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Solving of partial differential equations with distributed order in time using fractional-order Bernoulli-Legendre functions

Parisa Rahimkhani
Department of Mathematics,
Faculty of Mathematical Sciences,
Alzahra University, Tehran, Iran.
E-mail: P.rahimkhani@alzahra.ac.ir

Yadollah Ordokhani*
Department of Mathematics,
Faculty of Mathematical Sciences,
Alzahra University, Tehran, Iran.
E-mail: ordokhani@alzahra.ac.ir

Abstract In this paper, an efficient numerical method is used to provide the approximate solution of distributed-order fractional partial differential equations (DFPDEs). The proposed method is based on the fractional integral operator of fractional-order Bernoulli-Legendre functions and the collocation scheme. In our technique, by approximating functions that appear in the DFPDEs by fractional-order Bernoulli functions in space and fractional-order Legendre functions in time using Gauss-Legendre numerical integration, the under study problem is converted to a system of algebraic equations. This system is solved by using Newton's iterative scheme, and the numerical solution of DFPDEs is obtained. Finally, some numerical experiments are included to show the accuracy, efficiency, and applicability of the proposed method.

Keywords. Fractional-order functions, Distributed-order fractional derivative, Fractional integral operator, Numerical method.

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1. Introduction

Fractional differential equations (FDEs) and fractional integro-differential equations (FIDEs) were widely used in modeling many areas of physics and engineering, for instance fluid mechanics, electrical networks, signal processing, diffusion, reaction processes [4, 14, 21], non-linear oscillation of earthquake [12], fluid-dynamic traffic model [11] and fractional nonlinear complex model for seepage flow in porous media [10]. Thus, many researchers have been interested in finding accurate numerical schemes for solving FDEs and FIDEs for example Fourier transforms method [8], eigenvector expansion method [39], homotopy analysis method [5], variational iteration method [30], Adomian decomposition method [25], power series method [31], orthonormal Bernoulli polynomials method [38], orthonormal Bernstein polynomials

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* corresponding.

method [24], hybrid of block-pulse and parabolic functions method [23], Bernoulli wavelets method [32], fractional-order orthogonal Bernstein polynomials method [22], fractional-order Legendre wavelets method [36] and Müntz-Legendre wavelet method [37].

Distributed-order fractional derivatives indicate fractional derivatives that are integrated over the order of the differentiation within a given range. Distributed-order fractional differential equations (DFDEs) can be considered as a natural generalization of the single order and multi-term fractional different equations. DFDEs attracted a considerable attention in the modeling of various physical and engineering phenomena for example diffusion, viscoelasticity, oscillator, and wave phenomena [1, 2, 16].

Because of the computational complexity of distributed-order fractional derivatives, the exact analytical solution of DFDEs is hardly available. Therefore, in the last decades many researchers have been attracted to deal with the numerical solution of this class of problems. For example: in [28], the diffusion equation of distributed order with Dirichlet, Neumann and Cauchy boundary conditions was analyzed. In [17], Luchko studied the existence and uniqueness results of boundary value problems for the generalized DFDEs. Meerschaert et al. [20] presented explicit strong solutions and stochastic analogues for DFDEs. In [6], authors investigated the Langevin-like equations of distributed-order and considered their possible applications. Gorenflo et al. [9] obtained a representation of the fundamental solution of a distributed order time-fractional diffusion-wave equation by applying the Fourier and Laplace transforms. Mashayekhi and Razzaghi [18] used hybrid functions of block-pulse functions and Bernoulli polynomials for solving DFDEs. In [13], authors presented two spectrally schemes, namely the Petrov-Galerkin spectral scheme and the fractional spectral collocation scheme for DFDEs. Morgado et al. [27] used Chebyshev collocation scheme for solving DFDEs. Zaky [40] applied a Legendre collocation scheme for numerical solution of distributed-order fractional optimal control problems. Zaky and Machado [41] introduced an efficient numerical scheme based on the pseudo-spectral method and the Jacobi-Gauss-Lobatto integration formula for solving an unconstrained convex distributed optimal control problem governed by DFDE. Authors [15] solved the DFDEs by using the Gauss-Legendre quadrature formula and Laguerre Petrov-Galerkin spectral scheme. In this research, we used the fractional-order Legendre-Bernoulli functions for the numerical solution of partial differential equations with distributed order. Some of the most important advantages of the proposed scheme are listed in the following:

- Fractional-order functions can well reflect the properties of fractional-order differential equations.
- Well known Legendre-Bernoulli polynomials are a special case of fractional-order Legendre-Bernoulli functions.
- Fractional-order functions have two degrees of freedom but polynomials have one degree of freedom.

- A small value of fractional-order Legendre-Bernoulli functions is needed to achieve high accuracy and satisfactory results.
- By applying this scheme, the consideration problem is transformed into a system of algebraic equations that can be solved via a suitable numerical method.
- This method is very convenient, since the initial and boundary conditions are taken into account automatically.

The paper is organized as follows: in Section 2, we introduce some necessary definitions of fractional calculus. In Section 3, fractional-order Bernoulli, fractional-order Legendre functions and their properties are defined. In Section 4, we present the numerical scheme for solving the distributed-order fractional differential equations. Error bound of our approximation is obtained in Section 5. In section 6, we present and discuss some numerical examples. Finally, a conclusion is given in section 7.

2. Preliminaries

In this section, we recall some basic definitions and properties of fractional derivative and integral.

Definition 2.1. The Riemann-Liouville fractional integral of order ν is given as [35]

$$ {}_0^R\Im_t^\nu \zeta(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1}\zeta(\tau)d\tau, \qquad t>0. $$

Definition 2.2. The Caputo fractional derivative of order ν is given as [35]

$$ {}_0^C D_t^\nu \zeta(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t (t-\tau)^{n-\nu-1}\zeta^{(n)}(\tau)d\tau, \qquad n-1<\nu\le n. $$

Proposition 2.3. *The Caputo fractional derivative and Riemann-Liouville fractional integral satisfies the following properties* [35]*:*

(1) ${}_0^C D_t^\nu {}_0^R\Im_t^\nu \zeta(t) = \zeta(t)$,
(2) ${}_0^R\Im_t^\nu {}_0^C D_t^\nu \zeta(t) = \zeta(t) - \sum_{i=0}^{n-1} \zeta^{(i)}(0)\frac{t^i}{i!}$,
(3) ${}_0^C D_t^\nu \zeta(t) = 0^R\Im_t^{n-\nu} {}_0^C D_t^n \zeta(t)$,
(4) ${}_0^C D_t^\nu (\lambda\zeta_1(t) + \theta\zeta_2(t)) = \lambda {}_0^C D_t^\nu \zeta_1(t) + \theta {}_0^C D_t^\nu \zeta_2(t)$,
(5) ${}_0^C D_t^\nu t^\beta = \begin{cases} 0, & \nu \in N_0, \ \beta < \nu, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\nu)} t^{\beta-\nu}, & \text{otherwise}, \end{cases}$
(6) ${}_0^C D_t^\nu \lambda = 0$,

where λ, θ are real constants and $n-1<\nu\le n$.

3. Fractional-order functions

To solve DFPDEs, we use fractional-order Bernoulli-Legendre functions.

