where

$$\mathbf{F}_w(\mathbf{x}) = \bar{\omega}_{kJ,i}(\mathbf{x})_{\bar{m}\times N}, \tag{2.34}$$

$$\bar{\omega}_{kJ,i}(\mathbf{x}) = \begin{cases} w_{,i}(\mathbf{x},\mathbf{x}_J)\tilde{p}_k(\mathbf{x};\mathbf{x}_J) + w(\mathbf{x},\mathbf{x}_J)\tilde{p}_{k,i}(\mathbf{x};\mathbf{x}_J)(\mathbf{x}), & \mathbf{x} \neq \mathbf{x}_J, \\ \sum\limits_{I\in\tau_{\mathbf{x}}, I\neq J} w_{,i}(\mathbf{x},\mathbf{x}_I)\left[p_k(\mathbf{x}_J) - p_k(\mathbf{x}_I)\right], & \mathbf{x} = \mathbf{x}_J, \end{cases} \tag{2.35}$$

$$\mathbf{A}_{,i}^{-1}(\mathbf{x}) = -\mathbf{A}^{-1}(\mathbf{x})\mathbf{A}_{,i}(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x}). \tag{2.36}$$

(2.29) is the shape function of the IMLS method.
The matrix $\mathbf{A}(\mathbf{x})$ is defined in (2.20) is of order $\bar{m}\times\bar{m}$ and plays an important role in the IMLS method. The solution of (2.20) is unique if the rank of $\mathbf{A}(\mathbf{x})$ is $\bar{m}$. Therefore, we need the following assumption [10, 11].

Assumption 2.3. *For any $\mathbf{x}\in\Omega$, assume that the matrix $\mathbf{A}(\mathbf{x})$ is invertible,i.e., the data point set $\{\mathbf{x}_I\in\mathbf{X}, I\in\tau_{\mathbf{x}}\}$ is $\mathbf{P}_m(\mathbb{R}^n)$-unisolvent.*

3. The proposed method

In this section, we employ the IMLS method for solving the integral equations. The general form of these equations can be considered as :

$$u(x) = \mathcal{F}\left(x, \int_a^b K_1(x,t,u(\mathrm{t}))\,\mathrm{d}t, \int_a^x K_2(x,t,u(\mathrm{t}))\,\mathrm{d}t\right), \tag{3.1}$$

where u is the unknown function, a and b are real finite numbers, K_1 and K_2 are called kernel functions, $\mathcal{F}:\mathbb{R}\times C^p\times C^p\to C^p$and kernel functions $K_1, K_2:\mathbb{R}\times\mathbb{R}\times C^p\to C^p$are given continuous mappings and satisfying in the following conditions:

$$\|\mathcal{F}(x,u_1,u_2) - \mathcal{F}(x,v_1,v_2)\| \leqslant \beta_1\|u_1-v_1\| + \beta_2\|u_2-v_2\|, \quad \forall u_1,u_2,v_1,v_2\in C^p, \tag{3.2}$$

$$\|K_1(x,u(x),\theta_1) - K_1(x,u(x),\theta_2)\| \leqslant \gamma_1\|\theta_1-\theta_2\|, \quad \forall\theta_1,\theta_2\in C^p, \tag{3.3}$$

$$\|K_2(x,u(x),\theta_1) - K_2(x,u(x),\theta_2)\| \leqslant \gamma_2\|\theta_1-\theta_2\|, \quad \forall\theta_1,\theta_2\in C^p, \tag{3.4}$$

$$\|K_1(x,u(x),\theta)\| \leqslant c_1\|\theta\|, \quad \forall\theta\in C^p, \tag{3.5}$$

$$\|K_2(x,u(x),\theta)\| \leqslant c_2\|\theta\|, \quad \forall\theta\in C^p, \tag{3.6}$$

where β_1, β_2, γ_1, γ_2, c_1 and c_2 are real constant values.

Now, to employ the IMLS method let us consider n nodal points in the interval $[a,b]$ as $a = x_1 \leq x_2 \leq \cdots \leq x_n = b$. The distribution of nodes could be selected regularly or randomly. Then instead of u, we can replace $u^h(x)$ from (2.28). So (3.1)

becomes

$$u^h(x) = \mathcal{F}\left(x, \int_a^b K_1(x,t,u^h(\mathrm{t}))\,\mathrm{d}t, \int_a^x K_2(x,t,u^h(\mathrm{t}))\,\mathrm{d}t\right), \tag{3.7}$$

or equivalently

$$\sum_{i=1}^{n}\phi_i(x)u_i = \mathcal{F}\left(x, \int_a^b K_1(x,t,\sum_{i=1}^{n}\phi_i(t)u_i)\,\mathrm{d}t, \int_a^x K_2(x,t,\sum_{i=1}^{n}\phi_i(t)u_i)\,\mathrm{d}t\right). \tag{3.8}$$

