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Solving fractional optimal control problems using Genocchi polynomials

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Abstract In this paper, we solve a class of fractional optimal control problems in the sense of Caputo derivative using Genocchi polynomials. At first we present some properties of these polynomials and we make the Genocchi operational matrix for Caputo fractional derivatives. Then using them, we solve the problem by converting it to a system of algebraic equations. Some examples are presented to show the efficiency and accuracy of the method.

Keywords. Optimal control problems, Caputo fractional derivative, Genocchi polynomials, Operational matrix.

2010 Mathematics Subject Classification. 65L05, 34K06, 34K28.

1. Introduction

In recent years, optimal control problems(OCPs) have become the favorite problem of many researchers. However, in the field of optimal control of fractional order systems, there is very little work to be done. Therefore, numerical methods have been developed day by day to obtain better results. A linear solver is used to obtain the fractional optimal control problems (FOCPs) using the Volterra-type integral equations in [24]. Ashpazzadeh et. al [7] used Hermite spline multiwavelets for solving the FOCPs of the Caputo type. Also in [9] multidimensional fractional optimal control problems with inequality constraint have been solved by using multiwavelets. A formulation for the OCP to a class of fuzzy fractional differential systems relating to SIR and SEIR epidemic models are presented in [14]. Nemati et. al [39] gave an approach

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for solving FOCPs using modified hat functions. A Closed-form solution of OCP of a fractional order system presented in [12].

A pseudo-state-space-approach is presented in [11] to solve FOCPs. Ghomojani[16] used Bezier curves to solve FOCPs. Lotfi et al. [33] used Legendre basis to numerically solve FOCPs. Using an extended modal series method and linear programming strategy the authors in [38] solved FOCPs. In [43] a solution of a class of FOCPs by Legendre wavelets is presented. A Bernoulli polynomial method for solving FOCPs with vector components are given in [44]. The authors in [34] gave a LMI stability test for fractional order initialized control systems.

In this article fractional optimal control problems in the sense of Caputo derivative is solved by using Genocchi polynomials. Using the operational matrix of derivative and the properties of Genocchi polynomials, the fractional optimal control problem is changed to a nonlinear programming one which can be solved by a suitable algorithm to get the result.

This article is organized as the following: In section 2, we describe the preliminary integration and fractional-order derivative. In section 3, we introduce the Genocchi polynomials and approximate arbitrary functions using the Genocchi polynomials and presenting properties of the Genocchi polynomials. In section 4, we discuss the operational matrix for the Caputo derivative with Genocchi polynomials. In section 5, we use the Genocchi polynomials method to solve the fractional optimal control problems and in the 6th section, we will solve the numerical examples with the proposed method.

2. Fractional-order Integration and Derivative

In this section, we provide some basic definitions and results for the fractional integration and derivative that will be used in the following.

Definition 2.1. According to what is defined in the [17, 37, 42], suppose $\alpha \in R^+$, the ${}_aI_t^\alpha$ operator on $L[a,b]$ is defined as

$$ {}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \tag{2.1}$$

which is called the Riemann-Liouville fractional integration operator.
As a contract, we assume the zero-order integral of each function is the function itself. Mean, for $\alpha = 0$

$$ {}_aI_t^0 f(t) = f(t).$$

Definition 2.2. From [17, 37, 42], if we assume $\alpha \in R^+$, ${}_a^{Rl}D_t^\alpha$ operator, which is called Riemann-Liouville fractional derivative of α-order of function $f(t)$, defined as

$$ {}_a^{RI}D_t^\alpha f(t) \triangleq \frac{d^m}{dt^m}\left\{\frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\tau)^{m-\alpha-1} f(\tau)\, d\tau\right\} \tag{2.2}$$

where m $= \lceil \alpha \rceil$. a and t denote the lower and upper bounds of fractional integration, respectively.

