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Collocation method based on Chebyshev polynomials for solving distributed order fractional differential equations

Marzieh Pourbabaee
Department of Applied Mathematics,
Faculty of Mathematical Sciences,
University of Kashan 87317-53153, Kashan,Iran.
E-mail: m.pourbabaee@kashanu.ac.ir, mr.pourbabaee@gmail.com

Abbas Saadatmandi*
Department of Applied Mathematics,
Faculty of Mathematical Sciences,
University of Kashan 87317-53153, Kashan,Iran.
E-mail: saadatmandi@kashanu.ac.ir, a.saadatmandi@gmail.com

Abstract This work presents a new approximation approach to solve the linear/nonlinear distributed order fractional differential equations using the Chebyshev polynomials. Here, we use the Chebyshev polynomials combined with the idea of the collocation method for converting the distributed order fractional differential equation into a system of linear/nonlinear algebraic equations. Also, fractional differential equations with initial conditions can be solved by the present method. We also give the error bound of the modified equation for the present method. Moreover, four numerical tests are included to show the effectiveness and applicability of the suggested method.

Keywords. Distributed order, Caputo derivative, Chebyshev polynomials, Fractional differential equations, Collocation Method.

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1. Introduction

The theory of fractional differential operators has a large history and is considered by a lot of researchers [28, 31]. Over the last decade, fractional differential equations (FDEs) have many applications in several branches of science and areas in fluid flow, physics, mechanics and other applications, see, say [28, 31, 39]. The existence of a unique solution of FDE has been studied by many researchers [15, 24]. Generally, since most of FDEs do not have closed form solutions, therefore numerical algorithms should be applied (see e.g., [2, 13, 14, 27, 29, 30, 32, 34, 35, 36]).

In recent years, special attention has been paid to distributed order fractional differential equations (DFDEs), see, say [1, 23, 33, 41]. As pointed by [25], an important relation between integer order and fractional order operators can be expressed by the distributed order operator. In 1995, Caputo [10] applied this concept for generalization the stress-strain relation in dielectrics. In 2000, Bagley and Torvik [6, 7] used

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* corresponding.

DFDEs in linear time-variant system. Also, a time-fractional diffusion-like of distributed order equation is considered in [11] and the Bagley-Torvik equation with the distributed order fractional derivative, which is used in fluid mechanics, is considered in [4, 20]. Finally, DFDEs appear in modelling of many interdisciplinary areas, such as diffusion and wave phenomena [12, 16], control systems [42], viscoelastic model [5], dynamical system [20]. It is often more difficult to get a closed form solution than a numerical one for a given DFDE. Therefore, effective numerical techniques should be applied (for example, we refer the reader to see [1, 16, 18, 22, 34, 38], and the references therein).

In this work, we focus on the following DFDE [21, 26, 40],

$$\int_{\tau_1}^{\tau_2} H_1\left(\alpha, D^{\alpha}u(t)\right)d\alpha + H_2\left(t, u(t), D^{\gamma_i}u(t)\right) = g(t), \quad t \in [0, L] \tag{1.1}$$

where τ_1 and τ_2 are positive constants; D^{α} is the fractional derivative of Caputo type of order α; $\gamma_i\,(\gamma_1 < \gamma_2 < \ldots < \gamma_r)$ are positive real numbers. Also, both H_1, H_2 are linear/nonlinear functions. Moreover, Eq. (1.1) has the following initial conditions

$$u^{(j)}(0) = u_0^{(j)}, \quad j = 0, 1, \ldots, \ell - 1, \tag{1.2}$$

where $\ell = \max\{\lceil \tau_2 \rceil, \lceil \gamma_r \rceil\}$ in which, the ceiling function denotes by $\lceil . \rceil$.

Recently, some researchers have developed several approaches to approximate the solution of this equation. For example, the authors of [26] in 2016 solved the above problem by employing hybrid functions which consists of Bernoulli polynomials and block-pulse functions. Also, an approach that uses hybrid of Taylor polynomials and block-pulse functions can be found in [21]. Moreover, very recently, this equation solved in [40] with Legendre wavelets method.

In the present paper an effective computational algorithm for solving Eqs. (1.1) and (1.2) is proposed. In our technique $u(t)$ is extended by shifted Chebyshev polynomials with unknown coefficients. For approximation the integral in Eq. (1.1) the Gauss-Legendre quadrature is used. Also, the Caputo fractional derivation for shifted-Chebyshev polynomials is given. Finally, by using collocation method together with the properties of Chebyshev polynomials the solution of Eqs. (1.1) and (1.2) reduce to the solution of algebraic equations.

This paper has been organized as follows: In section 2 some mathematical preliminaries of the fractional calculus and Chebyshev polynomials which are required for our subsequent development are given. In section 3, we obtain numerical method for solving the DFDEs given in Eqs. (1.1) and (1.2). In section 4, the error bound of the modified equation for the present method is given. Finally in section 5, we solve some examples by the proposed method.

2. Notations and Mathematical preliminaries

2.1. **Preliminaries in fractional calculus.**

Definition 2.1. The Riemann-Liouville fractional integral operator of order $q \geq 0$ is defined as [32]

$$\begin{aligned} J^q u(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-z)^{q-1} u(z) dz = \frac{1}{\Gamma(q)} t^{q-1} * u(t), \ t > 0, \ q > 0, \\ J^0 u(t) &= 1, \end{aligned} \tag{2.1}$$

where $\Gamma(.)$ is the Gamma function and the symbol $*$ means the convolution product.

