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Alpert wavelet system for solving fractional nonlinear Fredholm integro-differential equations

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Abstract In this paper, we first construct Alpert wavelet system and propose a computational method for solving a fractional nonlinear Fredholm integro-differential equation. Then we create an operational matrix of fractional integration and use it to simplify the equation to a system of algebraic equations. By using Newtons iterative method, this system is solved, and then solution of the fractional nonlinear Fredholm integro-differential equations is achieved. Thresholding parameter is used to increase the sparsity of matrix coefficients and the speed of computations. Finally, the method is demonstrated by examples and the compared results with CAS wavelet method show that our proposed method is more effective and accurate.

Keywords. Alpert wavelet system, Fredholm integro-differential equation, Operational matrix, Fractional equation.

2010 Mathematics Subject Classification. 65Rxx, 65Txx, 45Bxx.

1. Introduction

Fractional calculus which deals with derivatives and integrals of arbitrary order has a significant role in modeling physical and engineering processes. Many authors have used fractional calculus to model the physical phenomenon like nonlinear oscillation of earthquakes [5] fluid-dynamic traffic [6] colored noise [12] solid machines [21] signal processing [19]. The reason of using fractional calculus is that many mathematical formulations contain nonlinear integro-differential equations with fractional order.
The fractional integro-differential equations are solved by different methods by several

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authors. For example, Momani and et. al. solved this equation by Adomian decomposition [13, 20]. Variationed iteration method and homotopy perturbation method are presented by Nawaz in [14]. Saeedi and et. al. proposed CAS wavelet method in [23]. Among these methods, the wavelet method is more accurate and fast, since wavelets simplify these problems to a system of algebraic equations by making the operational matrix and using it in the equation. Many kinds of wavelets like CAS wavelet [25], Haar wavelet [7], Legendre wavelet [11, 26], Chebyshev wavelet [3, 27], have been used to find a numerical solution of linear and nonlinear integral equations and differential equations. Also, we refer the interested reader in fractional integro-differential equations to see [8, 9, 15, 16, 17, 18, 22] for some recent works in the subject.

In this paper, our study focuses on a class of nonlinear Fredholm fractional integro-differential equation

$$D^{\alpha}f(x)-\lambda\int_0^1 k(x,t)[f(t)]^q dt = g(x), \quad q>1, \tag{1.1}$$

subject to the initial conditions

$$f^{(i)}(0)=\delta_i, \quad i=0,1,...,r-1, \qquad r-1<\alpha\le r,\;\; r\in\mathbb{N},$$

where $g\in L^2([0,1])$ and $k\in L^2([0,1])^2$ are given functions, $f(x)$ is the solution to be determined, D^{α} is the fractional derivative in the Caputo sense and q is a positive integer. We find a numerical solution of this problem by Alpert multiwavelet system. The wavelet numerical method has several advantages as follows:

- In classic bases like Chebyshev polynomials in spectral methods, as the degree of polynomials increases, the computational complexity increases, and these bases don't give good results for large numbers of the bases, but in this paper we can take large values for J to obtain better results.
- The solution is of multiresolution type.
- The main advantage is that after discretizing, the coefficient matrix of algebraic equations is sparse (We have shown this fact in the related Figures). Hence the method is easy to implement.

In this method the properties of Alpert multiwavelets are first given. The Riemann–Liouville fractional integral operator for Alpert multiwavelets is utilized to reduce the solution of nonlinear fractional Fredholm integro-differential equations to a system of algebraic equations. In order to save the memory requirement and computation time, a threshold procedure is applied to obtain sparse algebraic equations. The method is computationally very attractive and gives accurate results.

2. Preliminaries

In this section, we give some necessary preliminary definitions and preliminaries of the fractional calculus theory which will be used in this paper.

Definition 2.1. The Riemann- Liouville fractional integral operator I^α of order α on the usual Lebesgue space $L^1[a,b]$ is given by [24]

$$(I^\alpha f)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \quad \alpha > 0, \quad (I^0 f)(t) = f(t),$$

which I^α has the following properties:

(i) $I^\alpha I^\beta = I^\beta I^\alpha = I^{\alpha+\beta}$,

(ii) $I^\alpha (t-a)^v = \frac{\Gamma(v+1)}{\Gamma(\alpha+v+1)}(t-a)^{(\alpha+v)}$,

where $f \in L^1[a,b]$, $\alpha, \beta \geq 0$ and $v > -1$.