3.1. **Fractional-order Bernoulli functions.** The fractional-order Bernoulli functions (FBFs) $\beta_m^{(h,\gamma)}(x)$, on the interval $[0,h]$ are defined as [33]

$$ \beta_m^{(h,\gamma)}(x) = \sum_{i=0}^{m} \begin{pmatrix} m \\ i \end{pmatrix} \frac{\beta_{m-i}^\gamma}{h^{i\gamma}} x^{i\gamma}, \qquad 0\le x\le h, \tag{3.1} $$

where $\beta_i^\gamma := \beta_i^\gamma(0) = \beta_i,\ i = 0, 1, ..., m,$ are Bernoulli numbers. Therefore, the first four FBFs are
$\beta_0^{(h,\gamma)}(x) = 1,$
$\beta_1^{(h,\gamma)}(x) = (x/h)^\gamma - \frac{1}{2},$
$\beta_2^{(h,\gamma)}(x) = (x/h)^{2\gamma} - (x/h)^\gamma + \frac{1}{6},$
$\beta_3^{(h,\gamma)}(x) = (x/h)^{3\gamma} - \frac{3}{2}(x/h)^{2\gamma} + \frac{1}{2}(x/h)^\gamma.$
FBFs satisfy the following properties [33]

$$\int_0^h \beta_n^{(h,\gamma)}(x)\beta_m^{(h,\gamma)}(x)x^{\gamma-1}dx = \frac{h^\gamma}{\gamma}(-1)^{n-1}\frac{m!n!}{(m+n)!}\beta_{m+n}^\gamma, \qquad m, n \geq 1. \tag{3.2}$$

$$\int_0^h \beta_0^{(h,\gamma)^2}(x)x^{\gamma-1}dx = \frac{h^\gamma}{\gamma}. \tag{3.3}$$

3.2. **Fractional-order Legendre functions.** The fractional-order Legendre functions (FLFs) $L_m^{(h,\gamma)}(t)$, on the interval $[0, h]$ are defined as [33]

$$L_{m+1}^{(h,\gamma)}(t) = \tfrac{(2m+1)(2(t/h)^\gamma-1)}{m+1}L_m^{(h,\gamma)}(t) - \tfrac{m}{m+1}L_{m-1}^{(h,\gamma)}(t), \qquad m = 1, 2, \ldots,$$
$$L_0^{(h,\gamma)}(t) = 1, \qquad L_1^{(h,\gamma)}(t) = 2(t/h)^\gamma - 1.$$

The analytic form of $L_m^{(h,\gamma)}(t)$ of degree $m\gamma$ is defined by using

$$L_m^{(h,\gamma)}(t) = \sum_{i=0}^{m}\frac{b_{i,m}}{h^{i\gamma}}t^{i\gamma}, \qquad m = 0, 1, 2, \ldots, \tag{3.4}$$

where $b_{i,m} = \frac{(-1)^{m+i}(m+i)!}{(m-i)!(i!)^2}$, and $L_m^{(h,\gamma)}(0) = (-1)^m,\ \ L_m^{(h,\gamma)}(h) = 1.$
The FLFs are orthogonal with respect to the weight function $\omega(t) = t^{\gamma-1}$ on the interval $[0, h]$, then the orthogonality condition is [33]

$$\int_0^h L_n^{(h,\gamma)}(t)L_m^{(h,\gamma)}(t)t^{\gamma-1}dt = \frac{h^\gamma}{(2m+1)\gamma}\delta_{nm}, \qquad m \geq n, \tag{3.5}$$

where δ_{nm} is the Kronecker function.

4. Description of the scheme

In this section, we use fractional integral operators of FBFs ($\Re^{(h_1,\gamma)}(\nu, x)$) and FLFs ($\Theta^{(h_2,\gamma)}(\nu, t)$) of order ν which were derived in [33] for solving the following DFPDEs as:

$$\int_0^1 \mathcal{C}(\alpha)\frac{\partial^\alpha\zeta(x,t)}{\partial t^\alpha}d\alpha = \frac{\partial^2\zeta(x,t)}{\partial x^2} + \mathcal{G}(x, t, \zeta(x, t)), 0 \leq x \leq h_1, 0 \leq t \leq h_2, \tag{4.1}$$

where the function $\mathcal{C}(\alpha)$ is acting as weight for the order of differentiation such that as

$$\mathcal{C}(\alpha) \geq 0, \qquad \int_0^1 \mathcal{C}(\alpha)d\alpha = \varrho > 0,$$

with initial condition

$$\zeta(x,0) = f(x), \tag{4.2}$$

and boundary conditions

$$\zeta(0,t) = g_0(t), \qquad \zeta(h_1,t) = g_1(t). \tag{4.3}$$

For solving the above problem, we expand $\frac{\partial^3\zeta(x,t)}{\partial x^2 \partial t}$ as:

$$\frac{\partial^3\zeta(x,t)}{\partial x^2 \partial t} = B^{(h_1,\gamma)T}(x)\mathcal{A}L^{(h_2,\gamma)}(t), \tag{4.4}$$

where $\mathcal{A}$ is an unknown matrix of dimensional $M \times M'$ which should be obtained. Also, we have

$$B^{(h_1,\gamma)}(x) = [\beta_0^{(h_1,\gamma)}(x), \beta_1^{(h_1,\gamma)}(x), \dots, \beta_M^{(h_1,\gamma)}(x)]^T, \tag{4.5}$$

and

$$L^{(h_2,\gamma)}(t) = [L_0^{(h_2,\gamma)}(t), L_1^{(h_2,\gamma)}(t), \dots, L_{M'}^{(h_2,\gamma)}(t)]^T. \tag{4.6}$$

Integrating from Eq. (4.4) of order 2 with respect to x yields

$$\frac{\partial\zeta(x,t)}{\partial t} = \Re^{(h_1,\gamma)T}(2,x)\mathcal{A}L^{(h_2,\gamma)}(t) + \left.\frac{\partial\zeta(x,t)}{\partial t}\right|_{x=0} + x\frac{\partial}{\partial x}\left.\left(\frac{\partial\zeta(x,t)}{\partial t}\right)\right|_{x=0}. \tag{4.7}$$

Putting $x = h_1$ in Eq. (4.7) and considering Eq. (4.3), we have

$$\frac{\partial}{\partial x}\left.\left(\frac{\partial\zeta(x,t)}{\partial t}\right)\right|_{x=0} = \frac{1}{h_1}\left[\frac{\partial g_1(t)}{\partial t} - \Re^{(h_1,\gamma)T}(2,h_1)\mathcal{A}L^{(h_2,\gamma)}(t) - \frac{\partial g_0(t)}{\partial t}\right]. \tag{4.8}$$

Therefore, we get

$$\begin{aligned}\frac{\partial\zeta(x,t)}{\partial t} &= \Re^{(h_1,\gamma)T}(2,x)\mathcal{A}L^{(h_2,\gamma)}(t) - \frac{x}{h_1}(\Re^{(h_1,\gamma)T}(2,h_1)\mathcal{A}L^{(h_2,\gamma)}(t)) \\ &+ (1-\frac{x}{h_1})\frac{\partial g_0(t)}{\partial t} + \frac{x}{h_1}\frac{\partial g_1(t)}{\partial t}.\end{aligned} \tag{4.9}$$