Definition 2.2. Let $\alpha > 0, m \in \mathbb{N}$ and $m-1 < \alpha \leq m$. The Caputo fractional derivative with order α is defined as [32]:

$$D^\alpha u(t) = J^{m-\alpha} D^m u(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-z)^{m-\alpha-1} u^{(m)}(z) dz, \quad t > 0, \tag{2.2}$$

Clearly, D^α is a linear operator and satisfies the following properties [32]:

$$D^\alpha K = 0, \quad (K \text{ is a constant}). \tag{2.3}$$

$$D^\alpha t^j = \begin{cases} 0, & j < \lceil \alpha \rceil, \\ \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} t^{j-\alpha}, & j \geq \lceil \alpha \rceil. \end{cases}, \ j \in \mathbb{N}. \tag{2.4}$$

Definition 2.3. The fractional differential operator of distributed order for a function $u(t)$ with respect to nonnegative weight function $\rho(\alpha)$ is given by [20]

$$D^{\rho(\alpha)} u(t) = \int_{\beta_1}^{\beta_2} \rho(\alpha) D^\alpha u(t) d\alpha, \tag{2.5}$$

where β_1 and β_2 are positive numbers. Clearly, $D^{\rho(\alpha)}$ is a linear operator, and we have:

$$D^{\rho(\alpha)} K = 0, \quad (K \text{ is a constant}). \tag{2.6}$$

2.2. **Preliminary of Chebyshev polynomials.**

The Chebyshev polynomials can be determined from three-term recurrence formula as [9]:

$$\widetilde{T}_{i+1}(x) = 2x\widetilde{T}_i(x) - \widetilde{T}_{i-1}(x), \ x \in [-1,1], \quad i = 1, 2, \ldots$$

where $\widetilde{T}_0(x) = 1, \ \widetilde{T}_1(x) = x$. Alos, the analytic form of $\widetilde{T}_i(x)$ is given by

$$\widetilde{T}_i(x) = i \sum_{j=0}^{[\frac{i}{2}]} (-1)^j 2^{i-2j-1} \frac{(i-j-1)!}{j!(i-2j)!} x^{i-2j}. \tag{2.7}$$

For the sake of using $\widetilde{T}_i(x)$ on $[0, L]$, we use the transformation $x = \frac{2t}{L} - 1$. Let the shifted Chebyshev polynomials $\widetilde{T}_i(\frac{2t}{L} - 1)$ be denoted by $T_i^*(t)$. In this form $T_i^*(t)$ satisfy the recurrence relation:

$$T_{i+1}^*(t) = 2\left(\frac{2t}{L} - 1\right) T_i^*(t) - T_{i-1}^*(t), \ i = 1, 2, \ldots$$

where $T_0^*(t) = 1$ and $T_1^*(t) = \frac{2t}{L} - 1$. The orthogonality condition is

$$\int_0^L T_j^*(t)T_k^*(t)w_L(t)dt = \begin{cases} \pi, & j = k = 0, \\ \frac{\pi}{2} & j = k \neq 0 \\ 0 & j \neq k, \end{cases},$$

where $w_L(t) = \frac{1}{\sqrt{Lt-t^2}}$. Also, $T_i^*(t)$ has the following explicit expression [17]

$$T_i^*(t) = i\sum_{k=0}^{i}(-1)^{i-k}\frac{(i+k-1)!2^{2k}}{(i-k)!(2k)!L^k}t^k. \tag{2.8}$$

Note that $T_i^*(0) = (-1)^i$ and $T_i^*(L) = 1$. In this paper, for simplicity, we assume $L = 1$.

2.2.1. *Function approximation.*

An arbitrary function $u(t) \in L^2[0,1]$ may be approximated by shifted Chebyshev polynomials as [9]

$$u(t) \simeq \sum_{i=0}^{m} c_i T_i^*(t) = C^T\Phi(t), \tag{2.9}$$

where

$$C^T = [c_0, c_1, \dots, c_m], \quad \Phi(t) = [T_0^*(t), T_1^*(t), \dots, T_m^*(t)]^T. \tag{2.10}$$

The coefficients c_j are given by [9]

$$c_j = \frac{1}{h_j}\int_0^1 u(\vartheta)T_j^*(\vartheta)\frac{1}{\sqrt{\vartheta-\vartheta^2}}d\vartheta, \; j = 0,1,2,\dots,m, \tag{2.11}$$

where $h_j = \frac{\epsilon_j}{2}\pi, \epsilon_0 = 2, \epsilon_j = 1, j \geq 1$. Also, the derivative of $\Phi(t)$ is given by [17]

$$\frac{d\Phi(t)}{dt} = \mathbf{D}^{(1)}\Phi(t), \tag{2.12}$$

where

$$\mathbf{D}^{(1)} = d_{rs} = \begin{cases} \frac{4r}{\epsilon_s}, & \text{for } s = 0,1,\dots, r = s+\ell, \quad \begin{cases} \ell = 1,3,...,m, & \text{if } m \text{ odd}, \\ \ell = 1,3,...,m-1, & \text{if } m \text{ even}, \end{cases} \\ 0, & \text{otherwise}. \end{cases}$$

Here, $\mathbf{D}^{(1)}$ is the operational matrix of derivative. It is obvious that, by using Eq. (2.12), we have

$$\frac{d^n\Phi(t)}{dt^n} = \mathbf{D}^{(n)}\Phi(t), \quad n \in \mathbb{N}, \tag{2.13}$$

where $\mathbf{D}^{(n)} = (\mathbf{D}^{(1)})^n$. Also, for the the shifted Chebyshev vector $\Phi(t)$ we have [3]

$$\Phi(t) = \mathbf{A}T_m(t), \tag{2.14}$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & 0 & \cdots & 0 \\ 2(-1)^2 \frac{1!}{2!} & 2(-1)^1 \frac{2^2 2!}{2!} & 2(-1)^0 \frac{2^4 3!}{4!} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m(-1)^m \frac{(m-1)!}{m!} & m(-1)^{m-1} \frac{2^2 (m)!}{2!(m-1)!} & m(-1)^{m-2} \frac{2^4 (m+1)!}{4!(m-2)!} & \cdots & m(-1)^0 \frac{2^{2m}(2m-1)!}{(2m)!} \end{pmatrix}, \tag{2.15}$$

and $T_m(t) = [1, t, t^2, \dots, t^m]^T$.