Definition 2.2. The Caputo definition of fractional differential operator D^α is given by [24]

$$(D^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)d\tau, \quad n-1 < \alpha \leq n,$$

where $t > 0$, n is an integer. The operator D^α has the following properties for $n-1 < \alpha \leq n$ and $f \in L^1[a,b]$,

$$(D^\alpha I^\alpha f)(t) = f(t)$$

and

$$(I^\alpha D^\alpha f)(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0)\frac{t^k}{k!}, \quad t > 0. \tag{2.1}$$

Definition 2.3. (MRA) A Multiresolution analysis of the Lebesgue space $L^2(\mathbb{R})$ consists of a sequence of nested subspaces $\{V_j^r\}_{j\in\mathbb{Z}} \subset L^2(\mathbb{R})$ such that satisfies in the following conditions:

(i) $...V_{-1}^r \subset V_0^r \subset V_1^r \subset ....,$

(ii) $\overline{span}(\cup_{j\in\mathbb{Z}} V_j^r) = L^2(\mathbb{R})$.

(iii) $\cap_{j\in\mathbb{Z}} V_j^r = \{0\}$.

(iv) $f(x) \in V_j^r \iff f(x+2^j) \in V_j^r \iff f(2x) \in V_{j+1}^r$,

(v) There exist orthogonal functions $\{\phi^k\}_{k=0,1,...,r-1} \in L^2(\mathbb{R})$ such that

$$V_0^r = \overline{span}\{\phi^k; 0 \leq k \leq r-1\}.$$

3. Construction of Scaling Functions, Wavelets and Wavelet transform matrix

Suppose that P_r is the Legendre polynomial of order r where r is a fixed nonnegative integer number. Let τ_k for $k = 0,1,...,r-1$ denote the roots of P_r. The interpolating scaling function (ISF), for $k = 0,1,...r-1$ is given in [11]

$$\phi_k(t) = \begin{cases} \sqrt{\frac{2}{\omega_k}} L_k(2t-1), & t \in [0,1], \\ 0, & otherwise, \end{cases}$$

where ω_k, are the Gauss-Legendre quadrature weights as [4]

$$\omega_k = \frac{2}{rP'_r(\tau_k)P_{r-1}(\tau_k)},$$

and $L_k(t)$ are the lagrange interpolating polynomials as

$$L_k(t) = \prod_{i=0, i\neq k}^{r-1} (\frac{t-\tau_i}{\tau_k - \tau_i}),$$

that they have characterized by Kronecker property,

$$L_k(\tau_i) = \delta_{ik} = \begin{cases} 1 & i = k, \\ 0 & i \neq k. \end{cases}$$

By Considering $V_1^r = V_0^r \oplus W_0^r$ and $V_0^r \subset V_1^r$ in (MRA), there exist sequences of coefficients $\{g_{i,j}\}$ and $\{h_{i,j}\}$ in $L^2(\mathbb{R})$ such that for $k = 0, 1, ..., r-1$,

$$\phi^k(x) = \sum_{j=0}^{r-1} (g^0_{k+1,j+1}\phi^j(2x) + g^1_{k+1,j+1}\phi^j(2x-1)),$$

and

$$\psi^k(x) = \sum_{j=0}^{r-1} (h^0_{k+1,j+1}\phi^j(2x) + h^1_{k+1,j+1}\phi^j(2x-1)).$$

The above relations are called two scale relations for scaling functions and wavelets, respectively and the coefficients $g^l_{i,j}$ and $h^l_{i,j}$, $l = 0, 1$, are called filter coefficients. To show the filter coefficients in above representations, we use four $r \times r$ matrices as

$$G^0 = \begin{bmatrix} g^0_{11} & \cdots & g^0_{1r} \\ \vdots & & \vdots \\ g^0_{r1} & \cdots & g^0_{rr} \end{bmatrix} \qquad G^1 = \begin{bmatrix} g^1_{11} & \cdots & g^1_{1r} \\ \vdots & & \vdots \\ g^1_{r1} & \cdots & g^1_{rr} \end{bmatrix}$$

$$H^0 = \begin{bmatrix} h^0_{11} & \cdots & h^0_{1r} \\ \vdots & & \vdots \\ h^0_{r1} & \cdots & h^0_{rr} \end{bmatrix} \qquad H^1 = \begin{bmatrix} h^1_{11} & \cdots & h^1_{1r} \\ \vdots & & \vdots \\ h^1_{r1} & \cdots & h^1_{rr} \end{bmatrix}$$