The integral domain $[a,x]$ must be transferred to a fixed $[a,b]$. For this purpose, the following transformation has been considered

$$\xi(x,\theta) = \frac{x-a}{b-a}\theta + \frac{b-x}{b-a}a. \tag{3.9}$$

Employing this transformation, (3.8) becomes

$$\sum_{i=1}^{n}\phi_i(x)u_i = \mathcal{F}\left(x, \int_a^b K_1(x,t,\sum_{i=1}^{n}\phi_i(t)u_i)\,\mathrm{d}t\,, \int_a^b K_2^{\mathcal{T}}(x,\xi(x,\theta),\sum_{i=1}^{n}\phi_i(\xi(x,\theta))u_i)\,\mathrm{d}\theta\right), \tag{3.10}$$

or equivalently

$$\sum_{i=1}^{n}\phi_i(x)u_i = \mathcal{F}\left(x, \int_a^b K_1(x,\theta,\sum_{i=1}^{n}\phi_i(\theta)u_i)\,\mathrm{d}\theta\,, \int_a^b K_2^{\mathcal{T}}(x,\xi(x,\theta),\sum_{i=1}^{n}\phi_i(\xi(x,\theta))u_i)\,\mathrm{d}\theta\right), \tag{3.11}$$

where

$$K_2^{\mathcal{T}} = \frac{x-a}{b-a}K_2. \tag{3.12}$$

Assume that (3.11) holds at x_j,

$$\begin{aligned}\sum_{i=1}^{n}\phi_i(x_j)u_i = \mathcal{F}\Bigg(x_j, \int_a^b K_1(x_j,\theta,\sum_{i=1}^{n}\phi_i(\theta)u_i)\,\mathrm{d}\theta\,, \int_a^b K_2^{\mathcal{T}}(x_j,\xi(x_j,\theta),\\ \sum_{i=1}^{n}\phi_i(\xi(x_j,\theta))u_i)\,\mathrm{d}\theta\Bigg), \qquad j=1,2,\ldots,n.\end{aligned} \tag{3.13}$$

Using a m_1-points quadrature formula with the coefficients $\{\theta_k\}$ and weights $\{\omega_k\}$ in interval $[a,b]$ for numerically solving in (3.13) yields

$$\begin{aligned}\sum_{i=1}^{n}\phi_i(x_j)\hat{u}_i = \mathcal{F}\Bigg(x_j, \sum_{k=1}^{m_1}\omega_k K_1(x_j,\theta_k,\sum_{i=1}^{n}\phi_i(\theta_k)\hat{u}_i)\,, \sum_{k=1}^{m_1}\omega_k K_2^{\mathcal{T}}(x_j,\xi(x_j,\theta_k),\\ \sum_{i=1}^{n}\phi_i(\xi(x_j,\theta_k))\hat{u}_i)\Bigg), \qquad j=1,2,\ldots,n,\end{aligned} \tag{3.14}$$

Where $\hat{u}_i$ are the approximate quantities of u_i when we use a quadrature rule instead of the exact integral. By solving (3.14) with an appropriate numerical solver for

the nonlinear systems, we can obtain the values of $\hat{u}_i$. Then the value of $u(x)$ is approximated by

$$u(x) \simeq u^h(x) = \sum_{i=1}^{n} \phi_i(x)\hat{u}_i, \quad \forall x \in [a,b]\,. \tag{3.15}$$

4. The error analysis

In this section, the error estimate for the proposed method is obtained. Since the error estimate of the method is strictly connected to the error estimate of the IMLS method, first we introduce the error estimate of the IMLS method. The error estimate of the approximation function of the IMLS method was presented in a one-dimensional case, and the error estimates of the first and second order derivatives are also given [6]. In the current work, we employ the results for error estimates of the IMLS method in [24].

Let $B_x(y) = \{y|\,|y-x| \leqslant \mu\rho,\, y \in \Omega \subset \mathbb{R}\}$, where $1 < \mu < 2$ is a constant. Define the index set $H = \{I|x_I \in \mathbf{X}, x_I \in B_x(y)\}$. Suppose h is a sufficient small number such that $\tau_x \subset H$ and $\tau_{x+h} \subset H$. When $x \notin \mathbf{X}$ and $w(x,x_I) \neq 0$, there certainly exist constants c_β such that $\left|\frac{\mathrm{d}w(x,x_I)}{\mathrm{d}x^\beta}\right| \leqslant \frac{c_\beta}{\rho_I^\beta}$. In our subsequent discussion of this section, the basis functions are selected as $p_0(x) = 1, p_1(x) = x, \ldots, p_m(x) = x^m$.