2.3. **Legendre-Gauss quadrature.**

Let S be an arbitrary positive integer. The Legendre-Gauss quadrature rule on the interval (τ_1, τ_2) is [19, 34]

$$\int_{\tau_1}^{\tau_2} u(t)dt \simeq \sum_{q=1}^{S} \omega_q u(\sigma_q), \tag{2.16}$$

where

$$\sigma_q = \frac{\tau_2 - \tau_1}{2}\zeta_q + \frac{\tau_2 + \tau_1}{2}, \quad \omega_q = \frac{\tau_2 - \tau_1}{(1-\zeta_q^2)(L'_S(\zeta_q))^2}, \quad q = 1, ..., S.$$

Here, $\{\zeta_1, \zeta_2, \dots, \zeta_S\}$ denotes the S roots of the Legendre polynomial $L_S(x)$. Note that, if $u(t)$ is a polynomial of degree $\leq 2S - 1$ then the quadrature given in Eq. (2.16) is exact. [19].

3. Numerical solution of problem (1.1)-(1.2)

To solve problem (1.1)-(1.2), we use the shifted Chebyshev polynomials for approximation of $u(t)$ as:

$$u(t) \simeq \sum_{i=0}^{m} c_i T_i^*(t) = C^T \Phi(t), \tag{3.1}$$

where $C = [c_0, c_1, \dots, c_m]^T$ is unknown vector. Employing Eq. (2.14), Eq. (3.1) can be written as

$$u(t) \simeq C^T \mathbf{A} T_m(t). \tag{3.2}$$

Now, using Eqs. (2.4), (2.14) and (3.2) we get

$$\begin{aligned}
\frac{d^{\alpha}(u(t))}{dt^{\alpha}} &\simeq C^T\mathbf{A}\frac{d^{\alpha}}{dt^{\alpha}}(T_m(t)) \\
&= C^T\mathbf{A}\begin{bmatrix}0 & \frac{\Gamma(2)}{\Gamma(2-\alpha)}t^{1-\alpha} & \frac{\Gamma(3)}{\Gamma(3-\alpha)}t^{2-\alpha} & \cdots & \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}t^{m-\alpha}\end{bmatrix}^T \\
&= C^T\mathbf{A}\begin{bmatrix}0 & 0 & 0 & \cdots & 0\\ 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha)} & 0 & \cdots & 0\\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-\alpha)} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}\end{bmatrix}\begin{bmatrix}1\\ t^{1-\alpha}\\ t^{2-\alpha}\\ \vdots\\ t^{m-\alpha}\end{bmatrix} \\
&= C^T\mathbf{A}\mathbf{M}_{\alpha}\bar{T}_{m,\alpha}(t),
\end{aligned} \tag{3.3}$$

where

$$\mathbf{M}_{\alpha} = \begin{bmatrix}0 & 0 & 0 & \cdots & 0\\ 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha)} & 0 & \cdots & 0\\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-\alpha)} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}\end{bmatrix} \text{ and } \bar{T}_{m,\alpha}(t) = \begin{bmatrix}1\\ t^{1-\alpha}\\ t^{2-\alpha}\\ \vdots\\ t^{m-\alpha}\end{bmatrix}.$$

If we employ Eqs. (3.2) and (3.3), then Eq. (1.1) can be written as

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2} H_1(\alpha, C^T\mathbf{A}\mathbf{M}_{\alpha}\bar{T}_{m,\alpha}(t))d\alpha + \\
&H_2\left(t, C^T\mathbf{A}T_m(t), C^T\mathbf{A}\mathbf{M}_{\gamma_1}\bar{T}_{m,\gamma_1}(t), \cdots, C^T\mathbf{A}\mathbf{M}_{\gamma_r}\bar{T}_{m,\gamma_r}(t)\right) = g(t).
\end{aligned} \tag{3.4}$$

First, we evaluate the integral in Eq. (3.4) by using the Legendre-Gauss quadrature rule (2.16). Then, we collocate Eq. (3.4) at $m-\ell+1$ points t_i. Therefore, from Eq. (3.4), for $i = 1, 2, \dots, m-\ell+1$ we get

$$\begin{aligned}
&\sum_{j=1}^{S}\omega_j H_1\left(\sigma_j, C^T\mathbf{A}\mathbf{M}_{\sigma_j}\bar{T}_{m,\sigma_j}(t_i)\right) + \\
&H_2\left(t_i, C^T\mathbf{A}T_m(t_i), C^T\mathbf{A}\mathbf{M}_{\gamma_1}\bar{T}_{m,\gamma_1}(t_i), \cdots, C^T\mathbf{A}\mathbf{M}_{\gamma_r}\bar{T}_{m,\gamma_r}(t_i)\right) = g(t_i).
\end{aligned} \tag{3.5}$$

To do this, we use the zeros of $T^*_{m-\ell+1}(t)$ as a collocation points, i.e.

$$t_i = 0.5 + 0.5\cos\left(\frac{(2i-1)\pi}{2(m-\ell+1)}\right), \quad i = 1, 2, \dots, m-\ell+1.$$

Moreover, by substituting Eqs. (2.13), (2.14) and (3.2) in initial conditions (1.2), we obtain

$$C^T\mathbf{A}T_m(0) = u_0, \tag{3.6}$$

$$C^T\mathbf{D}^{(j)}\mathbf{A}T_m(0)=u_0^{(j)},\quad j=1,2,\dots \ell-1. \tag{3.7}$$

Eq. (3.5) together with Eqs. (3.6) and (3.7) generate a system of $(m+1)$ linear/nonlinear algebraic equations. The unknown vector C can be obtained by solving this system of algebraic equations. In this paper we used *fsolve* command in Maple for solving this system. Therefore, by Eq. (3.2), $u(t)$ can be calculated.