The matrices G^0 and G^1 are called filter coefficient matrix for scaling functions and H^0 and H^1 are called filter coefficient matrix for Alpert wavelets, which their coefficients are given by interpolating property of scaling functions as

$$g^0_{k,k\prime} = \frac{\sqrt{\omega_{k'}}}{2}\phi^k(\frac{1+\tau_{k'}}{4}) \qquad g^1_{k,k\prime} = \frac{\sqrt{\omega_{k'}}}{2}\phi^k(\frac{3+\tau_{k'}}{4})$$

$$h^0_{k,k\prime} = \frac{\sqrt{\omega_{k'}}}{2}\psi^k(\frac{1+\tau_{k'}}{4}) \qquad h^1_{k,k\prime} = \frac{\sqrt{\omega_{k'}}}{2}\psi^k(\frac{3+\tau_{k'}}{4}).$$

In general, the two scale relation for the sequent spaces V_J and V_{J+1} is given by

$$\Phi^r_J(x) = G_J \Phi^r_{J+1}(x),$$

where $\Phi_J^r(x)$ consists of $r2^J \times 1$ bases of the space V_J^r and G_J is called the transform matrix between scaling functions in two sequent spaces that is obtained in the following form:

$$G_J = \begin{bmatrix} G & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & G \end{bmatrix}_{r2^J, r2^{J+1}}$$

where $G = [G^0 G^1]$.
In the same way the two scale relation between the spaces W_J and V_{J+1} is obtained by

$$\Psi_J^r(x) = T_J \Phi_{J+1}^r(x), \tag{3.1}$$

where $\Psi_J^r(x)$ consists of $r2^J \times 1$ bases of the space W_J^r and T_J is called the wavelet transform matrix [1, 2, 10].
Consider

$$H_J = \begin{bmatrix} H & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & H \end{bmatrix}_{r2^J, r2^{J+1}}$$

where $H = [H^0 H^1]$. By using above matrices, we get the wavelet transform matrix

$$T_J = \begin{bmatrix} G_0 \times G_1 \times \cdots \times G_{J-1} \\ H_0 \times G_1 \times \cdots \times G_{J-1} \\ H_1 \times G_2 \times \cdots \times G_{J-1} \\ \vdots \\ H_{J-2} \times G_{J-1} \\ H_{J-1} \end{bmatrix}_{r2^J, r2^J}.$$

We will use this matrix to solve the fractional nonlinear integro-differential equations using wavelets.

4. Construction of operational matrix of fractional integration

The Riemann-Liouville fractional integral of scaling function ϕ^k is of the form

$$I_t^\alpha \phi^k(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \phi^k(t) \ \ dt. \tag{4.1}$$

For $0 < x < 1$, (4.1) can be written as

$$a(x) = I_t^\alpha \phi^k(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \phi^k(t), \tag{4.2}$$

and for $x \geq 1$ it can be written as

$$b(x) = \frac{1}{\Gamma(\alpha)} \int_0^1 (x-t)^{\alpha-1} \phi^k(t) \ dt. \tag{4.3}$$

By (4.2) and (4.3), the Riemann-Liouville fractional integral of the scaling function is obtained as follows:

$$I_t^{\alpha}\phi^k(x) = \Omega^k(x) = \begin{cases} a(x), & 0 < x < 1, \\ b(x), & x \geq 1, \\ 0, & otherwise. \end{cases} \tag{4.4}$$