Hypothesis 4.1. There exist constants c_ε and c_I such that $\rho \leqslant c_\varepsilon\varepsilon$ and $\rho \leqslant c_I\rho_I$ respectively. This hypothesis shows that, for $\forall x \in \Omega$, the number of nodes in the domain of influence of x is finite.

Hypothesis 4.2. For $\forall x \in \Omega$, there exists $m+1$ elements in the set $\tau_x \cap \tau_{x+h}$. Suppose d be a positive integer. Let $C^d(\Omega)$ be the set of all the functions whose derivative of order d exists and is continuous in Ω .

The convergence order of the local approximation $u^h(x,y)$ can be obtained from the known results on the best leastsquares approximation. Thus, from the error estimates of the Lagrange interpolation polynomial, we have the following theorem.

Theorem 4.3. [24]. *If $u \in C^{m+1}(\bar{\Omega})$, and Hypotheses 4.1 and 4.2 are satisfied, for each $x \in \Omega$, there exists a constant $\mathcal{C}$, which is independent of ρ, such that*

$$|u^h(x,y) - u(y)| \leqslant \mathcal{C}\|u^{(m+1)}\|_{L^\infty(\Omega)}\rho^{m+1}, \quad \forall y \in B_x(y),$$

In particular, when $y = x$, we have

$$\|u^h(x) - u(x)\|_{L^\infty(\Omega)} \leqslant \mathcal{C}\|u^{(m+1)}\|_{L^\infty(\Omega)}\rho^{m+1}. \tag{4.1}$$

Also, we assume the numerical quadrature satisfies the condition described in the following [2].

QA: There exists positive number η, small enough and independent of i and mesh-size, such that

$$\left|\int_{\omega} Q \, \mathrm{d}x - \int_{\omega}^{*} Q \, \mathrm{d}x\right| \leqslant \eta \, |\omega| \, \|Q\|_{L^{\infty}(\omega)}, \tag{4.2}$$

where the notation $\int^{*}(,)$ denotes the integrals which are computed using a quadrature formula and the constant η is a bound for the error of the numerical quadrature formula [4].

Let us consider the Volterra-Fredholm integral equation (3.1) with conditions (3.2)-(**??**) and $(b-a)(\beta_1\gamma_1+\beta_2\gamma_2) \in [0,1)$. By defining the integral operator $\pounds(u(x))$ in the Banach space C^p as:

$$\pounds(u(x)) = \mathcal{F}\left(x, \int_a^b K_1(x,t,u(\mathrm{t}))\,\mathrm{d}t, \int_a^x K_2(x,t,u(\mathrm{t}))\,\mathrm{d}t\right), \tag{4.3}$$

then the Banach fixed point theorem guarantees that, under certain assumptions ((3.2)- (**??**)), $\pounds$ has a unique fixed point, the Volterra-Fredholm integro-differential equation has exactly one solution. Also, assume that $u(x)$ is the solution of (3.1) , $u^h(x)$ is the solution of (3.7) and $u_*^h(x)$ is the solution of (3.14).

Theorem 4.4. *Let $u \in C^{m+1}(\bar{\Omega})$ where Ω is a bounded set in $\mathbb{R}$, and $\mathcal{F}$, K_1 and K_2 satisfy the conditions of equation of (3.1) and $[a,x] \subseteq \Omega = [a,b]$. Assume that the numerical quadrature formula satisfies (4.2). Also, suppose that Hypotheses* 4.1 *and* 4.2 *are satisfied. Then we have*

$$\|\pounds(u(x)) - \pounds(u_*^h(x))\|_{L^{\infty}(\Omega)} \leqslant \mathcal{C}_1\|u\|_{L^{\infty}(\Omega)} + \mathcal{C}_2\|u^{(m+1)}\|_{L^{\infty}(\Omega)} \tag{4.4}$$

Consequently, we obtain

$$\|u - u_*^h\|_{L^{\infty}(\Omega)} \leqslant \mathcal{C}_1\|u\|_{L^{\infty}(\Omega)} + \mathcal{C}_2\|u^{(m+1)}\|_{L^{\infty}(\Omega)} \tag{4.5}$$

where $\mathcal{C}_1 = (b-a)(\beta_1 c_1 + \beta_2 c_2)\eta$ *and* $\mathcal{C}_2 = (b-a)\left((\beta_1 c_1 + \beta_2 c_2)\eta + (\beta_1\gamma_1 + \beta_2\gamma_2)\right)\mathcal{C}\rho^{m+1}$.