Integrating Eq. (4.4) with respect to t we have

$$\begin{aligned}\frac{\partial^2\zeta(x,t)}{\partial x^2} &= B^{(h_1,\gamma)T}(x)\mathcal{A}\Theta^{(h_2,\gamma)}(1,t) + \left.\frac{\partial^2\zeta(x,t)}{\partial x^2}\right|_{t=0} \\ &= B^{(h_1,\gamma)T}(x)\mathcal{A}\Theta^{(h_2,\gamma)}(1,t) + \frac{\partial^2 f(x,t)}{\partial x^2}.\end{aligned} \tag{4.10}$$

Now, by integrating Eq. (4.9), with respect to t, we obtain

$$\begin{aligned}\zeta(x,t) &= \Re^{(h_1,\gamma)T}(2,x)\mathcal{A}\Theta^{(h_2,\gamma)}(1,t) \\ &- \frac{x}{h_1}(\Re^{(h_1,\gamma)T}(2,h_1)\mathcal{A}\Theta^{(h_2,\gamma)}(1,t)) \\ &+ (1-\frac{x}{h_1})(g_0(t)-g_0(0)) + \frac{x}{h_1}(g_1(t)-g_1(0)) + f(x).\end{aligned} \tag{4.11}$$

By differentiation of Eq. (4.11) of order α with respect to t, we have

$$\begin{aligned} \frac{\partial^{\alpha}\zeta(x,t)}{\partial t^{\alpha}} &= \Re^{(h_1,\gamma)T}(2,x)\mathcal{A}\Theta^{(h_2,\gamma)}(1-\alpha,t) \\ &- \frac{x}{h_1}(\Re^{(h_1,\gamma)T}(2,h_1)\mathcal{A}\Theta^{(h_2,\gamma)}(1-\alpha,t)) \\ &+ (1-\frac{x}{h_1}){}_0^C D_t^{\alpha} g_0(t) + \frac{x}{h_1}{}_0^C D_t^{\alpha} g_1(t). \end{aligned} \tag{4.12}$$

Substituting Eqs. (4.10), (4.11) and (4.12) in Eq. (4.1), yields

$$\begin{aligned} &\int_0^1 \mathcal{C}(\alpha)\Big(\Re^{(h_1,\gamma)T}(2,x)\mathcal{A}\Theta^{(h_2,\gamma)}(1-\alpha,t) \\ &- \frac{x}{h_1}(\Re^{(h_1,\gamma)T}(2,h_1)\mathcal{A}\Theta^{(h_2,\gamma)}(1-\alpha,t)) \\ &+ (1-\frac{x}{h_1}){}_0^C D_t^{\alpha} g_0(t) + \frac{x}{h_1}{}_0^C D_t^{\alpha} g_1(t)\Big)d\alpha = B^{(h_1,\gamma)T}(x)\mathcal{A}\Theta^{(h_2,\gamma)}(1,t) \\ &+ \frac{\partial^2 f(x,t)}{\partial x^2} + \mathcal{G}(x,t,\Re^{(h_1,\gamma)T}(2,x)\mathcal{A}\Theta^{(h_2,\gamma)}(1,t) \\ &- \frac{x}{h_1}(\Re^{(h_1,\gamma)T}(2,h_1)\mathcal{A}\Theta^{(h_2,\gamma)}(1,t)) \\ &+ (1-\frac{x}{h_1})(g_0(t)-g_0(0)) + \frac{x}{h_1}(g_1(t)-g_1(0)) + f(x)). \end{aligned} \tag{4.13}$$

By applying the Gauss-Legendre numerical integration, Eq. (4.13) transforms into the following equation:

$$\begin{aligned} &\frac{1}{2}\sum_{r=1}^{\tilde{n}} \omega_r \mathcal{C}(\frac{1+\eta_r}{2})\Big(\Re^{(h_1,\gamma)T}(2,x)\mathcal{A}\Theta^{(h_2,\gamma)}(1-\frac{1+\eta_r}{2},t) \\ &- \frac{x}{h_1}(\Re^{(h_1,\gamma)T}(2,h_1)\mathcal{A}\Theta^{(h_2,\gamma)}(1-\frac{1+\eta_r}{2},t)) \\ &+ (1-\frac{x}{h_1}){}_0^C D_t^{\frac{1+\eta_r}{2}} g_0(t) + \frac{x}{h_1}{}_0^C D_t^{\frac{1+\eta_r}{2}} g_1(t)\Big) = B^{(h_1,\gamma)T}(x)\mathcal{A}\Theta^{(h_2,\gamma)}(1,t) \\ &+ \frac{\partial^2 f(x,t)}{\partial x^2} + \mathcal{G}(x,t,\Re^{(h_1,\gamma)T}(2,x)\mathcal{A}\Theta^{(h_2,\gamma)}(1,t) \\ &- \frac{x}{h_1}(\Re^{(h_1,\gamma)T}(2,h_1)\mathcal{A}\Theta^{(h_2,\gamma)}(1,t)) \\ &+ (1-\frac{x}{h_1})(g_0(t)-g_0(0)) + \frac{x}{h_1}(g_1(t)-g_1(0)) + f(x)), \end{aligned} \tag{4.14}$$

where ω_r and η_r are weights and nods of Gauss-Legendre given in [3]. We collocate Eq. (4.14) at zeros of the shifted Legendre polynomial $L_{M+1}(x)$ and $L_{M'+1}(t)$;

respectively, we get

$$
\begin{aligned}
& \frac{1}{2}\sum_{r=1}^{\tilde{n}}\omega_r\mathcal{C}(\frac{1+\eta_r}{2})\Bigg(\Re^{(h_1,\gamma)T}(2,x_i)\mathcal{A}\Theta^{(h_2,\gamma)}(1-\frac{1+\eta_r}{2},t_j) \\
- \quad & \frac{x_i}{h_1}(\Re^{(h_1,\gamma)T}(2,h_1)\mathcal{A}\Theta^{(h_2,\gamma)}(1-\frac{1+\eta_r}{2},t_j)) \\
+ \quad & (1-\frac{x_i}{h_1}){}_0^C D_t^{\frac{1+\eta_r}{2}} g_0(t_j)+\frac{x_i}{h_1}{}_0^C D_t^{\frac{1+\eta_r}{2}} g_1(t_j)\Bigg)=B^{(h_1,\gamma)T}(x_i)\mathcal{A}\Theta^{(h_2,\gamma)}(1,t_j) \\
+ \quad & \frac{\partial^2 f(x_i,t_j)}{\partial x^2}+\mathcal{G}(x_i,t_j,\Re^{(h_1,\gamma)T}(2,x_i)\mathcal{A}\Theta^{(h_2,\gamma)}(1,t_j) \\
- \quad & \frac{x_i}{h_1}(\Re^{(h_1,\gamma)T}(2,h_1)\mathcal{A}\Theta^{(h_2,\gamma)}(1,t_j))+(1-\frac{x_i}{h_1})(g_0(t_j)-g_0(0)) \\
+ \quad & \frac{x_i}{h_1}(g_1(t_j)-g_1(0))+f(x_i)), \qquad 0\le i\le M, 0\le j\le M'.
\end{aligned}
\tag{4.15}
$$

This creates a system of $(M+1)\times(M'+1)$ algebraic equations which can be solved for the matrix $\mathcal{A}$ by applying Newton's iterative scheme. By determining $\mathcal{A}$, the unknown function $\zeta(x,t)$ can be calculated via Eq. (4.11).
In the following algorithm, we express the necessary steps for implementing of proposed scheme.