Now, by using (4.4) the Riemann-Liouville fractional integral of Φ_J is of the form

$$I_t^{\alpha}\Phi_J(x) = \begin{bmatrix} \Omega^0(2^J x) \\ \Omega^1(2^J x) \\ \vdots \\ \Omega^{r-1}(2^J x) \\ \vdots \\ \Omega^0(2^J x - 2^J + 1) \\ \vdots \\ \Omega^{r-1}(2^J x - 2^J + 1) \end{bmatrix}_{r2^J,1} = L(x) = I_{\alpha,\phi}\Phi_J(x), \tag{4.5}$$

where $I_{\alpha,\phi}$ is the Riemann-Liouville fractional integral operator matrix for the scaling function with dimension $r2^J \times r2^J$ and its coefficients are given by interpolation property as

$$[I_{\alpha,\phi}]_{i+1,rl+(k+1)} = 2^{-J-2}\sqrt{\frac{\omega_k}{2}}L_{i+1,1}(\frac{\tau_k + 2l + 1}{2^{J+1}}),$$

where

$$i = 0, 1, ..., r2^J - 1, \quad l = 0, 1, ..., 2^J - 1, \quad k = 0, 1, ..., r - 1.$$

Also, the Riemann-Liouville fractional integral of wavelet is obtained by (3.1) and (4.5) as

$$I_t^{\alpha}\Psi_J(x) \approx I_{\alpha,\psi}\Psi_J(x), \tag{4.6}$$

in which

$$\begin{aligned} I_t^{\alpha}\Psi_J(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\Psi_J(t) \;\; dt &= T_J\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\Phi_J(t) \;\; dt \\ &= T_J I_{\alpha,\phi}\Phi_J(x) \\ &= T_J I_{\alpha,\phi}T_J^{-1}\Psi_J(x). \end{aligned} \tag{4.7}$$

So by (4.6) and (4.7) we have

$$I_{\alpha,\psi} = T_J I_{\alpha,\phi} T_J^{-1}.$$

4.1. **Numerical method.** Here, the numerical solution is proposed to solve the fractional nonlinear Fredholm integro-differential equation using the Alpert wavelet system. By the wavelet compression property, it is shown that the usage of Alpert wavelets will increase the speed of calculations by transforming the equation to a system of algebraic equations. Using the larger thresholding parameter will increase the sparsity of the coefficient matrix in the Alpert wavelet system. Consider the fractional nonlinear-Fredholm integro-differential equation given in (1.1). If we approximate the function $D^\alpha f(x)$ with the Alpert wavelets, we have

$$D^\alpha f(x) \approx F^T \Psi_J^r(x), \tag{4.8}$$

where F^T is an unknown vector of order $1 \times r2^J$. Now, by Riemann-Liouville fractional integrating of the equation (4.8) and using (4.6), we have

$$I_t^\alpha D^\alpha f(x) \approx I_t^\alpha (F^T \Psi_J^r(x)) = F^T I_{\alpha,\psi} \Psi_J^r(x). \tag{4.9}$$

Applying (2.1) we can approximate $f(t)$ as

$$f(t) = F^T I_{\alpha,\psi} \Psi_J^r(t) + \sum_{i=0}^{n-1} f^i(0) \frac{t^i}{i!}. \tag{4.10}$$

On the other hand, we have

$$m(t) = \sum_{i=0}^{n-1} f^i(0) \frac{t^i}{i!} \approx \widehat{M}^T \Phi_J^r(t) = \underbrace{\widehat{M}^T T_J^{-1}}_{M^T} \Psi_J^r(t) = M^T \Psi_J^r(t). \tag{4.11}$$

Now by (4.9), (4.10), and (4.11), we have

$$f(t) = (F^T I_{\alpha,\psi} + M^T) \Psi_J^r(t). \tag{4.12}$$

Therefore

$$n(t) = [f(t)]^q = ((F^T I_{\alpha,\psi} + M^T) \Psi_J^r(t))^q \simeq N^T \Psi_J^r(t), \tag{4.13}$$

where N is an unknown vector of order $r2^J \times 1$.
Also, known functions $g(x)$ and $k(x,t)$ in (1.1) can be approximated in terms of the basic functions of the space V_J in the form

$$\begin{aligned} k(x,t) \approx {\Phi_J^r}^T(x) \widehat{K} \Phi_J^r(t) &= {\Psi_J^r}^T(t) \underbrace{{T_J^{-1}}^T \widehat{K} T_J^{-1}}_{K} \Psi_J^r(x) \\ &= {\Psi_J^r}^T(t) K \Psi_J^r(x), \end{aligned} \tag{4.14}$$