Proof. In the proof, we use Theorem.4.3 and the Lipschitz condition for $\mathcal{F}$, K_1 , K_2. From definition of $\pounds$, we have

$$\|\pounds(u(x)) - \pounds(u_*^h(x))\|_{L^{\infty}(\Omega)} \leqslant \|\pounds(u(x)) - \pounds(u^h(x))\|_{L^{\infty}(\Omega)} + \|\pounds(u^h(x)) - \pounds(u_*^h(x))\|_{L^{\infty}(\Omega)} \tag{4.6}$$

First, we obtain

$$
\begin{aligned}
&\|\mathcal{L}(u(x)) - \mathcal{L}(u^h(x))\|_{L^\infty(\Omega)} \\
&= \left\| \mathcal{F}\left(x, \int_a^b K_1(x,t,u(t))\mathrm{d}t, \int_a^x K_2(x,t,u(t))\mathrm{d}t\right)\right. \\
&\left. -\mathcal{F}\left(x, \int_a^b K_1(x,t,u^h(t))\mathrm{d}t, \int_a^x K_2(x,t,u^h(t))\mathrm{d}t\right)\right\|_{L^\infty(\Omega)} \\
&\leqslant \beta_1 \left\| \int_a^b K_1(x,t,u(t))\mathrm{d}t - \int_a^b K_1(x,t,u^h(t))\mathrm{d}t \right\|_{L^\infty(\Omega)} \\
&+ \beta_2 \left\| \int_a^x K_2(x,t,u(t))\mathrm{d}t - \int_a^x K_2(x,t,u^h(t))\mathrm{d}t \right\|_{L^\infty(\Omega)} \\
&\leqslant \beta_1(b-a) \left\| K_1(x,t,u(t)) - K_1(x,t,u^h(t)) \right\|_{L^\infty(\Omega)} \\
&+ \beta_2(b-a) \left\| K_2(x,t,u(t)) - K_2(x,t,u^h(t)) \right\|_{L^\infty(\Omega)} \\
&\leqslant (b-a)\beta_1\gamma_1 \|u(t) - u^h(t)\|_{L^\infty(\Omega)} + (b-a)\beta_2\gamma_2 \|u(t) - u^h(t)\|_{L^\infty(\Omega)} \\
&= (b-a)\left(\beta_1\gamma_1 + \beta_2\gamma_2\right) \|u(t) - u^h(t)\|_{L^\infty(\Omega)} \\
&\leqslant (b-a)\left(\beta_1\gamma_1 + \beta_2\gamma_2\right) \mathcal{C}\rho^{m+1} \|u^{(m+1)}\|_{L^\infty(\Omega)},
\end{aligned}
$$

In addition, we can write

$$
\begin{aligned}
&\|\mathcal{L}(u^h(x)) - \mathcal{L}(u_*^h(x))\|_{L^\infty(\Omega)} \\
&= \left\| \mathcal{F}\left(x, \int_a^b K_1(x,t,u^h(t))\mathrm{d}t, \int_a^x K_2(x,t,u^h(t))\mathrm{d}t\right)\right. \\
&\left. -\mathcal{F}\left(x, \int_a^{b*} K_1(x,t,u^h(t))\mathrm{d}t, \int_a^{x*} K_2(x,t,u^h(t))\mathrm{d}t\right)\right\|_{L^\infty(\Omega)} \\
&\leqslant \beta_1 \left\| \int_a^b K_1(x,t,u^h(t))\mathrm{d}t - \int_a^{b*} K_1(x,t,u^h(t))\mathrm{d}t \right\|_{L^\infty(\Omega)} \\
&+ \beta_2 \left\| \int_a^x K_2(x,t,u^h(t))\mathrm{d}t - \int_a^{x*} K_2(x,t,u^h(t))\mathrm{d}t \right\|_{L^\infty(\Omega)} \\
&\leqslant \beta_1(b-a)\eta \left\| K_1(x,t,u^h(t)) \right\|_{L^\infty(\Omega)} + \beta_2(b-a)\eta \left\| K_2(x,t,u^h(t)) \right\|_{L^\infty(\Omega)} \\
&\leqslant (b-a)\left(\beta_1 c_1 + \beta_2 c_2\right)\eta \|u^h\|_{L^\infty(\Omega)} \\
&= (b-a)\left(\beta_1 c_1 + \beta_2 c_2\right)\eta \left(\|u\|_{L^\infty(\Omega)} + \|u - u^h\|_{L^\infty(\Omega)}\right) \\
&\leqslant (b-a)\left(\beta_1 c_1 + \beta_2 c_2\right)\eta \left(\|u\|_{L^\infty(\Omega)} + \mathcal{C}\rho^{m+1}\|u^{(m+1)}\|_{L^\infty(\Omega)}\right),
\end{aligned}
$$