Algorithm

Input: M, M', γ.
Step 1: Define the FBFs $\beta_m^{(h,\gamma)}(x)$ in Eq. (3.1) and
FLFs $L_m^{(h,\gamma)}(t)$ in Eq. (3.4).
Step 2: Construct the FBFs and FLFs vectors $B^{(h_1,\gamma)}(x), \Theta^{(h_2,\gamma)}(t)$
from Eqs. (4.5) and (4.6).
Step 3: Compute the Riemann-Liouville fractional integral operators for FBFs
and FLFs $\Re^{(h_1,\gamma)}(\nu,x)$ and $\Theta^{(h_2,\gamma)}(\nu,t)$ from Ref. [33].
Step 4: Define the $(M+1)\times(M'+1)$ unknown matrix $\mathcal{A}$.
Step 5: Construct the function in Eq. (4.14).
Step 6: Collocating Eq. (4.14) at zeros of the shifted Legendre polynomials
$L_{M+1}(x)$ and $L_{M'+1}(t)$; respectively.
Step 7: Construct the system in Eq. (4.15).
Step 8: Solve the system of algebraic equations in step 7 by using
Newton's iterative method.
Output: The approximate solution $\zeta(x,t)$ by using Eq. (4.11).

5. Error bound

The purpose of this section is to obtain an estimate of the error bound of the approximation of a smooth function $u(x,t)\in\Delta=[0,h_1]\times[0,h_2]$ by its expansion in terms of the fractional-order Bernoulli-Legendre functions.
We assume that $u(x,t)$ is a sufficiently smooth function on $\Delta=[0,h_1]\times[0,h_2]$ and $P_{M,M'}^{(\gamma)}(x,t)$ is the interpolating function to u at points (x_i,t_j), where $x_i, i=$

$0, 1, \ldots, M$ are the roots of the $(M+1)$-degree shifted Chebyshev polynomial in $[0, h_1]$ and $t_j, j = 0, 1, \ldots, M'$ are the roots of the $(M'+1)$-degree shifted Chebyshev polynomial in $[0, h_2]$, then we obtain [29, 34]

$$\begin{aligned} u(x,t) - P^{(\gamma)}_{M,M'}(x,t) &= \frac{{}_0^C D_x^{(M+1)\gamma} u(\xi,t)}{\Gamma((M+1)\gamma+1)} \Pi_{i=0}^{M}(x-x_i)^{\gamma} \\ &+ \frac{{}_0^C D_t^{(M'+1)\gamma} u(x,\eta)}{\Gamma((M'+1)\gamma+1)} \Pi_{j=0}^{M'}(t-t_j)^{\gamma} \qquad (5.1) \\ &+ \frac{{}_0^C D_x^{(M+1)\gamma} {}_0^C D_t^{(M'+1)\gamma} u(\xi',\eta')}{\Gamma((M+1)\gamma+1)\Gamma((M'+1)\gamma+1)} \\ &\quad \Pi_{i=0}^{M}(x-x_i)^{\gamma} \Pi_{j=0}^{M'}(t-t_j)^{\gamma}, \end{aligned} \tag{5.2}$$

where $\xi, \xi' \in [0, h_1]$ and $\eta, \eta' \in [0, h_2]$. Therefore

$$\begin{aligned} |u(x,t) - P^{(\gamma)}_{M,M'}(x,t)| &\leq max_{(x,t)\in\Delta} |{}_0^C D_x^{(M+1)\gamma} u(\xi,t)| \frac{\Pi_{i=0}^{M}|x-x_i|^{\gamma}}{\Gamma((M+1)\gamma+1)} \\ &+ max_{(x,t)\in\Delta} |{}_0^C D_t^{(M'+1)\gamma} u(x,\eta)| \frac{\Pi_{j=0}^{M'}|t-t_j|^{\gamma}}{\Gamma((M'+1)\gamma+1)} \\ &+ max_{(x,t)\in\Delta} |{}_0^C D_x^{(M+1)\gamma} {}_0^C D_t^{(M'+1)\gamma} u(\xi',\eta')| \\ &\quad \frac{\Pi_{i=0}^{M}|x-x_i|^{\gamma} \Pi_{j=0}^{M'}|t-t_j|^{\gamma}}{\Gamma((M+1)\gamma+1)\Gamma((M'+1)\gamma+1)}. \end{aligned} \tag{5.3}$$

We assume that there are real numbers ϱ_1, ϱ_2 and ϱ_3 such that
$max_{(x,t)\in\Delta} |{}_0^C D_x^{(M+1)\gamma} u(\xi,t)| \leq \varrho_1$,
$max_{(x,t)\in\Delta} |{}_0^C D_t^{(M'+1)\gamma} u(x,\eta)| \leq \varrho_2$,
$max_{(x,t)\in\Delta} |{}_0^C D_x^{(M+1)\gamma} {}_0^C D_t^{(M'+1)\gamma} u(\xi',\eta')| \leq \varrho_3$.
By using above relations and taking into account the estimates for Chebyshev interpolation nodes [19], we get

$$\begin{aligned} |u(x,t) - P^{(\gamma)}_{M,M'}(x,t)| &\leq \varrho_1 \frac{(h_1/2)^{M\gamma}}{\Gamma((M+1)\gamma+1)2^{M\gamma}} \\ &+ \varrho_2 \frac{(h_2/2)^{M'\gamma}}{\Gamma((M'+1)\gamma+1)2^{M'\gamma}} \\ &+ \varrho_3 \frac{(h_1/2)^{M\gamma}(h_2/2)^{M'\gamma}}{\Gamma((M+1)\gamma+1)\Gamma((M'+1)\gamma+1)2^{(M+M')\gamma}}. \end{aligned} \tag{5.4}$$

Theorem 5.1. *Let $u^*(x,t)$ be the best approximation of the real sufficiently smooth function $u(x,t) \in \Delta$ by fractional-order Bernoulli-Legendre functions expansion, then,* *there exist real numbers* ϱ_1', ϱ_2' *and* ϱ_3' *such that*

$$\begin{aligned} \|u(x,t)-u^*(x,t)\|_2 &\leq \varrho_1' \frac{(h_1/2)^{M\gamma}}{\Gamma((M+1)\gamma+1)2^{M\gamma}} \\ &+ \varrho_2' \frac{(h_2/2)^{M'\gamma}}{\Gamma((M'+1)\gamma+1)2^{M'\gamma}} \\ &+ \varrho_3' \frac{(h_1/2)^{M\gamma}(h_2/2)^{M'\gamma}}{\Gamma((M+1)\gamma+1)\Gamma((M'+1)\gamma+1)2^{(M+M')\gamma}}. \end{aligned} \tag{5.5}$$