4. Error bounds

Let N be any positive integer, $I=(0,1)$ and set $P_N(I)=\text{span}\{T_0^*(t),T_1^*(t),\dots,T_N^*(t)\}$. Also,we define $\Pi_N u$ from $L^2(I)$ into $P_N(I)$ by

$$(\Pi_N u-u,z)=0,\quad \forall z\in P_N(I),$$

equivalently,

$$(\Pi_N u)(t)=\sum_{j=0}^{N}a_jT_j^*(t).$$

In fact, $\Pi_N u$ is the best approximation of u out of $P_N(I)$ [37]. Following [8, 37], to obtain the truncation error $u(t)-\Pi_N u(t)$, for each $m\in\mathbb{N}$, we define the Chebyshev-weighted Sobolev space $B^m(I)$ as:

$$B^m(I)=\left\{u:\frac{\partial^k u}{\partial t^k}\in L^2(I),\quad k=0,1,...,m\right\}.$$

The inner product, semi-norm and norm associated with $B^m(I)$ are

$$(u,z)_{B^m}=\sum_{j=0}^{m}\left(\frac{\partial^j u}{\partial t^j},\frac{\partial^j z}{\partial t^j}\right),\quad |u|_{B^m}=\left\|\frac{\partial^m u}{\partial t^m}\right\|,\quad \|u\|_{B^m}=(u,u)_{B^m}^{\frac{1}{2}}.$$

As pointed by [37], this space identifies itself from the ordinary weighted Sobolev space $H^m(I)$ along with distinct weight functions for derivatives of various orders. Also, $H^m(I)$ is a subspace of $B^m(I)$, and we have

$$\|u\|_{B^m}\le c\|u\|_{H^m},\qquad m\ge 0.$$

The error $u(t)-\Pi_N u(t)$ can be estimated as follows:

Theorem 4.1. *([9]) For $m\ge 0$ and all $u\in H^m(I)$ we have*

$$\|u(t)-\Pi_N u(t)\|_{L^2(I)}\le CN^{-m}\|u\|_{H^m}.$$

Also, in the sequel, we need the following theorem:

Theorem 4.2. *([8]) If $0\le p<m\le N+1$, then for any $u\in B^m(I)$*

$$\begin{aligned}\|\partial_t^p(u-\Pi_N u)\| &\le C_1\sqrt{\frac{(N-m+1)!}{(N-p+1)!}}(N+m)^{\frac{(p-m)}{2}}\|\partial_t^m u\|\\ &\le C_1\sqrt{\frac{(N-m+1)!}{(N-p+1)!}}(N+m)^{\frac{(p-m)}{2}}\|u\|_{B^m}.\end{aligned}$$

Moreover, in Hilbert space

$$\|\partial_t^p(u-\Pi_N)\| \le C\sqrt{\frac{(N-m+1)!}{(N-p+1)!}}(m+N)^{\frac{(p-m)}{2}}\|u\|_{H^m}.$$

Here, C and C_1 are some constant numbers.

We next estimate the error in the approximation of D^α.

Theorem 4.3. *If $n_\alpha - 1 < \alpha \le n_\alpha$, $n_\alpha < r \le N+1$ and $u \in H^r(I)$ where $r \in \mathbb{N}$, then*

$$\|D^\alpha u - D^\alpha(\Pi_N(u))\|_{L^2(I)} \le \left(\frac{C_\alpha}{\Gamma(n_\alpha+1-\alpha)}\right)\left(\sqrt{\frac{(N+1-r)!}{(N+1-n_\alpha)!}}(N+r)^{\frac{(n_\alpha-r)}{2}}\right)\|u\|_{H^r}.$$

Proof. Employing Eq. (2.1), Theorem 4.2 and the following relation [8]

$$\|f*g\|_p \le \|f\|_1\|g\|_p,$$

we obtain

$$\begin{aligned}
&\|D^\alpha u - D^\alpha(\Pi_N u)\|^2_{L^2(\Omega)} = \|I^{n_\alpha-\alpha}\big(D^{n_\alpha}u - D^{n_\alpha}(\Pi_N u)\big)\|^2_{L^2(I)} \\
= &\left\|\frac{t^{n_\alpha-\alpha-1}}{\Gamma(n_\alpha-\alpha)} * (D^{n_\alpha}u - D^{n_\alpha}(\Pi_N u)\right\|^2_{L^2(I)} \\
\le &\left\|\frac{t^{n_\alpha-\alpha-1}}{\Gamma(n_\alpha-\alpha)}\right\|^2_1 \|D^{n_\alpha}u - D^{n_\alpha}(\Pi_N)\|^2_{L^2(I)} \\
\le &\left(\frac{1}{\Gamma(n_\alpha+1-\alpha)}\right)^2\left(C_\alpha\sqrt{\frac{(N+1-r)!}{(N+1-n_\alpha)!}}(N+r)^{\frac{(n_\alpha-r)}{2}}\right)^2\|u\|^2_{H^r}.
\end{aligned}$$

