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Mixed reproducing kernel-based iterative approach for nonlinear boundary value problems with nonlocal conditions

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Abstract In this paper, a mixed reproducing kernel function (RKF) is introduced. The kernel function consists of piecewise polynomial kernels and polynomial kernels. On the basis of the mixed RKF, a new numerical technique is put forward for solving nonlinear boundary value problems (BVPs) with nonlocal conditions. Compared with the classical RKF-based methods, our method is simpler since it is unnecessary to convert the original equation to an equivalent equation with homogeneous boundary conditions. Also, it is not required to satisfy the homogeneous boundary conditions for the used RKF. Finally, the higher accuracy of the method is shown via several numerical tests.

Keywords. Reproducing kernel method, Nonlocal conditions, Iterative methods.

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1. Introduction

In this paper, we take into account the nonlinear BVPs with nonlocal conditions:

$$\begin{cases} u''(x) + p(x)u'(x) + q(x)u(x) = f(x, u), & 0 < x < 1, \\ B_1(u) = \mu_1, \quad B_2(u) = \mu_2, \end{cases} \tag{1.1}$$

where $B_1(u)$ and $B_2(u)$ are boundary operators.

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A wide variety of problems from physics, chemistry and engineering are modelled by BVPs with nonlocal conditions. It is interesting to develop effective technique for obtaining the accurate approximate solutions of nonlocal BVPs. The theory of the reproducing kernel Hilbert space (RKHS) and its RKF have significant application in quantum mechanics, signal analysis, statistical learning theory and pattern recognition. RKHS is a ideal space for function approximation since the approximation of a function in a RKHS leads to the uniform approximation. In RKHSs, by employing the related theory, the authors in [8,11] developed a new numerical approach called the reproducing kernel method (RKM) for solving linear and nonlinear operator equations. The approach and related improvement have been successfully employed to many different fields by some researchers [1–5, 7, 9, 10, 12–20, 22, 23, 25, 26]. On the basis of the Sobolev RKF, the authors developed some numerical techniques for some BVPs with nonlocal conditions [9,10,12,13,16,17,22,26]. In [13], by employing Sobolev RKF and polynomial RKF, combined with idea of optimization, Geng and Qian presented an optical RKM to solve linear BVPs with nonlocal conditions. The approximate solutions yielded via the method have higher accuracy when compared with classical RKM. However, the method is difficult to handle nonlinear cases.

In this paper, based on the mixed RKF which consist of piecewise polynomial kernels and polynomial kernels, we present a new iterative RKM for nonlinear nonlocal problem (1.1).

2. Reproducing kernel theory

In this section, we firstly introduce some theory on RKHS and RKF, then introduce the RKF with the form of combination of piecewise polynomial kernels and polynomial kernels, which will be employed for the approximate solutions of BVPs (1.1). Let I be a nonempty abstract set.

Definition 2.1. A function $G : I \times I \to R$ is said to be a RKF of the Hilbert space H if and only if

$$\begin{aligned} &1)\forall\ s \in \Omega,\quad G(\cdot,\ s) \in H,\\ &2)\forall\ s \in \Omega, \forall\ \phi \in H,\quad (\phi(\cdot),\ G(\cdot,\ s)) = \phi(s). \end{aligned}$$

If there exists a RKF in a Hilbert space, then the space is a RKHS.

Definition 2.2. For a symmetric function $G : I\times I \to R$, for any $n \in N$, $x_1, x_2, \dots, x_n \in I, c_1, c_2, \dots, c_n \in R$, we have

$$\sum_{i,j=1}^{n} c_i c_j G(x_i, x_j) \geq 0.$$

Then function G is a positive definite kernel function (PDKF) on I.

Theorem 2.3. *[6] The RKF is positive definite, and every PDKF defines a unique RKHS, of which it is the unique RKF.*

From Theorem 2.3, we can see that there is a one-to-one correspondence between the RKF and RKHS.

Definition 2.4. The space $W^4[0,1]$ consists of functions $v(x)$ such that $v^{(3)}(x)$ is absolutely continuous and $v^{(4)}(x) \in L^2[0,1]$. The inner product of this space is given by

$$(v_1, v_2)_4 = \sum_{i=0}^{3} v_1^{(i)}(0) v_2^{(i)}(0) + \int_0^1 v_1^{(4)}(x) v_2^{(4)}(s) ds.$$

Theorem 2.5. *[8] $W^4[0,1]$ is a RKHS and its RKF $K_1(x,y)$ is provided by*

$$K_1(x,y) = \begin{cases} \tau(x,y), & y \le x, \\ \tau(y,x), & y > x, \end{cases} \tag{2.1}$$

where $\tau(x,y) = \frac{35x^3(y+4)y^3 - 21x^2\left(y^3-60\right)y^2 + 7x\left(y^5+720\right)y - y^7 + 5040}{5040}$.