Proof. By considering the definition of the best approximation and Eq. (5.4), we have

$$\begin{aligned} & \|u(x,t)-u^*(x,t)\|_2^2 \\ \leq\ & \|u(x,t)-P^{(\gamma)}_{M,M'}(x,t)\|_2^2 \\ \leq\ & \int_0^{h_2}\int_0^{h_1} |u(x,t)-P^{(\gamma)}_{M,M'}(x,t)|^2 dxdt \\ \leq\ & \int_0^{h_2}\int_0^{h_1} \varrho_1 \frac{(h_1/2)^{M\gamma}}{\Gamma((M+1)\gamma+1)2^{M\gamma}} dxdt \\ +\ & \int_0^{h_2}\int_0^{h_1} \varrho_2 \frac{(h_2/2)^{M'\gamma}}{\Gamma((M'+1)\gamma+1)2^{M'\gamma}} dxdt \\ +\ & \int_0^{h_2}\int_0^{h_1} \varrho_3 \frac{(h_1/2)^{M\gamma}(h_2/2)^{M'\gamma}}{\Gamma((M+1)\gamma+1)\Gamma((M'+1)\gamma+1)2^{(M+M')\gamma}} dxdt. \\ =\ & h_1h_2[\varrho_1 \frac{(h_1/2)^{M\gamma}}{\Gamma((M+1)\gamma+1)2^{M\gamma}} + \varrho_2 \frac{(h_2/2)^{M'\gamma}}{\Gamma((M'+1)\gamma+1)2^{M'\gamma}} \\ +\ & \varrho_3 \frac{(h_1/2)^{M\gamma}(h_2/2)^{M'\gamma}}{\Gamma((M+1)\gamma+1)\Gamma((M'+1)\gamma+1)2^{(M+M')\gamma}}]^2. \end{aligned} \tag{5.6}$$

From (5.6) we conclude that (5.5) is valid, with

$$\varrho_1' = \sqrt{h_1h_2}\varrho_1, \qquad \varrho_2' = \sqrt{h_1h_2}\varrho_2, \qquad \varrho_3' = \sqrt{h_1h_2}\varrho_3.$$

□

Now, we obtain the distance $\|u^*(x,t)-\bar{u}(x,t)\|_2$ between the best approximation $u^*(x,t)$ of the exact solution $u(x,t)$ and our computed solution $\bar{u}(x,t)$. For this purpose, we need the following theorem.

Theorem 5.2. *Suppose that*

$$u^*(x,t) = \sum_{i=0}^{M}\sum_{j=0}^{M'} u^*_{ij}\beta_i^{(h_1,\gamma)}(x)L_j^{(h_2,\gamma)}(t) = B^{(h_1,\gamma)T}(x)U^*L^{(h_2,\gamma)}(t),$$

and

$$\bar{u}(x,t) = \sum_{i=0}^{M}\sum_{j=0}^{M'} \bar{u}_{ij}\beta_i^{(h_1,\gamma)}(x)L_j^{(h_2,\gamma)}(t) = B^{(h_1,\gamma)T}(x)\bar{U}L^{(h_2,\gamma)}(t),$$

be two functions of space Δ. Then

$$\|u^*(x,t)-\bar{u}(x,t)\|_2 \leq \sqrt{\sum_{j=0}^{M'}\frac{h_1^\gamma h_2^\gamma}{(2j+1)\gamma^2}+\sum_{i=1}^{M}\sum_{j=0}^{M'}\frac{(-1)^{i-1}h_1^\gamma h_2^\gamma (i!)^2\beta_{2i}^\gamma}{(2j+1)(2i)!\gamma^2}}\, \|U^*-\bar{U}\|_2, \tag{5.7}$$

where

$$U^*=[u^*_{i,j}]_{(M+1)\times(M'+1)}, \qquad \bar{U}=[\bar{u}_{i,j}]_{(M+1)\times(M'+1)}.$$

Proof. We have

$$\begin{aligned}
\|u^*(x,t)-\bar{u}(x,t)\|_2 &= \left(\int_0^{h_1}\int_0^{h_2}|u^*(x,t)-\bar{u}(x,t)|^2 x^{\gamma-1}t^{\gamma-1}dxdt\right)^{\frac{1}{2}} \\
&= \left(\int_0^{h_1}\int_0^{h_2}|\sum_{i=0}^{M}\sum_{j=0}^{M'}(u^*_{i,j}-\bar{u}_{i,j})\beta_i^{h_1\gamma}(x)L_j^{h_2\gamma}(t)|^2 x^{\gamma-1}t^{\gamma-1}dxdt\right)^{\frac{1}{2}} \\
&\leq \int_0^{h_1}\int_0^{h_2}\left[\sum_{i=0}^{M}\sum_{j=0}^{M'}|u^*_{i,j}-\bar{u}_{i,j}|^2\right]\left[\sum_{i=0}^{M}\sum_{j=0}^{M'}|\beta_i^{h_1\gamma}(x)L_j^{h_2\gamma}(t)|^2\right]x^{\gamma-1}t^{\gamma-1}dxdt\Bigg)^{\frac{1}{2}} \\
&= \left(\sum_{i=0}^{M}\sum_{j=0}^{M'}|u^*_{i,j}-\bar{u}_{i,j}|^2\right)^{\frac{1}{2}}\left(\sum_{i=0}^{M}\sum_{j=0}^{M'}\int_0^{h_1}\int_0^{h_2}|\beta_i^{h_1\gamma}(x)L_j^{h_2\gamma}(t)|^2 x^{\gamma-1}t^{\gamma-1}dxdt\right)^{\frac{1}{2}} \\
&= \|U^*-\bar{U}\|_2\left(\sum_{j=0}^{M'}\frac{h_1^\gamma h_2^\gamma}{(2j+1)\gamma^2}+\sum_{i=1}^{M}\sum_{j=0}^{M'}\frac{(-1)^{i-1}h_1^\gamma h_2^\gamma(i!)^2\beta_{2i}^\gamma}{(2j+1)(2i)!\gamma^2}\right)^{\frac{1}{2}}.
\end{aligned} \tag{5.8}$$

So, the proof of this theorem is completed. □

By attention to the above theorems, we find the distance $\|u(x,t)-\bar{u}(x,t)\|_2$ between the exact solution $u(x,t)$ and our numerical solution $\bar{u}(x,t)$. For this aim, we present the following theorem.

Theorem 5.3. *Let $\bar{u}(x,t)$ be our numerical approximation of space Δ. Then we have*

$$\begin{aligned}
\|u(x,t)-\bar{u}(x,t)\|_2 &\leq \varrho_1'\frac{(h_1/2)^{M\gamma}}{\Gamma((M+1)\gamma+1)2^{M\gamma}} \\
&+ \varrho_2'\frac{(h_2/2)^{M'\gamma}}{\Gamma((M'+1)\gamma+1)2^{M'\gamma}} \\
&+ \varrho_3'\frac{(h_1/2)^{M\gamma}(h_2/2)^{M'\gamma}}{\Gamma((M+1)\gamma+1)\Gamma((M'+1)\gamma+1)2^{(M+M')\gamma}} \\
&+ \sqrt{\sum_{j=0}^{M'}\frac{h_1^\gamma h_2^\gamma}{(2j+1)\gamma^2}+\sum_{i=1}^{M}\sum_{j=0}^{M'}\frac{(-1)^{i-1}h_1^\gamma h_2^\gamma(i!)^2\beta_{2i}^\gamma}{(2j+1)(2i)!\gamma^2}} \\
&\|U^*-\bar{U}\|_2.
\end{aligned} \tag{5.9}$$