This completes the proof. □

We are now ready to obtain the bound of the modified equation for our technique. Let us define

$$\Theta_1(u(t)) = \int_{\tau_1}^{\tau_2} H_1(\alpha, D^\alpha u(t))d\alpha + H_2(t,u(t),D^{\gamma_i}u(t)) - g(t), \tag{4.1}$$

$$\Theta_2(u(t)) = \sum_{j=1}^{S} w_j H_1(\sigma_j, D^{\sigma_j}u(t)) + H_2(t,u(t),D^{\gamma_i}u(t)) - g(t). \tag{4.2}$$

Moreover, we define the residual function $\text{Res}_N^S(u)$ of the approximation $\Pi_N u$ for the exact solution u in Eq. (1.1) as:

$$\text{Res}_N^S(u) = \Theta_2(\Pi_N u). \tag{4.3}$$

Theorem 4.4. *Let $u \in B^q(I)$ with $q > 0$. Also, assume that both H_1 and H_2 in Eq. (1.1) are lipschitz functions, with constants μ_1 and μ_2 respectively, then*

$$\begin{aligned} \|Res_N^S(u)\|_2 &\leq \frac{C_2\pi}{4^S} + \mu_2 C N^{-q}\|u\|_{H^q} \\ &+ \mu_1 \sum_{j=1}^{S} \frac{w_j C_{\sigma_j}}{\Gamma(n_{\sigma_j}-\sigma_j+1)} \left\{ \sqrt{\frac{(N-q+1)!}{(N-n_{\sigma_j}+1)!}} (N+q)^{\frac{(n_{\sigma_j}-q)}{2}} \right\} \|u\|_{H^q} \\ &+ \mu_2 \sum_{i=1}^{r} \frac{C_{\gamma_i}}{\Gamma(n_{\gamma_i}-\gamma_i+1)} \left\{ \sqrt{\frac{(N-q+1)!}{(N-n_{\gamma_i}+1)!}} (N+q)^{\frac{(n_{\gamma_i}-q)}{2}} \right\} \|u\|_{H^q}, \end{aligned}$$

where $n_{\sigma_j} - 1 < \sigma_j \leq n_{\sigma_j}$, $j = 1, ..., S$ and $n_{\gamma_i} - 1 < \gamma_i \leq n_{\gamma_i}$, $i = 1, ..., r$. Also, C, C_2, C_{σ_j} and C_{γ_i} are constant numbers.

Proof. From Eqs. (4.1)-(4.3) we have

$$\begin{aligned} \|\text{Res}_N^S(u)\|_2 &= \|0 - \text{Res}_N^S(u)\|_2 = \left\|\Theta_1(u) - \text{Res}_N^S(u)\right\|_2 \\ &\leq \|\Theta_1(u) - \Theta_2(u)\|_2 + \left\|\Theta_2(u) - \text{Res}_N^S(u)\right\|_2. \end{aligned} \tag{4.4}$$

$\Theta_1(u) - \Theta_2(u)$ is the error for applying quadrature rule and we have (see [34])

$$\| \Theta_1(u) - \Theta_2(u) \|_2 \leq \frac{C_2\pi}{4^S}, \tag{4.5}$$

where

$$C_2 = \max\left\{ \left| \frac{\partial^{\,2S}}{\partial\alpha^{2S}} H_1(\alpha, D^\alpha u(t)) \right|, \quad t \in [0,1], \ \tau_1 < \alpha < \tau_2 \right\}.$$

Also, since H_1 and H_2 satisfy a Lipschitz condition, we get

$$\begin{aligned} \|\Theta_2(u) - \text{Res}_N^S(u)\|_{L^2(I)} &\leq \sum_{j=1}^{S} w_j \mu_1 \|D^{\sigma_j}(u - u_N)\|_{L^2(I)} \\ &+ \mu_2 \|u - u_N\|_{L^2(I)} \\ &+ \mu_2 \sum_{i=1}^{r} \|D^{\gamma_i}(u - u_N)\|_{L^2(I)}. \end{aligned} \tag{4.6}$$

Let

$$\max\{n_{\sigma_j}, j = 1, ..., S\} \leq q_1 < N+1, \text{ and } \max\{n_{\gamma_i}, i = 1, ..., r\} \leq q_2 < N+1.$$

Also, let $q = \max\{q_1, q_2\}$. By using Theorems 4.1, 4.2, 4.3 and Eq. (4.6), we obtain

$$\begin{aligned} &\|\Theta_2(u) - \text{Res}_N^S(u)\|_{L^2(I)} \leq \mu_2 C N^{-q}\|u\|_{H^q} \\ &+ \mu_1 \sum_{j=1}^{S} \frac{w_j C_{\sigma_j}}{\Gamma(n_{\sigma_j}-\sigma_j+1)} \left\{ \sqrt{\frac{(N-q+1)!}{(N-n_{\sigma_j}+1)!}} (N+q)^{\frac{(n_{\sigma_j}-q)}{2}} \right\} \|u\|_{H^q} \\ &+ \mu_2 \sum_{i=1}^{r} \frac{C_{\gamma_i}}{\Gamma(n_{\gamma_i}-\gamma_i+1)} \left\{ \sqrt{\frac{(N-q+1)!}{(N-n_{\gamma_i}+1)!}} (N+q)^{\frac{(n_{\gamma_i}-q)}{2}} \right\} \|u\|_{H^q}. \end{aligned} \tag{4.7}$$

Then, the desired result follows from Eqs. (4.4), (4.5) and (4.7). ☐

5. Illustrative examples

In this part, we consider four test problems in order to illustrate the efficiency of our numerical approach.