Consequently, from the two last obtained inequalities and (4.6), we can get the following inequality:

$$\begin{aligned}
&\|\pounds(u(x)) - \pounds(u_*^h(x))\|_{L^\infty(\Omega)} \\
&\leqslant \|\pounds(u(x)) - \pounds(u^h(x))\|_{L^\infty(\Omega)} + \|\pounds(u^h(x)) - \pounds(u_*^h(x))\|_{L^\infty(\Omega)} \\
&\leqslant (b-a)\,(\beta_1\gamma_1 + \beta_2\gamma_2)\,\mathcal{C}\rho^{m+1}\|u^{(m+1)}\|_{L^\infty(\Omega)} \\
&+ (b-a)\,(\beta_1 c_1 + \beta_2 c_2)\,\eta\left(\|u\|_{L^\infty(\Omega)} + \mathcal{C}\rho^{m+1}\|u^{(m+1)}\|_{L^\infty(\Omega)}\right), \\
&\leqslant (b-a)\,(\beta_1 c_1 + \beta_2 c_2)\,\eta\|u\|_{L^\infty(\Omega)} \\
&+ (b-a)\mathcal{C}\rho^{m+1}\left((\beta_1 c_1 + \beta_2 c_2)\eta + (\beta_1\gamma_1 + \beta_2\gamma_2)\right)\|u^{(m+1)}\|_{L^\infty(\Omega)} \\
&= \mathcal{C}_1\|u\|_{L^\infty(\Omega)} + \mathcal{C}_2\|u^{(m+1)}\|_{L^\infty(\Omega)}
\end{aligned}$$

Using from the fixed point property of u and u_*^h, we can write

$$\begin{aligned}
&\|u - u_*^h\|_{L^\infty(\Omega)} \\
&= \|\pounds(u(x)) - \pounds(u_*^h(x))\|_{L^\infty(\Omega)} \\
&\leqslant (b-a)\,(\beta_1 c_1 + \beta_2 c_2)\,\eta\|u\|_{L^\infty(\Omega)} \\
&+ (b-a)\mathcal{C}\rho^{m+1}\left((\beta_1 c_1 + \beta_2 c_2)\eta + (\beta_1\gamma_1 + \beta_2\gamma_2)\right)\|u^{(m+1)}\|_{L^\infty(\Omega)} \\
&= \mathcal{C}_1\|u\|_{L^\infty(\Omega)} + \mathcal{C}_2\|u^{(m+1)}\|_{L^\infty(\Omega)}
\end{aligned}$$

It completes the proof. □

5. Numerical results

In this section, the numerical results for the one-dimensional Fredholm ,Volterra and VolterraFredholm integral equations have been depicted by the IMLS method. To measure the accuracy of the method, the maximum error has been used with the following definition:

$$\text{Maximum error} : \|e\|_\infty = \max|u(j) - u_{exact}(j)|.$$

To show the rate of convergence of the new method, the values of ratio with the following formula have been reported

$$\text{Ratio} = \frac{\|e^{N-1}\|_\infty}{\|e^N\|_\infty}.$$

In numerical computation of this section, the nodes are arranged regularly, and the radius of the domain of influence of node x_i is determined by $\rho_I = d_{\max}\,|X_I - X_{I-1}|$, $d_{\max}$ is a positive scalar. The value of $d_{\max}$ must be chosen so that the matrix of (2.23) is invertible.

For the tests we used the linear and the quadratic basis and weight function (2.3). Also, for the numerical quadrature rule we used the five or seven -point Gauss Legendre quadrature formula. All routines are written in Matlab 2017 and run on a Intel Core i5 PC Laptop with 1.8 GHz of CPU and 4 GB of RAM.

Example 5.1. [9].Consider the following Fredholm integral equation:

$$\lambda u(x) - \int_0^1 u(x)\,\mathrm{d}x = g(x), \qquad 0 \le x \le 1,$$

where $\lambda = 5$. Suppose the exact solution

$$u(x) = x^2\sqrt{x},$$

and appropriate right hand side. The maximum error values are given for different values of N in Table 1 for both linear and quadratic cases for the collocation node points. The numerical and exact values solution of this equation is shown in Fig 1.

TABLE 1. Numerical results Example 5.1.