Theorem 2.6. *[24] For $c > 0, m \in N$, $K_2(s,t) = (st+c)^m$ is a PDKF.*

By employing Theorem 2.3, there exists an associated RKHS Q_m with K_2 as a RKF.

Theorem 2.7. *[6] If $F_1(s,t)$ and $F_2(s,t)$ are PDKFs defined in the same set, then $F(s,t) = F_1(s,t) + F_2(s,t)$ is also a PDKF.*

Define

$$K(x,y) = K_1(x,y) + K_2(x,y),$$

where $K_1(x,y)$ is given in (2.1) and $K_2(x,y)$ is a polynomial RKF.

From Theorem 2.7, $K(x,y)$ is a PDKF and there exists an associated RKHS Q with K as a RKF.

3. Iterative RKM for (1.1)

Put $\psi_i(x) = L_s K(x,s)|_{s=x_i}, (i = 1,2,...,N)$, $\psi_{-1}(x) = B_{1s}K(x,s)$, $\psi_0(x) = B_{2s}K(x,s)$. Denote by U_N the space generated by $\{\psi_i(x)\}_{i=-1}^{N}$. Application of Gram-Schmidt orthogonalization to $\{\psi_i(x)\}_{i=-1}^{N}$ yields orthonormal basis functions $\{\overline{\psi}_i(x)\}_{i=-1}^{N}$,

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik} \psi_k(x), \tag{3.1}$$

for $i = -1, 2, ..., N$.

Theorem 3.1. *For equation (1.1), its solution can be approximated by*

$$\overline{u}_N(x) = \sum_{i=-1}^{N} \sum_{k=-1}^{i} \beta_{ik} F_k \overline{\psi}_i, \tag{3.2}$$

where

$$F_k = \begin{cases} f(x_k, u(x_k)), & 1 \le k \le N, \\ \mu_1, & k = -1, \\ \mu_2, & k = 0. \end{cases}$$

Also, it is the best approximation in space U_N.

Proof. Due to the fact that $\{\overline{\psi}_i(x)\}_{i=-1}^{N}$ are orthonormal basis functions in space U_N, we can get the best approximation to the solution to equation (1.1)

$$\overline{u}_N(x) = \sum_{i=-1}^{N} (u(x), \overline{\psi}_i(x))\overline{\psi}_i(x) = \sum_{i=-1}^{N} \sum_{k=-1}^{i} \beta_{ik}(u(x), \psi_k(x))\overline{\psi}_i(x).$$

The application of reproducing property of the RKF $K(x, s)$ yields

$$\overline{u}_N(x) = \sum_{i=-1}^{N} \sum_{k=-1}^{i} \beta_{ik} F_k \overline{\psi}_i,$$

where

$$F_k = \begin{cases} f(x_k, u(x_k)), & 1 \leq k \leq N, \\ \mu_1, & k = -1, \\ \mu_2, & k = 0. \end{cases}$$

□

Suppose that $\{x_i\}_{i=1}^{\infty}$ is dense on $[0, 1]$. Put

$$u(x) = \sum_{i=-1}^{\infty} A_i \overline{\psi}_i,$$

where

$$A_i = \sum_{k=-1}^{i} \beta_{ik} F_k.$$

Since space RKHS Q is a Hilbert space, therefore, $\overline{u}_N(x)$ converges to $u(x)$ uniformly.
Remark:
If $f(x, u)$ is independent of u, $\overline{u}_N(x)$ gives the approximated solution of (1.1) directly.
If $f(x, u)$ is dependent on u, $\overline{u}_N(x)$ is not known, we will give the approximation to the solution of (1.1) by an iterative way.

Theorem 3.2. *If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0, 1]$, then $u(x)$ is a solution of (1.1), in other words, $u(x) = L^{-1}f(x, u)$.*

Proof. Put $Lu(x) = u''(x) + p(x)u'(x) + q(x)u(x)$. From [13], we get

$$Lu(x_j) = f(x_j, u(x_j)), B_1u(x) = \mu_1, \quad B_2u(x) = \mu_2. \tag{3.3}$$

From the fact that $\{x_i\}_{i=1}^{\infty}$ is dense on [0,1], it follows that, for $\forall x \in [0, 1]$, there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ satisfying

$$x_{n_j} \to x (j \to \infty).$$

Letting $j \to \infty$, one obtains

$$Lu(x) = f(x, u(x)). \tag{3.4}$$

The combination of (3.3) and (3.4) show that $u(x)$ satisfies equation (1.1) and its boundary conditions, and therefore the proof is complete. □

If equation (1.1) is linear, then F_k in (3.2) is known and the approximate solution of (1.1) is given by (3.2) directly.