Proof. By using Theorems 5.1 and 5.2, we have

$$\begin{aligned}
\|u(x,t)-\bar{u}(x,t)\|_2 &\leq \|u(x,t)-u^*(x,t)\|_2+\|u^*(x,t)-\bar{u}(x,t)\|_2 \\
&\leq \varrho_1' \frac{(h_1/2)^{M\gamma}}{\Gamma((M+1)\gamma+1)2^{M\gamma}} \\
&+ \varrho_2' \frac{(h_2/2)^{M'\gamma}}{\Gamma((M'+1)\gamma+1)2^{M'\gamma}} \\
&+ \varrho_3' \frac{(h_1/2)^{M\gamma}(h_2/2)^{M'\gamma}}{\Gamma((M+1)\gamma+1)\Gamma((M'+1)\gamma+1)2^{(M+M')\gamma}} \\
&+ \sqrt{\sum_{j=0}^{M'} \frac{h_1^\gamma h_2^\gamma}{(2j+1)\gamma^2}+\sum_{i=1}^{M}\sum_{j=0}^{M'}\frac{(-1)^{i-1}h_1^\gamma h_2^\gamma (i!)^2\beta_{2i}^\gamma}{(2j+1)(2i)!\gamma^2}} \\
&\|U^*-\bar{U}\|_2.
\end{aligned} \tag{5.10}$$

□

6. Numerical results

We show the effectiveness of our scheme with four examples. The computations associated with the examples were performed using Mathematica 10. Also, we report the CPU time (seconds) in all examples, which have been obtained on a 2.67 GHz Core i5 personal computer with 4 GB of RAM.

Example 1. Consider the distributed-order fractional partial differential equation as [26]:

$$\begin{aligned}
&\int_0^1 \Gamma(\frac{7}{2}-\alpha)\frac{\partial^\alpha \zeta(x,t)}{\partial t^\alpha}d\alpha = \frac{\partial^2\zeta(x,t)}{\partial x^2}+\zeta^2(x,t) \\
+&\frac{15\sqrt{\pi}(t-1)t^{\frac{3}{2}}}{8Lnt}x(x-1)-2t^{\frac{5}{2}}-t^5x^2(x-1)^2,
\end{aligned} \tag{6.1}$$

with the initial and boundary conditions

$$\zeta(x,0)=0, \qquad \zeta(0,t)=0, \qquad \zeta(1,t)=0, \tag{6.2}$$

where the analytical solution is $\zeta(x,t)=t^{\frac{5}{2}}x(x-1)$.

In Table 1, we display the maximum absolute error using the fractional-order Bernoulli-Legendre functions for $\gamma=\frac{1}{2}$ with various values of M and M' together with the finite difference method [26]. Also, Table 2 lists the absolute error of our scheme for various values of γ.

Example 2. Consider the distributed-order fractional partial differential equation as [7]

$$\int_0^1 \Gamma(3-\alpha)\frac{\partial^\alpha\zeta(x,t)}{\partial t^\alpha}d\alpha = \frac{\partial^2\zeta(x,t)}{\partial x^2}+2t^2+\frac{2(t-1)t(2-x)x}{Lnt}, \tag{6.3}$$

TABLE 1. L_∞ errors of the present method for $\gamma = \frac{1}{2}$, different values of M, M' and Ref. [26] for Example 1.

Ref. [26]	$L_\infty errors$	CPU
$\Delta t = 0.0625$	3.22×10^{-2}	–
$\Delta t = 0.015625$	8.99×10^{-3}	–
$\Delta t = 0.00390625$	2.31×10^{-3}	–
$\Delta t = 0.000976563$	5.83×10^{-4}	–
Present method		
$M = M' = 2$	8.28×10^{-4}	0.047
$M = M' = 3$	2.93×10^{-15}	0.219
$M = M' = 4$	1.94×10^{-16}	0.515

TABLE 2. Absolute error of the present method with $M = M' = 4$ for different values of γ for Example 1.

(x,t)	$\gamma = \frac{1}{4}$	$\gamma = \frac{1}{3}$	$\gamma = \frac{1}{2}$	$\gamma = \frac{2}{3}$	$\gamma = 1$
$(0.1, 0.1)$	8.05×10^{-6}	1.46×10^{-6}	2.71×10^{-18}	9.58×10^{-7}	5.81×10^{-6}
$(0.2, 0.2)$	1.51×10^{-5}	2.55×10^{-6}	6.07×10^{-18}	1.79×10^{-6}	1.30×10^{-5}
$(0.3, 0.3)$	9.29×10^{-6}	1.43×10^{-6}	8.67×10^{-18}	1.15×10^{-6}	1.13×10^{-5}
$(0.4, 0.4)$	1.80×10^{-5}	2.99×10^{-6}	2.78×10^{-17}	2.78×10^{-6}	2.97×10^{-5}
$(0.5, 0.5)$	6.88×10^{-6}	1.12×10^{-6}	3.47×10^{-17}	1.07×10^{-6}	1.23×10^{-5}
$(0.6, 0.6)$	6.04×10^{-6}	1.18×10^{-6}	4.16×10^{-17}	1.54×10^{-6}	2.13×10^{-5}
$(0.7, 0.7)$	7.34×10^{-6}	1.41×10^{-6}	1.39×10^{-17}	1.91×10^{-6}	2.83×10^{-5}
$(0.8, 0.8)$	2.11×10^{-7}	1.43×10^{-8}	0	5.41×10^{-9}	4.24×10^{-7}
$(0.9, 0.9)$	3.07×10^{-6}	5.80×10^{-7}	1.94×10^{-16}	8.74×10^{-7}	1.53×10^{-5}
$(1, 1)$	0	1.11×10^{-16}	0	2.22×10^{-16}	0
CPU	0.985	0.953	0.516	0.843	0.407

with the initial and boundary conditions

$$\zeta(x,0) = 0, \qquad \zeta(0,t) = 0, \qquad \zeta(2,t) = 0, \tag{6.4}$$

where the analytical solution is $\zeta(x,t) = t^2 x(2-x)$.
The L_∞ errors of our scheme for $\gamma = \frac{1}{2}$ and various values of M, M' and the finite difference method [7] are reported in Table 3. So, the results demonstrate that our scheme is more accurate. In addition, the absolute errors of the numerical solution for $M = M' = 4$ and different values of γ are shown in Table 4. In Figs. 1 and 2, we represent absolute errors of the present method with $M = M' = 4$, $\gamma = \frac{1}{2}$ and the finite difference method [7] at $(x, 0.25)$ and $(x, 0.75)$.

TABLE 3. L_∞ errors of the present method for $\gamma = \frac{1}{2}$, different values of M, M' and Ref. [7] for Example 2.

Ref. [7]	$L_\infty errors$	CPU
$\Delta t = 0.0625$	2.39×10^{-2}	$-$
$\Delta t = 0.015625$	4.66×10^{-3}	$-$
$\Delta t = 0.00390625$	9.10×10^{-4}	$-$
$\Delta t = 0.000976563$	1.84×10^{-4}	$-$
Present method		
$M = M' = 2$	1.91×10^{-9}	0.062
$M = M' = 3$	4.09×10^{-14}	0.109
$M = M' = 4$	4.44×10^{-16}	0.297

TABLE 4. Absolute error with $M = M' = 4$ for different values of γ for Example 2.