N	*Linear*$(m=1)$			*Quadratic*$(m=2)$		
	$\|e\|_\infty$	Ratio	Time	$\|e\|_\infty$	Ratio	Time
5	1.10×10^{-3}	–	1.64	7.49×10^{-5}	–	1.70
9	7.24×10^{-4}	1.52	1.66	2.10×10^{-5}	3.57	1.71
17	2.43×10^{-4}	2.98	1.74	2.13×10^{-6}	9.86	1.77
33	9.07×10^{-5}	2.68	1.83	4.70×10^{-7}	4.53	1.92
65	1.00×10^{-5}	9.07	2.15	3.02×10^{-7}	1.56	2.17
129	4.47×10^{-6}	2.24	2.73	3.14×10^{-7}	0.96	2.80

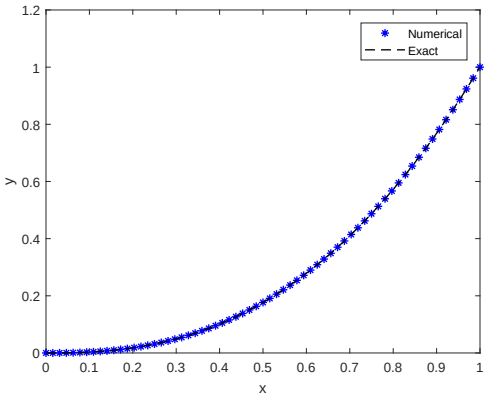

FIGURE 1. The IMLS approximation of degree 2 with 65 points for Example 5.1.

We compare the obtained numerical results in Table 1 with obtained ones in [9].It is seen that the maximum error $\|e\|_\infty$ of the proposed method in scattered points reported in Table 1, is lower then ones in[9].

Example 5.2. As the second example, consider the following VolterraFredholm integral equation:

$$u(x)=\mathrm{e}^x - 1 - x + \int_0^x u(\mathrm{t})\,\mathrm{d}t + \int_0^1 xu(\mathrm{t})\,\mathrm{d}t.$$

The exact solution of this equation is $u(x) = x\,\mathrm{e}^x$. The results for different m, N s, are given in Table 2 and for the numerical quadrature rule we used the seven-point Gauss Legendre quadrature formula. also Fig 2. shows the absolute error for $N = 201$, $m = 2$.

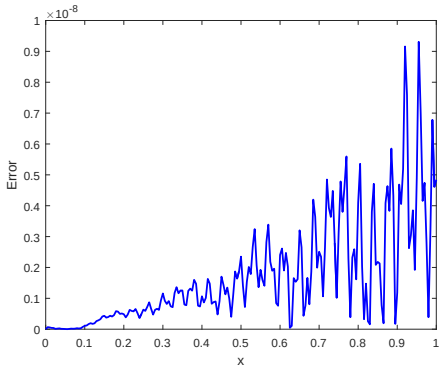

FIGURE 2. The IMLS approximation error of degree 2 with 201 points for Example 5.2.

TABLE 2. Numerical results Example 5.2.

N	*Linear*$(m = 1)$			*Quadratic*$(m = 2)$		
	$\|e\|_\infty$	Ratio	Time	$\|e\|_\infty$	Ratio	Time
5	3.53×10^{-1}	–	1.80	7.90×10^{-3}	–	1.85
11	5.41×10^{-2}	6.52	1.88	5.21×10^{-4}	15.16	1.98
21	1.57×10^{-2}	3.45	2.07	1.02×10^{-4}	5.11	2.07
51	3.10×10^{-3}	5.06	2.28	4.40×10^{-5}	2.32	2.41
101	5.56×10^{-4}	5.58	2.94	2.44×10^{-7}	180.33	3.05
201	1.40×10^{-4}	3.97	4.61	9.16×10^{-9}	26.64	4.71

Example 5.3. [5].In this example, we solve the following nonlinear VolterraFredholm integral equation:

$$u(x) = f(x) + \frac{1}{4}\int_0^x (x-t)\,u^2(t)\,\mathrm{d}t + \int_0^1 (1+t)u(t)\,\mathrm{d}t.$$

The analytic solution of this problem is $u(x) = x^2 \sin x$ and $f(x)$ is chosen accordingly. The results for different m, N s are given in Table 3.