For nonlinear equation (1.1) , we give the approximation to solution of (1.1) by the following iterative way.

First, choosing an appropriate initial approximation $u_0(x)$.

Put

$$u_{n,N}(x) = \sum_{i=-1}^{N} \sum_{k=-1}^{i} \beta_{ik} G_k \overline{\psi}_i \tag{3.5}$$

and

$$u_n(x) = \sum_{i=-1}^{\infty} \sum_{k=-1}^{i} \beta_{ik} \overline{G}_k \overline{\psi}_i, \tag{3.6}$$

where

$$G_k = \begin{cases} f(x_k, u_{n-1,N}(x_k)), & 1 \le k \le N, \\ \mu_1, & k = -1, \\ \mu_2, & k = 0, \end{cases}$$

$$\overline{G}_k = \begin{cases} f(x_k, u_{n-1}(x_k)), & 1 \le k \le N, \\ \mu_1, & k = -1, \\ \mu_2, & k = 0. \end{cases}$$

Note that $u_n(x)$ is the iterative solution, while $u_{n,N}$ is the iterative approximate solution.

Put

$$u(x) = L^{-1} f(x, u) = g(u).$$

Clearly, $u_n(x) = g(u_{n-1}(x))$.

Theorem 3.3. *Suppose that $\|g(v_1) - g(v_2)\| \le \rho \|v_1 - v_2\|$ and $\rho < 1$. Then $u_{n,N}(x)$ converges to $u(x)$.*

Proof. In view of $\|g(v_1) - g(v_2)\| \le \rho \|v_1 - v_2\|$, one gets

$$\|u_n(x) - u(x)\| = \|g(u_{n-1}) - g(u)\| \le \rho \|u_{n-1} - u\| \tag{3.7}$$

and

$$\|u_n(x) - u_{n-1}(x)\| = \|g(u_{n-1}) - g(u_{n-2})\| \le \rho \|u_{n-1} - u_{n-2}\|. \tag{3.8}$$

We can use formula (3.7) and (3.8) and obtain

$$\begin{aligned} \|u_n(x) - u(x)\| &\le \rho \|u_{n-1} - u\| \\ &= \rho \|u_n - u - (u_n - u_{n-1})\| \\ &\le \rho \|u_n - u\| + \rho \|u_n - u_{n-1}\|. \end{aligned} \tag{3.9}$$

From (3.9), we have

$$\begin{aligned} \|u_n(x) - u(x)\| &\le \tfrac{\rho}{1-\rho} \|u_n - u_{n-1}\| \\ &\le \tfrac{\rho^n}{1-\rho} \|u_1 - u_0\|. \end{aligned} \tag{3.10}$$

Clearly,

$$\| u_n(x) - u(x) \| \to 0, \quad n \to \infty,$$

and

$$\| u_{n,N}(x) - u_n(x) \| \to 0, \quad N \to \infty.$$

Note that

$$\begin{aligned} \| u_{n,N}(x) - u(x) \| &= \| u_{n,N}(x) - u_n(x) + u_n(x) - u(x) \| \\ &\leq \| u_{n,N}(x) - u_n(x) \| + \| u_n(x) - u(x) \| . \end{aligned}$$

Therefore,

$$\| u_{n,N}(x) - u(x) \| \to 0, \quad N \to \infty, \quad n \to \infty.$$

□

Theorem 3.4. *If* $p(x), q(x), f(x) \in C^2[0,1]$, $\|g(u) - g(v)\| \leq \rho \|u - v\|$ *and* $\rho < 1$, *then*

$$\| u_{n,N}(x) - u(x) \| \leq ch^2 + d\, \rho^n,$$

where c *and* d *are constants,* $h = \max\limits_{1 \leq j \leq n-1} | x_{j+1} - x_j |$.