Definition 2.3. If $\alpha \in R^{+}$ is non-integer and $m = \lceil \alpha \rceil$, then the Caputo fractional derivative of order α is defined as [19]

$$ {}_a^C D_t^\alpha f(t) = {}_a I_t^{m-\alpha} \{ f^{(m)}(t) \}. \tag{2.3} $$

Using relation (2.1) in (2.3), the definition of the Caputo derivative is obtained as follows[13, 15]:

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\tau)^{m-\alpha-1} f^{(\alpha)}(\tau) d\tau. \tag{2.4} $$

It should be noted that the Caputo fractional-order derivative for a constant number is zero. It means that, if c is the constant number, then ${}_a^C D_t^\alpha c = 0$. Also, the Caputo fractional-order derivative operator is a linear operator. That is, for all real scalars λ and μ, and for all functions $f(t)$ and $g(t)$, we have

$$ {}_a^C D_t^\alpha (\lambda f(t) + \mu g(t)) = \lambda {}_a^C D_t^\alpha f(t) + \mu {}_a^C D_t^\alpha g(t). $$

3. Genocchi numbers and polynomials

The Genocchi numbers and polynomials are widely studied in mathematics and physics such as homotopy theory (stable Homotopy groups of spheres)[5], differential topology (differential structures on spheres), theory of modular forms (Eisenstein) and quantum physics (quantum Groups).
Genocchi numbers, G_n, and polynomials, $G_n(x)$, are respectively defined, by using exponential generating functions [3, 4, 6, 20] as:

$$ Q(t) := \frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \tag{3.1} $$

$$ Q(t,x) := \frac{2te^{tx}}{e^t+1} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, (|t| < \pi). \tag{3.2} $$

Therefore, the Genocchi polynomials of degree n is given by

$$ G_n(x) = \sum_{k=0}^{n} \binom{n}{k} G_{n-k} x^k, \tag{3.3} $$

where $G_{n-k},\ k = 0, 1, \ldots, n$ in Eq.(3.3) are the Genocchi numbers, that can be found [20, 21, 22] as

$$ \begin{aligned} &G_0 = 0, \\ &G_1 = 1, \\ &G_{2i} = 2iE_{2i-1}(0), \\ &G_{2i+1} = 0, (i \geq 1), \end{aligned} $$

where E_i is the Euler's numbers, which is defined [1, 4, 27] as:

$$E_{2i-1}(0) = \left(2^{1-2i}.\sum_{j=1}^{2i-1}\left((-1)^{2i+j}.j^{2i-1}.\left(\sum_{k=0}^{2i-1-j}\left(\begin{array}{c}2i\\k\end{array}\right)\right)\right)\right).$$

Using Eq. (3.3) and Genocchi numbers, we can construct the Genocchi polynomials. Some of them are as:

$$\begin{aligned}&G_0(x) = 0,\\&G_1(x) = 1,\\&G_2(x) = 2x - 1,\\&G_3(x) = 3x^2 - 3x,\\&G_4(x) = 5x^4 - 10x^3 + 5x.\end{aligned}$$

The Genocchi polynomials satisfies in the following relations [20]:

(1)

$$\int_0^1 G_n(x)\,G_m(x)\,dx = \frac{2(-1)^n n!m!}{(n+m)!}G_{n+m} \qquad , \qquad m,n \geq 1,$$

(2)

$$\frac{d}{dx}G_i(x) = iG_{i-1}(x), \qquad i \geq 1,$$

and

(3)

$$G_i(x+1) + G_i(x) = 2ix^{i-1}.$$

3.1. **Function approximation.** Suppose $\{G_0(t), G_1(t), \dots, G_N(t)\}$ is the set of Genocchi polynomials. From best approximation theorem in [8, 29] we can approximate an arbitrary function $f(t)$ on $L^2[0,1]$ as

$$f(t) \approx \sum_{n=0}^{N} c_n G_n(t) = C^T G(t), \tag{3.4}$$

where

$$G(t) = [G_0(t), G_1(t), \dots, G_N(t)]^T. \tag{3.5}$$

$$C = [c_1, c_2, \dots, c_N]^T,$$

and [23]

$$C = P^{-1} < f(t), G(t) >, \tag{3.6}$$

where

$$P =< G(t), G(t) >= \int_0^1 G(t)G^T(t)dx. \tag{3.7}$$

3.2. **Error bound of Genocchi polynomials approximation.** Let $f \in C^{n+1}[0,1]$. for the approximation relation (3.4), we have [23]