(x,t)	$\gamma = \frac{1}{4}$	$\gamma = \frac{1}{3}$	$\gamma = \frac{1}{2}$	$\gamma = \frac{2}{3}$	$\gamma = 1$
$(0.1, 0.1)$	3.34×10^{-17}	6.94×10^{-18}	1.02×10^{-17}	2.41×10^{-6}	8.89×10^{-18}
$(0.2, 0.2)$	7.98×10^{-17}	4.68×10^{-17}	5.20×10^{-18}	1.09×10^{-5}	2.26×10^{-17}
$(0.3, 0.3)$	1.18×10^{-16}	5.55×10^{-17}	4.86×10^{-17}	2.96×10^{-6}	5.55×10^{-17}
$(0.4, 0.4)$	1.25×10^{-16}	6.94×10^{-17}	4.16×10^{-17}	1.57×10^{-5}	2.78×10^{-17}
$(0.5, 0.5)$	2.50×10^{-16}	1.39×10^{-16}	1.39×10^{-16}	1.09×10^{-5}	8.33×10^{-17}
$(0.6, 0.6)$	5.00×10^{-16}	1.67×10^{-16}	2.78×10^{-16}	5.57×10^{-6}	1.11×10^{-16}
$(0.7, 0.7)$	1.05×10^{-15}	1.11×10^{-16}	2.78×10^{-16}	1.51×10^{-5}	1.67×10^{-16}
$(0.8, 0.8)$	3.33×10^{-16}	5.55×10^{-16}	2.22×10^{-16}	5.03×10^{-6}	2.22×10^{-16}
$(0.9, 0.9)$	3.33×10^{-16}	3.33×10^{-16}	1.11×10^{-16}	1.30×10^{-5}	2.22×10^{-16}
$(1, 1)$	5.55×10^{-16}	4.44×10^{-16}	4.44×10^{-16}	8.04×10^{-6}	1.11×10^{-15}
CPU	0.343	0.375	0.375	0.359	0.202

Example 3. Consider the distributed-order fractional partial differential equation [27]

$$\int_0^1 \Gamma(\frac{7}{2} - \alpha)\frac{\partial^\alpha \zeta(x,t)}{\partial t^\alpha} d\alpha = \frac{\partial^2 \zeta(x,t)}{\partial x^2} + \frac{t^{\frac{3}{2}}\left(15\sqrt{\pi}(t-1)(x-1)^2 x + 16t(2-3x)Lnt\right)}{8Lnt}, \tag{6.5}$$

with the initial and boundary conditions

$$\zeta(x,0) = 0, \qquad \zeta(0,t) = 0, \qquad \zeta(1,t) = 0, \tag{6.6}$$

where the analytical solution is $\zeta(x,t) = t^{\frac{5}{2}} x(1-x)^2$.
In Table 5 we give the L_∞ errors of our approach for different values of M and M' with

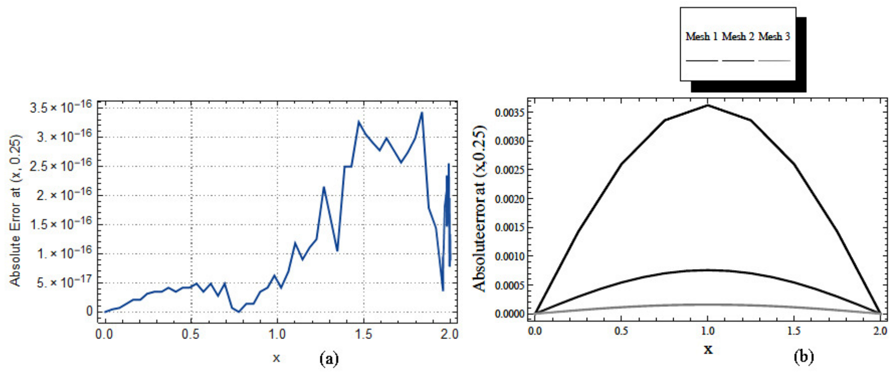

FIGURE 1. Absolute errors of (a) : our scheme for $M = M' = 4$ and $\gamma = \frac{1}{2}$, (b) : Ref. [7] at $(x, 0.25)$ for Example 2.

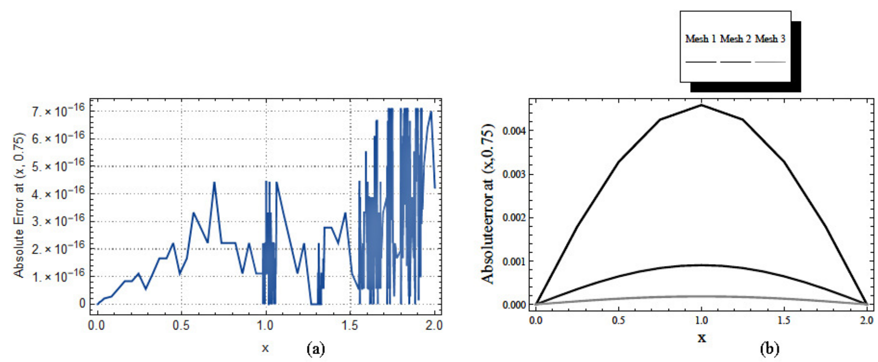

FIGURE 2. Absolute errors of (a) : our scheme for $M = M' = 4$ and $\gamma = \frac{1}{2}$, (b) : Ref. [7] at $(x, 0.75)$ for Example 2.

Chebyshev collocation method [27]. In Table 6, we express the comparison between the absolute error of exact and approximate solutions together with CPU time for $M = M' = 5$ with different values of γ. Furthermore, the graphs of absolute errors of our scheme and the Chebyshev collocation method [27] are illustrated in Figure 3.

Example 4. Consider the distributed-order fractional partial differential equation [27]

$$\int_0^1 \Gamma(\frac{5}{2}-\alpha)\frac{\partial^\alpha \zeta(x,t)}{\partial t^\alpha}d\alpha = \frac{\partial^2 \zeta(x,t)}{\partial x^2} + \frac{\sqrt{t}(x-1)^2(3\sqrt{\pi}(t-1)(x-1)^2x^2 - 8t(5x(3x-2)+1)Lnt)}{4Lnt}, \tag{6.7}$$

TABLE 5. L_∞ errors of the present method for $\gamma = \frac{1}{2}$, various values of M, M' and Ref. [27] for Example 3.

Ref. [27]	$L_\infty errors$	CPU
$n = m = 5$	2.40×10^{-5}	–
$n = m = 7$	4.76×10^{-6}	–
$n = m = 9$	1.43×10^{-6}	–
$n = m = 11$	4.40×10^{-7}	–
Present method		
$M = M' = 2$	5.77×10^{-4}	0.125
$M = M' = 3$	9.12×10^{-16}	0.219
$M = M' = 4$	2.22×10^{-16}	0.297
$M = M' = 5$	2.16×10^{-16}	0.735

TABLE 6. Absolute error with $M = M' = 5$ for various values of γ for Example 3.