Proof. The use of Theorem 3.3 gives

$$\| u_n(x) - u(x) \| \leq \frac{\rho^n}{1-\rho} \|u_1 - u_0\| = d\, \rho^n.$$

From [21], we have the following estimate

$$\| u_{n,N}(x) - u_n(x) \| \leq c\, h^2,$$

where c is a positive real number. Hence,

$$\begin{aligned} \| u_{n,N}(x) - u(x) \| &= \| u_{n,N}(x) - u_n(x) + u_n(x) - u(x) \| \\ &\leq \| u_{n,N}(x) - u_n(x) \| + \| u_n(x) - u(x) \| \\ &\leq c\, h^2 + d\, \rho^n. \end{aligned}$$

□

4. Numerical examples

Test 4.1

We apply the present method(PM) to a two points BVPs in [21]

$$u''(x) + 200 e^x u'(x) + 300 \sin(x) u(x) = f(x), \ \ x \in (0,1)$$

with two points boundary conditions $u(0) = 1$ and $u(1) = \frac{\sqrt{3}}{2}$, where $f(x)$ is selected such that its true solution is $u(x) = \sinh(x)$. Take $m = 10, N = 10$, $x_i = \frac{i-1}{N-1}$, $i = 1, 2, \ldots, N$ when we use the PM. Figure 1 show the absolute errors of our new approach.

Test 4.2

We apply the PM to a three points BVPs in [9,12]

$$x(1-x)u''(x) + (1-x)u'(x) + u(x) = f(x), \ \ x \in (0,1)$$

with three points boundary conditions $u(0) = 0$ and $u(1) + \frac{1}{2}u(\frac{4}{5}) = \frac{\sinh \frac{4}{5}}{2} + \sinh 1$, where $f(x)$ is selected such that its true solution is $u(x) = \sinh x$.

Take $m = 12, N = 10$, $x_i = \frac{i-1}{N-1}$, $i = 1, 2, \ldots, N$ in our method. The numerical results yielded by the PM and the methods in [9,12] are listed in Table 1.

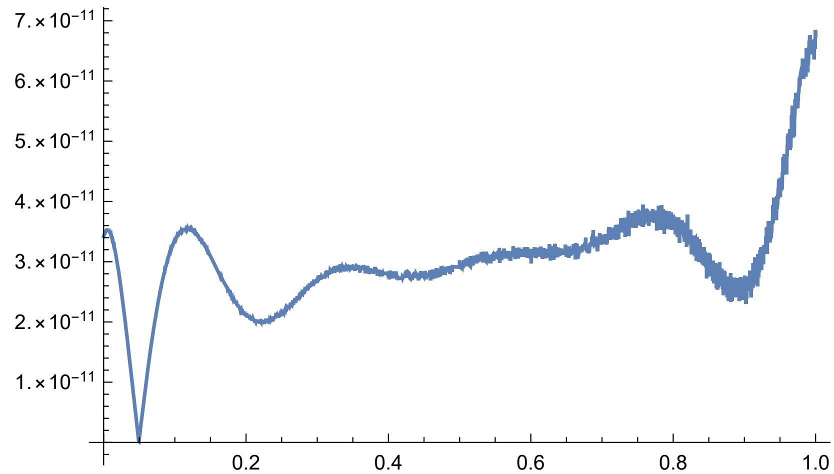

FIGURE 1. Absolute errors yielded by our method.

TABLE 1. Relative errors of different numerical techniques for Test 4.2

Nodes x	Relative error(PM)	Relative error in [9]	Relative error in [12]
0.08	5.8E-10	3.6E-04	8.9E-05
0.16	5.2E-10	2.0E-04	5.2E-05
0.24	4.1E-10	1.4E-04	3.1E-05
0.40	2.7E-10	9.5E-05	1.7E-05
0.48	2.2E-10	7.5E-05	2.4E-05
0.56	1.8E-10	5.5E-05	1.6E-05
0.64	1.5E-10	3.6E-05	1.1E-05
0.72	1.2E-10	1.9E-05	6.2E-06
0.80	9.8E-11	1.8E-15	5.7E-13
0.88	8.0E-11	4.2E-04	2.6E-07
0.96	5.2E-11	5.2E-04	2.0E-06

Test 4.3

We apply the PM to a nonlinear nonlocal BVP in [9, 10]

$$\begin{cases} x(1-x)u''(x) + 6u'(x) + 2u(x) + u^2(x) = g(x), \quad 0 \le x \le 1, \\ u(0) + u(\frac{2}{3}) = \sinh\frac{2}{3}, u(1) + \frac{1}{2}u(\frac{4}{5}) = \frac{\sinh\frac{4}{5}}{2} + \sinh 1, \end{cases} \tag{4.1}$$

where $g(x) = 6\cosh x + \sinh x\,\left(2 + x - x^2 + \sinh x\right)$ and the true solution is $u(x) = \sinh x$.