$$\left\|f(t) - C^T G(t)\right\|_2 \leq \frac{h^{\frac{2n+3}{2}} R}{(n+1)!\sqrt{2n+3}}, \tag{3.8}$$

where $R = \max_{t\in[t_i,t_{i+1}]} \left|f^{(n+1)}(t)\right|$, $h = t_{i+1} - t_i$, and $t \in [t_i, t_{i+1}] \subseteq [0,1]$. Therefore, at each sub interval $t \in [t_i, t_{i+1}]$, $i = 1, 2, ..., n$, local error bound from $f(t)$, is $O(h^{\frac{2n+3}{2}})$ and global error is $O(h^{\frac{2n+1}{2}})$ on the whole interval $[0,1]$.

4. Operational matrices of Caputo derivatives for Genocchi polynomials

In [23], it is shown that the derivative of the vector $G(t)$ defined as relation (3.5), is as

$$\frac{dG(t)}{dt} = D_1 G(t)$$

where D_1 is an $N \times N$ matrix as

$$\mathbf{D_1} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 3 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & N-1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & N & 0 \end{pmatrix}$$

and called operational matrix of derivative.
In general, the k^{th} derivative of vector G can be obtained as

$$\frac{d^k G(t)}{dt} = (D_1)^k G(t), \;\; k = 1, 2, \ldots .$$

Also, for non-integer $\alpha > 0$, the Caputo derivative of the vector G of order α can be approximated as [23]

$${}_0^C D_t^{\alpha} G(t) \approx D_{\alpha} G(t), \tag{4.1}$$

where D_α is an $N \times N$ matrix and called operational matrix of Caputo fractional derivative of order α, and is defined as:

$$D_\alpha = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \eta_{\lceil\alpha\rceil,k,1} & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \eta_{\lceil\alpha\rceil,k,2} & \cdots & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \eta_{\lceil\alpha\rceil,k,N} \\ \sum_{k=\lceil\alpha\rceil}^{i} \eta_{i,k,1} & \sum_{k=\lceil\alpha\rceil}^{i} \eta_{i,k,2} & \cdots & \sum_{k=\lceil\alpha\rceil}^{i} \eta_{i,k,N} \\ \sum_{k=\lceil\alpha\rceil}^{N} \eta_{N,k,1} & \sum_{k=\lceil\alpha\rceil}^{N} \eta_{N,k,2} & & \sum_{k=\lceil\alpha\rceil}^{N} \eta_{N,k,N} \end{pmatrix},$$

where $\eta_{i,k,j}$ is given by:

$$\eta_{i,k,j} = \frac{i!G_{i-k}}{(i-k)!\Gamma(k+1-\alpha)} c_j. \tag{4.2}$$

5. The proposed method

Consider the fractional optimal control problem with the Caputo derivatives as follows:

$$\min \quad J = \int_0^1 h(x(t), u(t), t)dt, \tag{5.1}$$

subject to the fractional dynamic system

$${}_0^C D_t^\alpha x(t) = f(x(t), u(t), t), \ \ 0 < \alpha \leq 1, \tag{5.2}$$

and the boundary condition

$$x(0) = x_0, \tag{5.3}$$

where ${}_0^C D_t^\alpha$ represents the Caputo derivative of order α, h and f are known functions, and $x(t) = [x_1(t), ..., x_m(t)]^T$ and $u(t) = [u_1(t), u_2(t), ..., u_n(t)]^T$ are unknown state and control functions, respectively.

In this section, we use the Genocchi polynomials for solving problems given in Eqs. (5.1)-(5.3). To approximate the objective function Eqs. (5.1) and the constraints of the optimal control problem Eqs. (5.2) using (3.4) we have