Table 3. Numerical results Example 5.3

N	*Linear*$(m=1)$			*Quadratic*$(m=2)$		
	$\|e\|_\infty$	Ratio	Time	$\|e\|_\infty$	Ratio	Time
5	1.34×10^{-2}	–	1.96	1.14×10^{-3}	–	1.98
9	8.20×10^{-3}	1.63	2.00	3.76×10^{-4}	3.72	2.01
13	4.70×10^{-3}	1.74	2.04	1.19×10^{-4}	3.16	2.05
23	3.04×10^{-4}	1.55	2.16	5.64×10^{-6}	21.10	2.17
45	1.71×10^{-4}	1.78	2.42	1.26×10^{-6}	4.48	2.48
65	1.12×10^{-4}	1.53	2.74	3.62×10^{-7}	3.48	2.77

Example 5.4. In this example consider the following integral equation

$$u(x) = 1 - \int_0^x (x-t)\,u(t)\,\mathrm{d}t + \int_0^\pi u(t)\,\mathrm{d}t,$$

with the exact solution $u(x)=\cos x$. Table 4 shows the numerical results for Example 4.

Table 4. Numerical results Example 5.4

N	*Linear*$(m=1)$			*Quadratic*$(m=2)$		
	$\|e\|_\infty$	Ratio	Time	$\|e\|_\infty$	Ratio	Time
7	3.75×10^{-2}	–	1.86	5.10×10^{-3}	–	1.95
11	2.32×10^{-2}	1.62	1.87	1.60×10^{-3}	3.19	1.98
21	5.40×10^{-3}	4.30	1.88	1.36×10^{-4}	11.76	2.00
41	1.40×10^{-3}	3.86	1.97	1.82×10^{-5}	7.47	2.06
51	1.00×10^{-3}	1.40	1.99	1.75×10^{-5}	1.04	2.11
101	3.68×10^{-4}	2.72	2.36	1.80×10^{-6}	9.72	2.58

Example 5.5. As the last example, we consider the following Volterra integral equation

$$u(x) = 2x\,\mathrm{e}^{-x}(x+1) - 2x + \sqrt{x} + \int_0^x 2tx\mathrm{e}^{-u^2(t)}\,\mathrm{d}t,$$

where the unknown solution is $u(x)=\sqrt{x}$. Table 5, shows the maximum errors for different values of N, also Fig 3 shows the absolute error for $N=401$ and $m=2$. The results show the efficiency of the method to approximate the nonlinear integral equations.

6. Conclusion

In this paper, a numerical scheme based on interpolating moving least square method has been used for the approximate solution of the linear and nonlinear Fredholm, Volterra and Volterra- Fredholm integral equations. The method is a meshless

TABLE 5. Numerical results Example 5.5

N	*Linear*$(m=1)$			*Quadratic*$(m=2)$		
	$\|e\|_\infty$	Ratio	Time	$\|e\|_\infty$	Ratio	Time
5	4.90×10^{-3}	–	2.01	7.90×10^{-4}	–	2.06
9	1.80×10^{-3}	2.72	2.02	3.50×10^{-4}	2.26	2.08
21	2.03×10^{-4}	8.87	2.19	2.33×10^{-5}	15.02	2.27
51	3.38×10^{-5}	6.01	2.55	3.71×10^{-6}	6.28	2.65
101	1.02×10^{-5}	3.31	3.45	2.48×10^{-7}	14.96	3.32
201	2.54×10^{-6}	4.02	5.73	9.07×10^{-8}	2.73	5.58
401	6.14×10^{-7}	4.14	29.54	2.42×10^{-9}	37.48	30.36

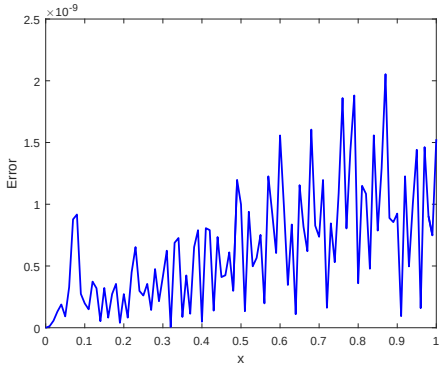

FIGURE 3. The IMLS approximation error of degree 2 with 401 points for Example 5.5.

method, because it requires no domain elements for the interpolation or approximation. The numerical results given in the previous section demonstrate the efficiency and accuracy of this scheme.

REFERENCES

[1] H. Adibi and A. M. Rismani, *Numerical solution to a functional integral equations using the Legendre-spectral method*, Australian Journal of Basic and Applied Sciences., *4* (2010), 481–486.
[2] I. Babuška, U. Banerjee, J. E. Osborn, and Q. Zhang, *Effect of numerical integration on meshless methods*, Computer Methods in Applied Mechanics and Engineering., *198* (2009), 2886–2897.
[3] I. Kaljević and S. Saigal, *An improved element free Galerkin formulation*, International Journal for numerical methods in engineering., *40* (1997), 2953–2974.
[4] N. Kambo, *Error of the Newton-Cotes and Gauss-Legendre quadrature formulas*, Mathematics of Computation., *24* (1970), 261–269.
[5] H. Laeli Dastjerdi and F. M. Maalek Ghaini, *Numerical solution of Volterra–Fredholm integral equations by moving least square method and Chebyshev polynomials*, Applied Mathematical Modelling., *36* (2012), 3283–3288.
[6] P. Lancaster and K. Salkauskas, *Surfaces generated by moving least squares methods*, Mathematics of computation., *37* (1981), 141–158.