When we use our method, we take $n = 5, m = 12,\ x_i = \frac{i-1}{N-1}, i = 1, 2, \cdots, N,\ N = 10$ and choose $u_0(x) = \frac{1}{54}((-45\sinh(\frac{2}{3}) + 30(\sinh(\frac{4}{5}) + 2\sinh(1)))x - 10(\sinh(\frac{4}{5}) + 2\sinh(1)) + 42\sinh(\frac{2}{3}))$. The numerical results obtained by different numerical approaches are listed in Table 2. The absolute errors for iteration times $n = 5,\ 7$ are shown in Figures 2,3.

TABLE 2. Numerical results obtained by different numerical techniques for Test 4.3.

x	Exact solution	Relative error([10])	Relative error([9])	Relative error (PM)
0.08	0.080085	7.9E-05	3.0E-08	1.4E-09
0.24	0.242311	1.9E-05	2.6E-07	8.3E-11
0.40	0.410752	7.0E-06	3.1E-07	5.6E-10
0.48	0.498646	3.6E-06	4.0E-07	6.4E-10
0.64	0.684594	6.6E-07	3.0E-07	4.8E-10
0.72	0.783840	1.7E-06	5.2E-09	1.5E-10
0.80	0.888106	2.2E-06	1.8E-06	1.2E-10
0.88	0.998058	2.2E-06	2.7E-06	2.0E-11
0.96	1.114400	2.8E-06	2.2E-05	1.8E-11

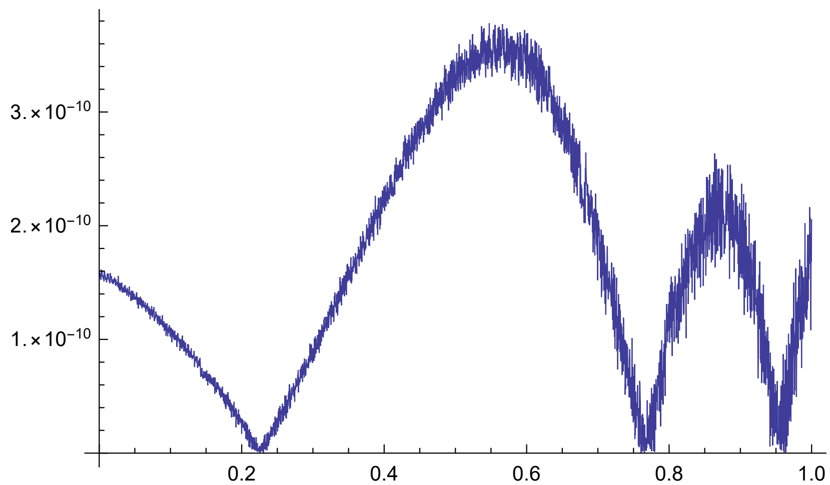

FIGURE 2. Absolute errors yielded by the PM for $n = 5$.

5. CONCLUSION

In this paper, a mixed RKF-based iterative method is proposed for nonlinear BVPs with nonlocal conditions. The new approach has two advantages. The first one is that it is not required to construct the RKF satisfying homogeneous boundary conditions. The second one is that it can gives higher accurate approximate solutions by taking fewer nodes.

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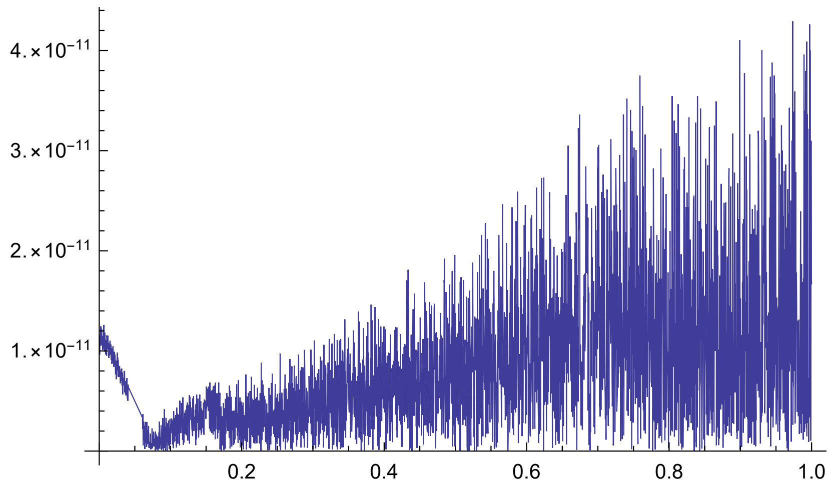

FIGURE 3. Absolute errors yielded by the PM for $n = 7$.

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