$$x_i(t) \approx {X_i}^T G(t), \quad i = 1, ..., m \tag{5.4}$$

$$u_j(t) \approx {U_j}^T G(t), \quad j = 1, ..., n \tag{5.5}$$

$${}_0^C D_t^\alpha x(t) \approx {X_i}^T D_\alpha G(t). \tag{5.6}$$

So we can write

$$x(t) \approx X^T \mathcal{G}_1(t), \tag{5.7}$$

$$u(t) \approx U^T \mathcal{G}_2(t), \tag{5.8}$$

$${}_0^C D_t^\alpha x(t)) \approx X^T D_\alpha \mathcal{G}_1(t). \tag{5.9}$$

where X and U are vectors of order mN and nN, respectively, given by

$$X = \left[X_1^T, X_2^T, \dots, X_m^T\right]^T, \tag{5.10}$$

$$U = \left[U_1^T, U_2^T, \dots, U_n^T\right]^T, \tag{5.11}$$

and $\mathcal{G}_1$ and $\mathcal{G}_2$ are $mN \times mN$ and $nN \times nN$ matrices, respectively defined as

$$\mathcal{G}_1 = I_n \otimes G, \tag{5.12}$$

$$\mathcal{G}_2 = I_m \otimes G, \tag{5.13}$$

in which I_n and I_m are $n\times n$ and $m\times m$ identity matrices, respectively, and '$\otimes$' denotes the Kronecker product [30]. Applying relations (5.7)-(5.9) in Eqs. (5.1)-(5.3), we get

$$Min \;\; J = \int_0^1 h(X^T\mathcal{G}_1(t), U^T\mathcal{G}_2(t), t)dt, \tag{5.14}$$

$$X^T D_\alpha \mathcal{G}_1(t) = f(X^T\mathcal{G}_1(t), U^T\mathcal{G}_2(t), t), \tag{5.15}$$

$$X^T\mathcal{G}_1(0) = x_0. \tag{5.16}$$

In order to evaluate the integral term in Eq. (5.14), we assume two cases:

(1) $h(x(t), u(t), t)$ is quadratic function as

$$h(x(t), u(t), t) = x^T(t)Qx(t) + u^T(t)Ru(t),$$

where Q is positive semi-definite matrix and R is the positive definite matrix. In this case using relation (3.7), we get [8]

$$J \simeq \mathcal{J}(X, U) = X^T QPX + U^T RPU. \tag{5.17}$$

(2) For non-quadratic case, we evaluate objective function J by a suitable Newton-Cots numerical integration as

$$J \simeq \mathcal{J}(X, U) = \sum_{k=0}^{\ltimes} \omega_k h(X^T\mathcal{G}_1(t_k), U^T\mathcal{G}_2(t_k), t_k), \tag{5.18}$$

where $\omega_k,\ k = 0, 1, \dots, \ltimes$ are the Newton-Cots integration weight functions.

By collocating Eq. (5.15) at the points $t_i = \frac{1}{2}(1 + \cos(\frac{(i-1)\pi}{m(N-1)})),\ i = 1, 2, \dots, mN$ we get

$$X^T D_\alpha \mathcal{G}_1(t_i) = f(X^T\mathcal{G}_1(t_i), U^T\mathcal{G}_2(t_i), t_i),\ \ i = 1, 2, \dots, mN. \tag{5.19}$$

Now the problem changed to find the minimum solution of Eq. (5.17) or (5.18) with the conditions (5.19) and (5.16). Using Lagrange multiplier method we have

$$\begin{aligned} Min \quad & J^*(X, U, \lambda, \mu) = \\ & \mathcal{J}(X, U) + \textstyle\sum_{i=1}^{mN} \lambda_i \left(X^T D_\alpha \mathcal{G}_1(t_i) - f(X^T\mathcal{G}_1(t_i), U^T\mathcal{G}_2(t_i), t_i)\right) \\ & +\mu(X^T\mathcal{G}_1(0) - x_0) \end{aligned} \tag{5.20}$$

where $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_{mN}]^T$ and μ are the Lagrange multipliers. To find the optimal solution of (5.20) we put

$$\frac{\partial J^*}{\partial X} = 0, \quad \frac{\partial J^*}{\partial U} = 0, \quad \frac{\partial J^*}{\partial \lambda} = 0, \quad \frac{\partial J^*}{\partial \mu} = 0. \tag{5.21}$$

Eq. (5.21) gives an algebraic system of equations which can be solved to find the values X, U, λ and μ.

6. Numerical solution

In this section, four examples are given to show the efficiency and accuracy of the presented method.