Example 1. Consider the following DFDE [21, 22]

$$\int_{0.2}^{1.5} \Gamma(3-\alpha) D^{\alpha} u(t) d\alpha = 2\left(\frac{t^{1.8}-t^{0.5}}{\ln t}\right), \; u(0)=u'(0)=0.$$

The exact solution is $u_{exact}(t) = t^2$. This problem was solved for $m = 2$ and $S = 7$ using the method described in section 3. So, we approximate $u(t)$ as

$$u(t) \simeq c_0 T_0^*(t) + c_1 T_1^*(t) + c_2 T_2^*(t) = C^T \Phi(t),$$

where $C^T = [c_0, c_1, c_2]$ is unknown vector and $\Phi(t) = [T_0^*(t), T_1^*(t), T_2^*(t)]^T$. From Eq. (2.14) we obtain

$$u(t) = C^T A T_2(t), \tag{5.1}$$

and $D^{\alpha}(u(t)) = C^T A M_{\alpha} \bar{T}_{2,\alpha}(t)$, where

$$T_2(t) = \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix}, \; \bar{T}_{2,\alpha}(t) = \begin{bmatrix} 1 \\ t^{1-\alpha} \\ t^{2-\alpha} \end{bmatrix}, \; M_{\alpha} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\Gamma(2-\alpha)} & 0 \\ 0 & 0 & \frac{2}{\Gamma(3-\alpha)} \end{bmatrix}, \; A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & -8 & 8 \end{bmatrix}.$$

By collocating Eq. (3.5) at $t_1 = 0.5$, we get

$$2.5534156946354227\, c_1 - 0.52031731391918894\, c_2 - 1.2116681830778127 = 0, \tag{5.2}$$

also, applying the initial conditions we have

$$c_0 - c_1 + c_2 = 0, \tag{5.3}$$

$$2c_1 - 8c_2 = 0. \tag{5.4}$$

Finally, by solving Eqs. (5.2)-(5.4) we obtain

$$c_0 \simeq 0.37500000000000000005, \; c_1 \simeq 0.50000000000000000007,$$

$$c_2 \simeq 0.12500000000000000001.$$

Thereby, using Eq. (5.1), we get $u(t) \simeq 1.00000000000000000001\, t^2$. The curve of absolute error function $|u(t) - u_{exact}(t)|$ is shown in Figure 1. Also, in Table 1 we give the values of L^2-errors for $m = 2$ and various S.

Table 1. L^2-errors with $m = 2$ and various S for Example 1.

S	2	3	4	5	6	7
L^2-error	6.65×10^{-5}	1.15×10^{-7}	1.06×10^{-10}	6.04×10^{-14}	2.33×10^{-17}	6.55×10^{-21}

Example 2. Consider the nonlinear DFDE [21, 40]:

$$\int_0^1 (\Gamma(4-\alpha) D^{\alpha} u(t))^2 d\alpha = \frac{18t^4(t^2-1)}{(\ln t)}, \quad u(0) = 0.$$

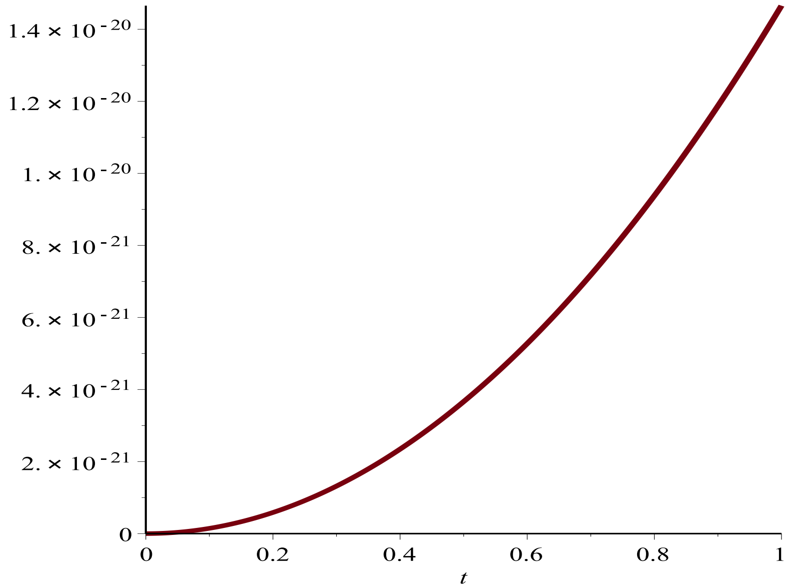

FIGURE 1. Plot of $|u(t) - u_{exact}(t)|$ with $S = 7$ and $m = 2$, for Example 1.

The exact solution is $u_{exact}(t) = t^3$. Applying the method of Section 3 with $m = 4$ and $S = 3$, we obtain

$$c_0 \simeq 0.312502,\ c_1 \simeq 0.468750,\ c_2 \simeq 0.187498,$$

$$c_3 \simeq 0.031250,\ c_4 \simeq -1.319035 \times 10^{-7}.$$

Thus using Eq. (3.2) we obtain $u(t) \simeq 1.00005661\, t^3$. Table 2 shows the L^2-errors for $m = 4, 7$ and various S.

TABLE 2. L^2-errors for $m = 4, 7$ and various S for Example 2.