[7] G. G. Maisuradze, D. L. Thompson, A. F. Wagner, and M. Minkoff, *Interpolating moving least-squares methods for fitting potential energy surfaces: Detailed analysis of one-dimensional applications*, The Journal of chemical physics., *119* (2003), 10002–10014.
[8] K. Maleknejad, H. Almasieh, and M. Roodaki, *Triangular functions (TF) method for the solution of nonlinear Volterra–Fredholm integral equations*, Communications in Nonlinear Science and Numerical Simulation., *15* (2010), 3293–3298.
[9] D. Mirzaei and M. Dehghan, *A meshless based method for solution of integral equations*, Applied Numerical Mathematics., *60* (2010), 245–262.
[10] D. Mirzaei, R. Schaback, and M. Dehghan, *On generalized moving least squares and diffuse derivatives*, IMA Journal of Numerical Analysis., *32* (2012), 983–1000.
[11] D. Mirzaei, *Analysis of moving least squares approximation revisited*, Journal of Computational and Applied Mathematics., *282* (2015), 237–250.
[12] M. M. Mustafa and I. N. Ghanim, *Numerical solution of linear Volterra-Fredholm integral equations using Lagrange polynomials*, Mathematical Theory and Modeling., *4* (2014), 137–146.
[13] H. Netuzhylov, *Enforcement of boundary conditions in meshfree methods using interpolating moving least squares*, Engineering analysis with boundary elements., *32* (2008), 512–516.
[14] H. P. Ren and Z. Wu, *An improved boundary element-free method (IBEFM) for two-dimensional potential problems*, Chinese Physics B., *18* (2009), 4065–4073.
[15] H. P. Ren, Y. M. Cheng, and W. Zhang, *An interpolating boundary element-free method (IBEFM) for elasticity problems*, Science China Physics, Mechanics and Astronomy., *53* (2010), 758–766.
[16] H. P. Ren and Y. M. Cheng, *The interpolating element-free Galerkin (IEFG) method for two-dimensional elasticity problems*, International Journal of Applied Mechanics., *3* (2011), 735–758.
[17] H. Ren and Y. M. Cheng, *The interpolating element-free Galerkin (IEFG) method for two-dimensional potential problems*, Engineering Analysis with Boundary Elements., *36* (2012), 873–880.
[18] H. P. Ren, J. Cheng, and A. Huang, *The complex variable interpolating moving least-squares method*, Applied mathematics and computation., *219* (2012), 1724–1736.
[19] H. P. Ren and Y. M. Cheng, *A new element-free Galerkin method based on improved complex variable moving least-squares approximation for elasticity*, International Journal of Computational Materials Science and Engineering., *1* (2012), 1250011.
[20] F. Sun, J. Wang, and Y. Cheng, *An improved interpolating element-free Galerkin method for elasticity*, Chinese Physics B., *22* (2013), 120203.
[21] F. Sun, C. Liu, and Y.Cheng, *An improved interpolating element-free Galerkin method based on nonsingular weight functions*, Mathematical Problems in Engineering., *2014* (2014), 1–13.
[22] J. Wang, F. Sun, and Y. Cheng, *An improved interpolating element-free Galerkin method with a nonsingular weight function for two-dimensional potential problems*, Chinese Physics B., *21* (2012), 090204.
[23] J. Wang, J. Wang, F. Sun, and Y. Cheng, *An interpolating boundary element-free method with nonsingular weight function for two-dimensional potential problems*, International Journal of Computational Methods., *10* (2013), 1350043.
[24] J. Wang, F. Sun, Y. Cheng, and A. Huang, *Error estimates for the interpolating moving least-squares method*, Applied Mathematics and Computation., *245* (2014), 321–342.
[25] S. Yalçinbaş, *Taylor polynomial solutions of nonlinear Volterra–Fredholm integral equations*, Applied Mathematics and Computation., *127* (2002), 195–206.
[26] S. A. Yousefi, A. Lotfi, and M. Dehghan, *Hes variational iteration method for solving nonlinear mixed Volterra–Fredholm integral equations*, Computers & Mathematics with Applications., *58* (2009), 2172–2176.