Remark 6.1. Consider the following fractional optimal control problem [40]

$$\min \quad J = \frac{1}{2}\int_0^1 ({x_1}^2(t) + {x_2}^2(t) + u^2(t))dt \tag{6.1}$$

Subject to

$${}_a^C D_t^\alpha x_1(t) = -x_1(t) + x_2(t) + u(t), \tag{6.2}$$

$${}_a^C D_t^\alpha x_2(t) = -2x_2(t), \tag{6.3}$$

$$x_1(0) = 1, \ x_2(0) = 1. \tag{6.4}$$

For this problem, the exact solutions for the state vector, control function and J in the case $\alpha = 1$ are

$$x_1(t) = -\frac{3}{2}e^{-2t} + 2.48164e^{-\sqrt{2t}} + 0.018352e^{\sqrt{2t}},$$

$$x_2(t) = e^{-2t},$$

$$u(t) = \frac{1}{2}e^{-2t} - 1.02793e^{-\sqrt{2t}} + 0.0443056e^{\sqrt{2t}},$$

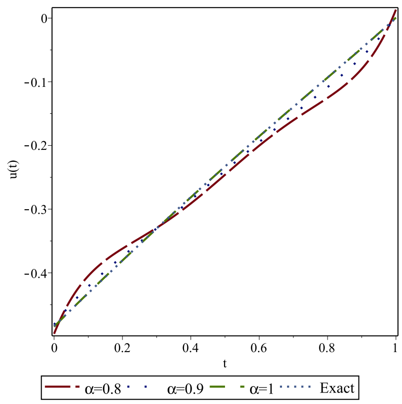
$$J = 0.43198.$$

Table 1 shows the values of performance index J obtained by the presented method for some values of α when it approaches to 1. We see that as α approach to 1, the values of J approach to the exact solution of the problem in the case $\alpha = 1$. Figure 6.1 (a,b,c) shows the plots of $u(t), x_1(t)$ and $x_2(t)$ obtained by the presented method, with $N = 6, 7, 8, 9$, respectively, for the values of α, that approaches to 1.

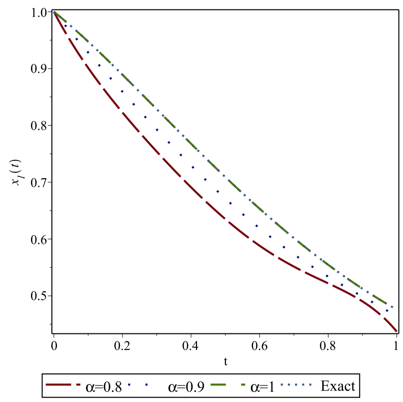
Table 1. Values of J when α approaches to 1, for Example 6.1.

α	0.8	0.9	0.99	0.999	0.9999	0.99999	1
$N = 5$	0.37833	0.40398	0.42909	0.43178	0.43204	0.43207	0.43207
$N = 6$	0.37833	0.40398	0.42909	0.43170	0.43196	0.43199	0.43199
$N = 7$	0.37756	0.40363	0.42905	0.43169	0.43196	0.43198	0.43198
$N = 8$	0.37717	0.40346	0.42904	0.43169	0.43196	0.43198	0.43198

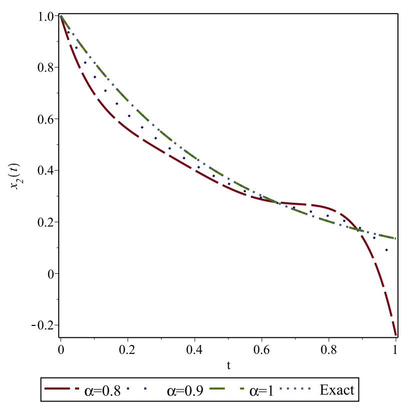
FIGURE 1. Plots of $u(t), x_1(t)$ and $x_2(t)$ for $\alpha = 0.8, 0.9, 1$, obtained by the presented method with$N = 8$, for Example 6.1.



(a) Curves of u(t).



(b) Curves of $x_1(t)$.



(c) Curves of $x_2(t)$.