S	2	3	4	5	6
$m = 4$	1.50×10^{-4}	2.69×10^{-6}	7.03×10^{-8}	1.85×10^{-9}	3.61×10^{-11}
$m = 7$	1.53×10^{-4}	3.22×10^{-6}	6.17×10^{-8}	1.17×10^{-9}	2.53×10^{-11}

Example 3. Consider the Bagly-Torvik equation of the following form [33, 40]

$$D^{(2)}u(t) + D^{\rho(\alpha)}u(t) + u(t) = g(t), \quad u(0) = u'(0) = 0,\ t \in [0, \tau]. \tag{5.5}$$

Here,

$$g(t) = \begin{cases} 8, & 0 \leq t \leq 1 \\ 0, & t > 1 \end{cases}, \qquad \text{and} \quad \rho(\alpha) = 6\alpha(1-\alpha), \quad 0 \leq \alpha \leq 1.$$

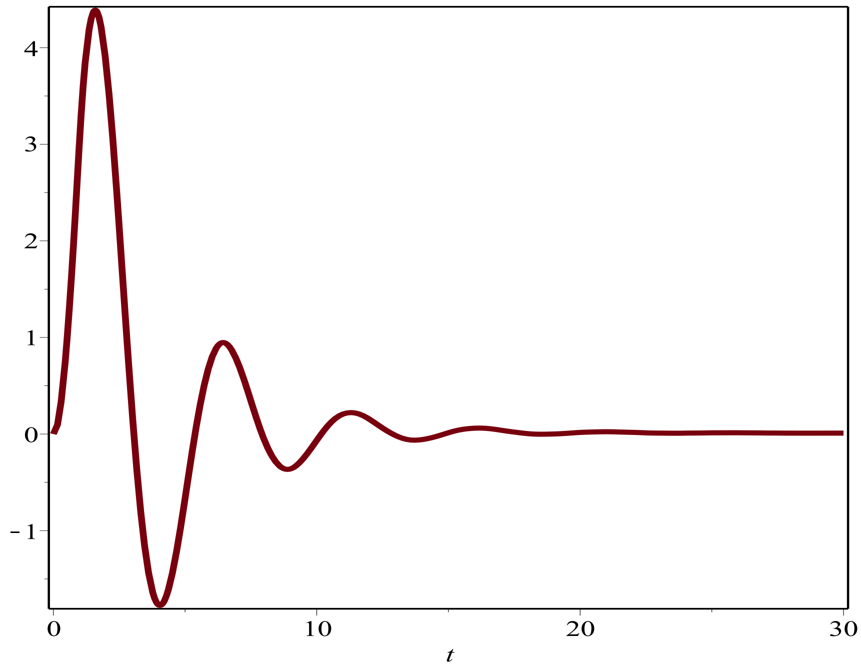

FIGURE 2. The curve of $u(t)$ for $t \in [0, 30]$, for Example 3.

By virtue of Eq. (2.5), we rewrite Eq. (5.5) as:

$$u^{''}(t) + \int_0^1 \rho(\alpha) D^{\alpha} u(t) d\alpha + u(t) = g(t). \tag{5.6}$$

Also, by using the transformation $s = t/\tau$, Eq. (5.6) may then be restated as

$$\frac{1}{\tau^2} u^{''}(s) + \int_0^1 \rho(\alpha) \frac{1}{\tau^{\alpha}} D^{\alpha} u(s) d\alpha + u(s) = g(s), \quad s \in [0, 1]$$

where

$$g(s) = \begin{cases} 8, & 0 \leq s \leq \frac{1}{\tau} \\ 0, & s > \frac{1}{\tau} \end{cases}.$$

In Figure 2 the curve of $u(t)$ with $m = 80$ is plotted. Figure 2 has very good agrement with the result obtained in [33, 40].

Example 4. In this test, we consider the mathematical model that relates the fractional distributed order oscillator [21, 22, 23, 26, 38, 40]

$$\frac{d^{(2)} u(t)}{dt^2} + w^2 u(t) + \sigma(t) = G(t), \quad u(0) = u^{'}(0) = 0, \tag{5.7}$$

$$\int_0^1 \Phi(\alpha) D^{\alpha} \sigma(t) d\alpha = \lambda \int_0^1 \Psi(\alpha) D^{\alpha} u(t) d\alpha. \tag{5.8}$$

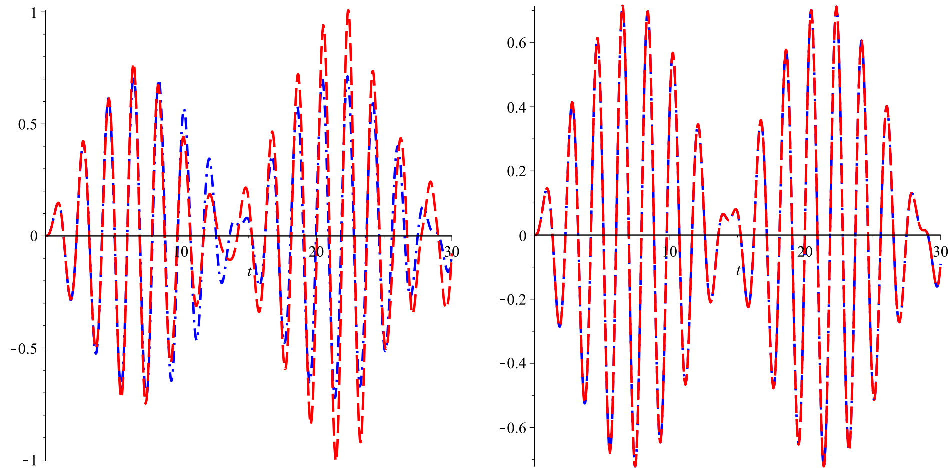

FIGURE 3. Plot of the exact and approximation solution with $m = 55$ (left) and $m = 65$ (right) and $S = 2$ for Example 4.