(x,t)	$\gamma = \frac{1}{4}$	$\gamma = \frac{1}{3}$	$\gamma = \frac{1}{2}$	$\gamma = \frac{2}{3}$	$\gamma = 1$
$(0.1, 0.1)$	6.30×10^{-7}	6.86×10^{-8}	1.00×10^{-17}	3.70×10^{-7}	3.27×10^{-6}
$(0.2, 0.2)$	2.10×10^{-7}	1.45×10^{-8}	3.04×10^{-18}	4.43×10^{-8}	1.03×10^{-6}
$(0.3, 0.3)$	5.13×10^{-7}	5.08×10^{-8}	3.73×10^{-17}	4.10×10^{-7}	6.89×10^{-6}
$(0.4, 0.4)$	1.24×10^{-7}	9.41×10^{-9}	1.39×10^{-17}	1.69×10^{-8}	8.67×10^{-7}
$(0.5, 0.5)$	8.18×10^{-8}	1.38×10^{-8}	3.47×10^{-18}	4.31×10^{-7}	5.41×10^{-6}
$(0.6, 0.6)$	2.63×10^{-8}	6.12×10^{-9}	9.02×10^{-17}	3.41×10^{-7}	3.18×10^{-6}
$(0.7, 0.7)$	3.87×10^{-8}	3.09×10^{-9}	6.25×10^{-17}	5.01×10^{-8}	1.16×10^{-6}
$(0.8, 0.8)$	3.15×10^{-8}	2.97×10^{-9}	3.12×10^{-17}	1.42×10^{-8}	1.43×10^{-6}
$(0.9, 0.9)$	1.42×10^{-8}	1.32×10^{-9}	2.16×10^{-16}	1.45×10^{-7}	6.64×10^{-7}
$(1, 1)$	5.55×10^{-17}	5.55×10^{-17}	0	1.09×10^{-16}	1.11×10^{-16}
CPU	0.906	0.859	0.735	0.890	0.531

with the initial and boundary conditions

$$\zeta(x, 0) = 0, \qquad \zeta(0, t) = 0, \qquad \zeta(1, t) = 0, \tag{6.8}$$

where the analytical solution is $\zeta(x, t) = t^{\frac{3}{2}} x^2 (1 - x)^4$.
Table 7 demonstrates a comparison between the numerical results given by our scheme for $\gamma = \frac{1}{2}$ and different values of M and M' and Ref. [27]. It is observed from these results that our approach is more accurate than Ref. [27]. Table 8 shows the comparison between the absolute error of exact and approximate solutions together with CPU time for $M = 8, M' = 3$ with different values of γ. Also, the graph of absolute errors of the present scheme for $M = 8, M' = 3$ and $\gamma = \frac{1}{2}$ (left), and the graph of absolute errors of Ref. [27] (right) are illustrated in Figure 4.

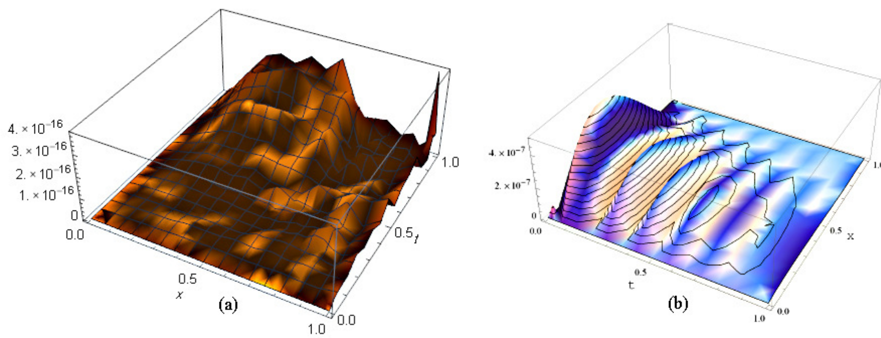

FIGURE 3. Absolute errors of (a) : our scheme for $M = M' = 5$ and $\gamma = \frac{1}{2}$, (b) : Ref. [27] for Example 3.

TABLE 7. L_∞ errors of the present method for $\gamma = \frac{1}{2}$, various values of M, M' and Ref. [27] for Example 4.

Ref. [27]	$L_\infty errors$	CPU
$n = m = 5$	1.21×10^{-3}	–
$n = m = 7$	1.06×10^{-5}	–
$n = m = 9$	4.47×10^{-6}	–
$n = m = 11$	1.69×10^{-6}	–
Present method		
$M = 5, M' = 3$	7.78×10^{-5}	0.328
$M = 6, M' = 3$	9.21×10^{-6}	0.532
$M = 7, M' = 3$	4.98×10^{-7}	0.796
$M = 8, M' = 3$	2.40×10^{-15}	1.157

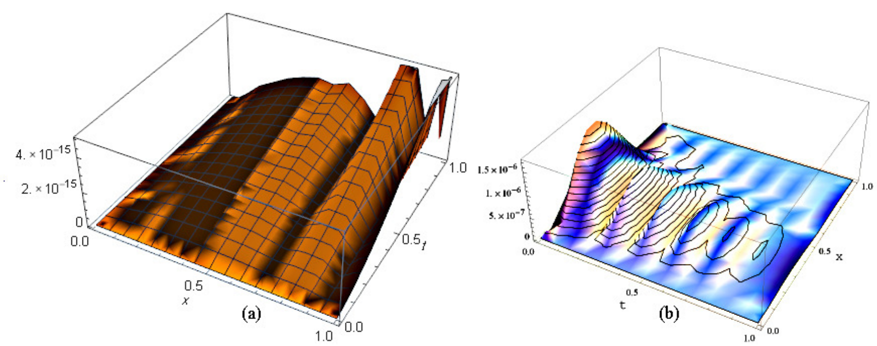

FIGURE 4. Absolute errors of (a) : our scheme for $M = 8, M' = 3$ and $\gamma = \frac{1}{2}$, (b) : Ref. [27] for Example 4.

Table 8. Absolute error with $M = 8, M' = 3$ for various values of γ for Example 4.

(x,t)	$\gamma=\frac{1}{4}$	$\gamma=\frac{1}{3}$	$\gamma=\frac{1}{2}$	$\gamma=\frac{2}{3}$	$\gamma=1$
$(0.1, 0.1)$	2.07×10^{-4}	1.40×10^{-7}	1.98×10^{-16}	3.58×10^{-7}	2.33×10^{-6}
$(0.2, 0.2)$	1.47×10^{-3}	2.91×10^{-7}	5.99×10^{-16}	4.28×10^{-6}	2.92×10^{-5}
$(0.3, 0.3)$	3.55×10^{-3}	4.32×10^{-7}	1.02×10^{-15}	2.26×10^{-6}	1.78×10^{-5}
$(0.4, 0.4)$	5.25×10^{-3}	9.62×10^{-8}	1.37×10^{-15}	2.14×10^{-6}	1.84×10^{-5}
$(0.5, 0.5)$	5.52×10^{-3}	1.14×10^{-7}	1.62×10^{-15}	3.13×10^{-6}	2.90×10^{-5}
$(0.6, 0.6)$	4.28×10^{-3}	1.20×10^{-7}	1.78×10^{-15}	1.51×10^{-6}	1.43×10^{-5}
$(0.7, 0.7)$	2.32×10^{-3}	9.75×10^{-8}	2.10×10^{-15}	3.92×10^{-7}	3.27×10^{-6}
$(0.8, 0.8)$	7.33×10^{-4}	7.47×10^{-8}	1.51×10^{-15}	2.77×10^{-7}	2.49×10^{-6}
$(0.9, 0.9)$	6.92×10^{-5}	7.93×10^{-8}	2.40×10^{-15}	3.14×10^{-9}	4.52×10^{-7}
CPU	1.469	1.516	1.125	1.504	0.735

7. Conclusion

In this paper, a numerical scheme for computing approximate solution of distributed-order fractional partial differential equations is described. Our method is based on expanding the existing functions in terms of the fractional-order Bernoulli-Legendre functions. We have obtained the error bound of the proposed approximation. Also, a set of numerical examples has been presented. Our numerical findings are compared with exact solutions and with previous schemes. The comparison of the obtained results demonstrates that this scheme is very accurate.

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