Remark 6.2. Consider the following FOCP [36]

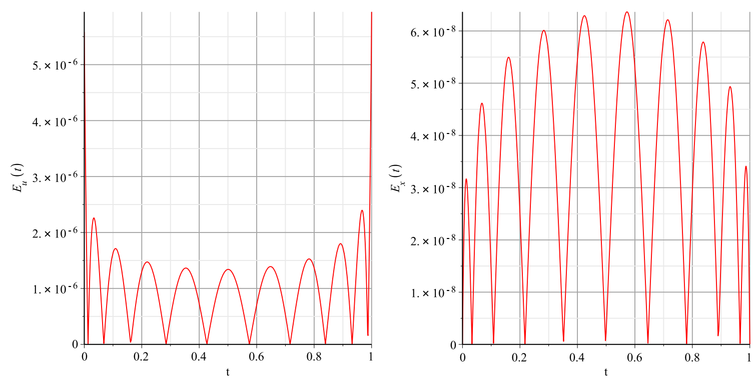
$$
\begin{aligned}
\min \ J &= \frac{1}{2}\int_0^1 (x^2(t) + u^2(t))dt,\\
D^1 x(t) &= -x(t) + u(t),\\
x(0) &= 1.
\end{aligned}
$$

It's exact solution is

$$
\begin{aligned}
x(t) &= \cosh(\sqrt{2}t) + \beta \sinh(\sqrt{2}t),\\
u(t) &= (1+\sqrt{2}\beta)\cosh(\sqrt{2}t) + (\sqrt{2}+\beta)\sinh(\sqrt{2}t),\\
\beta &= -\frac{\cosh(\sqrt{2}) + \sqrt{2}\sinh(\sqrt{2})}{\sqrt{2}\cosh(\sqrt{2}) + \sinh(\sqrt{2})},\\
J &= 0.19290929809316938745415 44.
\end{aligned}
$$

Figure 2 (a,b) shows the error plots for $x(t)$ and $u(t)$ for $N = 12$, where a is plot for $E_u(t)$ and b is plot for $E_x(t)$. Also, in Table 2, we give the values of performance index J obtained using the present method, the Epsilon Ritz method [32] and method given in [36].

FIGURE 2. Error plots of $u(t)$ (left) and $x(t)$ (right) with $N = 12$, for Example 6.2



Remark 6.3. Consider the following minimization problem [40]

$$
\min \quad J(x,u) = \int_0^1 (0.625x^2(t) + 0.5x(t)u(t) + 0.5u^2(t))dt,
$$

subject to

$$
\begin{aligned}
&D_t^\alpha x(t) = 0.5x(t) + u(t), \quad t \in [0,1], \quad 0 < \alpha \leq 1,\\
&x(0) = 1.
\end{aligned}
$$

For $\alpha = 1$, the exact value of the state and control functions are

$$
\begin{aligned}
x(t) &= \cosh(\sqrt{2}t) + \beta \sinh(\sqrt{2}t),\\
u(t) &= \frac{-(\tanh(1-t) + 0.5)\cosh(1-t)}{\cosh(1)},
\end{aligned}
$$

and the minimum value is $J = 0.3807970780$.

Table 3 shows the values of J for $\alpha = 0.5, 0.8, 0.9, 0.99$ and 1 and gives a comparison between our results with the results obtained by [40]. Figure 3 shows the control functions curves for $\alpha = 0.8, 0.9, 0.99, 1$ with $N = 8$ and the error plot of $u(t)$ for $\alpha = 1$ and $N = 8$.

TABLE 2. Comparison of the value of J for $\alpha = 1$, for Example 6.2.

Method	J	Error of J
Epsilon-Ritz method [32]		
N=8	0.192909	3.0×10^{-7}
Hybrid functions [36]		
M=3, N=1	0.1929094450245	1.5×10^{-7}
M=5, N=1	0.1929092980929	2.7×10^{-13}
Present method		
N=7	0.19290929809570833695557418	2.5×10^{-12}
N=8	0.1929092980931770283741513	7.6×10^{-15}
N=9	0.1929092980931694000830882	1.3×10^{-17}
N=10	0.1929092980931693874771643	2.3×10^{-20}
N=12	0.1929092980931693874541544	4.9×10^{-26}
Exact	0.1929092980931693874541544	

TABLE 3. Values of J for our method and the method presented in [40], for Example 6.3.