Here, $u(t), \sigma(t)$ and $G(t)$ respectively represent the displacement, the dissipation force and the external forcing function. Also, w represents the eigenfrequency of the undamped system. We also assume that $\Phi(\alpha) = a^\alpha,\ \Psi(\alpha) = b^\alpha$ and $G(t) = G_0 \sin(\Omega t)$, where G_0, a, b, λ are some constants. If $a = b$, the solution $u(t)$ of this problem is identical to the elastic with $w_{el} = \sqrt{w^2+1} = \sqrt{10}$ and is given by [40]

$$u_{exact}(t) = \frac{G_0}{w_{el}^2 - \Omega}(\sin(\Omega t) - \frac{\Omega}{w_{el}} \sin(w_{el} t)).$$

Following [23], we can convert the system of Eqs. (5.7) and (5.8) into a single DFDE as:

$$\int_0^1 \left\{ \Phi(\alpha) D^{\alpha+2} u(t) + Z(\alpha) D^\alpha u(t) \right\} d\alpha = f(t), \quad t \in [0, \tau],$$

where

$$Z(\alpha) = w^2 \Phi(\alpha) + \lambda \Psi(\alpha), \quad \text{and } f(t) = \int_0^1 \Phi(\alpha) D^\alpha G(t) d\alpha.$$

Using the transformation $s = t/\tau$ yield

$$\int_0^1 \{\Phi(\alpha) \frac{D^{\alpha+2}}{\tau^{\alpha+2}} u(s) + Z(\alpha) \frac{1}{\tau^\alpha} D^\alpha u(s)\} d\alpha = \int_0^1 \Phi(\alpha) \frac{1}{\tau^\alpha} D^\alpha G(s) d\alpha,\ s \in [0,1].$$

We use particular values of $w = 3, \Omega = 1.2w, G_0 = 1$ and $\lambda = 1$. Figure 3 shows the exact and approximation solution for $m = 55, 65$ and $S = 2$. Also, in Figure 4 we display the absolute error for $m = 65, 75$ and $S = 2$.

Moreover, in Table 3, we compare absolute error of the the presented method by selecting $S = 2$ and $\hat{m} = m + 1$ number of bases together with the result obtained by using the Legendre wavelets method given in [40] (witch we denote as Method 1) and

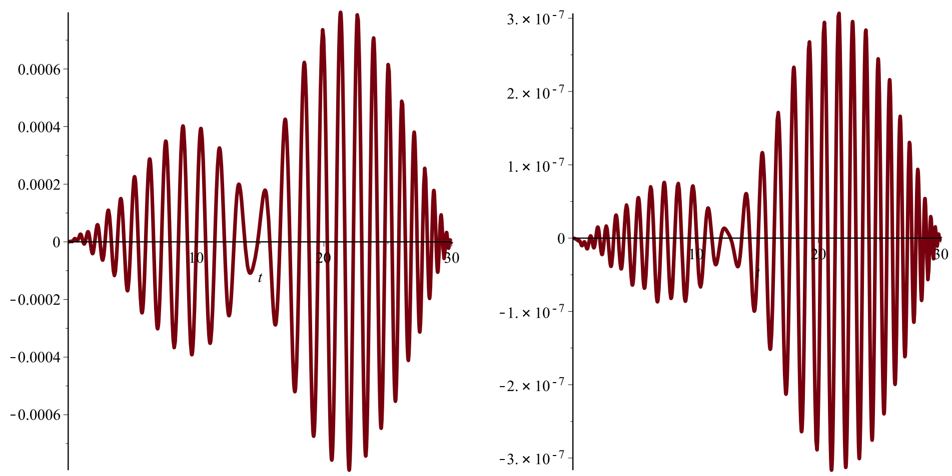

FIGURE 4. Graph of $|u(t)-u_{exact}(t)|$ with $m=65$ (left) and $m=75$ (right) and $S=2$ for Example 4.

TABLE 3. Comparison of absolute error of u in [0, 10] in Example 4

t	Present method			Method 1([40])	Method 2([21])
	$\hat{m}=41$	$\hat{m}=51$	$\hat{m}=71$	$k=5, M=10$ $(\hat{m}=2^{k-1}M=160)$	$N=10, M=10$ $(\hat{m}=N(M+1)=110)$
1	1.25×10^{-12}	1.72×10^{-19}	6.79×10^{-33}	3.3×10^{-12}	1.3×10^{-11}
2	2.15×10^{-12}	3.84×10^{-19}	1.73×10^{-34}	4.1×10^{-11}	1.3×10^{-10}
3	6.12×10^{-12}	1.31×10^{-19}	6.46×10^{-33}	4.9×10^{-11}	6.8×10^{-10}
4	4.03×10^{-12}	8.25×10^{-19}	5.16×10^{-36}	5.6×10^{-11}	1.1×10^{-9}
5	1.17×10^{-11}	1.55×10^{-18}	6.94×10^{-33}	6.4×10^{-11}	2.1×10^{-9}
6	1.54×10^{-11}	1.95×10^{-18}	2.94×10^{-34}	7.1×10^{-11}	4.1×10^{-9}
7	1.52×10^{-11}	2.38×10^{-18}	6.57×10^{-33}	8.3×10^{-11}	7.6×10^{-10}
8	1.18×10^{-12}	3.60×10^{-19}	1.10×10^{-34}	8.2×10^{-11}	1.2×10^{-8}
9	1.03×10^{-11}	1.06×10^{-18}	6.70×10^{-33}	9.1×10^{-11}	1.0×10^{-7}

the hybrid of block-pulse function with Taylor polynomials given in [21] (witch we denote as Method 2). It is important to notice that, in Table 3, the number of basis for the methods given in [40] and [21] are $(\hat{m}=2^{k-1}M=160)$ and $(\hat{m}=N(M+1)=110)$ respectively. From this table we see that the present method is clearly reliable if compared with the Legendre wavelets and hybrid methods.

6. Conclusion

In the present paper, the fundamental aim is to apply Chebyshev polynomials to reduce the solution of linear and nonlinear DFDEs with initial conditions to the solution of algebraic equations. In Section 4 the error bounds for fractional derivative and residual function was obtained. Also, the results obtained from our technique show in figures and tables, presented this approach is efficient and reliable.

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