α	J [40]	J (Presented method)			Exact solution
	$N=7$	$N=5$	$N=7$	$N=8$	
0.5	0.312246	0.3130525	0.31109335391277	0.310730358461472	
0.8	0.352311	0.3527545	0.35247167450799	0.352626269938225	
0.9	0.366705	0.3667988	0.36673617924282	0.366726576996071	
0.99	0.379407	0.3794081	0.37940751469864	0.379407437904958	
1	0.380797	0.38079708	0.38079707797789	0.380797077977882	0.380797077977882

Remark 6.4. This example has been chosen from [40]. It also has been studied by [2, 26]. The problem is

$$\begin{aligned} &\min && J(x,u) = \frac{1}{2}\int_0^1 (x^2(t) + u^2(t))dt, \\ &\text{subject to} && D_t^{\alpha} x(t) = tx(t) + u(t), \\ & && x(0) = 1. \end{aligned}$$

Table 6.4 gives the results reported in [40, 26] and the presented method. Figure 4 shows the plots of $x(t)$ and $u(t)$ for $\alpha = 0.9, 0.9, 0.99, 1$.

FIGURE 3. Curves of $u(t)$ for $\alpha = 0.8, 0.9, 0.99, 1$(left), and error plot of $u(t)$ (right), for Example 6.3

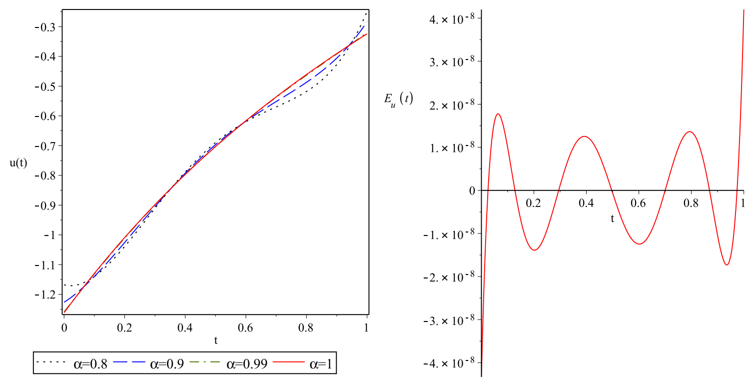
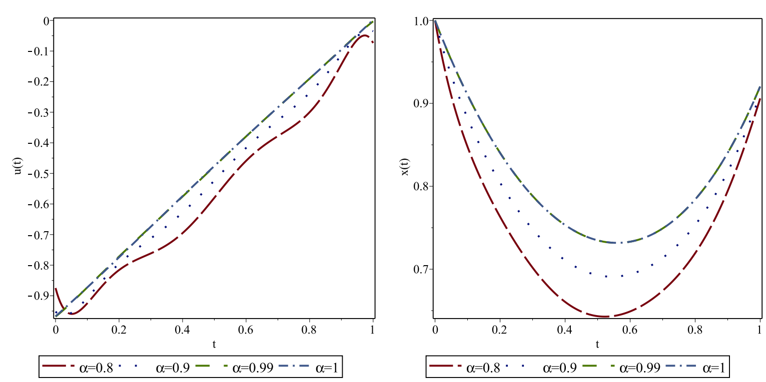


TABLE 4. Value of J obtained for example 6.4.

α	Present method $N = 5$	Method in [40] $N = 4$	Method in[26] $N = 5$
1	0.484268	0.484268	0.484268
0.99	0.483468	0.483463	0.483463
0.9	0.476067	0.476024	0.475883
0.8	0.467491	0.467669	0.466978
0.5	0.446738	0.447038	

FIGURE 4. Curves of $x(t)$ and $u(t)$ for $\alpha = 0.8, 0.9, 0.99, 1$ with $N = 5$, for Example 6.4



7. Conclusion

In this paper, a numerical solution of a class of fractional optimal control problems in the sense of Caputo derivative using Genocchi polynomials was presented. The Genocchi operational matrix for Caputo fractional derivatives was given. Then using them, the problem was changed to a system of algebraic equations. Presented examples show the efficiency and accuracy of the method.

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