and

$$g(x) \approx \widehat{G}^T \Phi_J^r(x) = \underbrace{\widehat{G}^T T_J^{-1}}_{G^T} \Psi_J^r(x) = G^T \Psi_J^r(x), \tag{4.15}$$

where K is a matrix of order $r2^J \times r2^J$ and G is a vector of order $r2^J \times 1$, which can be obtained as

$$K_{ij} = 2^{-J} \sqrt{\frac{w_k}{2}}. \tag{4.16}$$

Considering the orthogonality property of Alpert wavelets, we have

$$I = \int_0^1 {\Psi^r}_J^T(x)\Psi_J^r(x)\ dx, \tag{4.17}$$

where I is the identity matrix of order of $r2^J \times r2^J$.
Now, by (4.17) and putting (4.12)-(4.15) in (1.1), we have

$$F^T\Psi^J(x) - \lambda N^T I K^T \Psi_J^r(x) = G^T\Psi^J(x),$$

From the above equation, we have

$$(F^T - \lambda N^T K^T - G^T)\Psi_J^r(x) = 0,$$

Because of independency of entries of vector $\Psi(x)$, we get the nonlinear system of algebraic equations as

$$F^T - \lambda N^T K^T - G^T = 0,$$

which can be solved by iterative Newton method.

5. Illustrative examples

To show the efficiency and the accuracy of the proposed method based on Alpert multiwavelets, we consider the following two examples chosen from the [23, 27] and compare our method with the CAS wavelet method and second kind Chebyshev wavelet. To increase the computational speed, we use the thresholding method, based on the compression property of the wavelets. For this purpose, the parameter ϵ is chosen as the thresholding parameter, so that all elements of the coefficient matrix which are smaller than ϵ are considered to be zero. This work increases the sparsity of the coefficient matrix and also increases the computational speed.

Example 1. Consider the fractional nonlinear Fredholm integro-differential equation with $\alpha = \frac{1}{2}$

$$D^{1/2}f(x) - \int_0^1 xt[f(t)]^4\ \ dt = \frac{1}{\Gamma(\frac{1}{2})}(\frac{8}{3}x^{3/2} - 2x^{1/2}) - \frac{x}{1260} \qquad 0 < x < 1,$$

with exact solution $f(x) = x^2 - x$.
In Table 1, the L_2 error reported by the proposed method and CAS wavelet bases [23] together with CPU time for different numbers of bases are listed. From this Table, we can observe the convergence of numerical solutions as J increases.
In order to demonstrate the validity of our numerical findings, we show the values of L_2 error with J = 5, 6, $r = 2$ and different values of thresholding parameter ε, using the method presented in the previous section by multiwavelets, in Table 2.
Example 2. Consider the fractional nonlinear Fredholm integro-differential equation with $\alpha = 5/6$

$$D^{5/6}f(x) - \int_0^1 xe^t[f(t)]^2\ \ dt = \frac{3}{\Gamma(1/6)}(2x^{1/6} - \frac{432}{91}x^{13/6}) + x(248e - 674),$$

with exact solution $f(x) = x - x^3$.
For the purpose of comparison in Table 3, we compare the L_2 error of our method with $J = 5, 6$ and $r = 3, 4$ together with CAS wavelet method given in [23] and

TABLE 1. Computational results for Example 1.

Methods	L_2 *Error*	CPU times
CAS wavelet [23]		
$k = 2, M = 1$	$7.711e-04$	3.911
$k = 3, M = 1$	$2.0755e-05$	5.962
Alpert multiwavelets		
$r = 3$, J = 5	$4.1661e-04$	7.863
$r = 3$, J = 6	$1.2463e-04$	9.041
$r = 4$, J = 6	$1.7295e-05$	10.151

TABLE 2. Compression percentage and L_2 error with the use of Alpert wavelets. Example 1.

	Parameter States(ε)	Compression S_ε	Error L_2
	0	% 0	0.001429705
$J = 5$	10^{-4}	% 78.24	0.001429721
	10^{-3}	% 78.90	0.001429878
	10^{-2}	% 83.59	0.001512517
	0	% 0	0.00071406
$J = 6$	10^{-4}	% 87.56	0.00071452
	10^{-3}	% 88.34	0.00071729
	10^{-2}	% 91.61	0.000961306

second kind Chebyshev wavelet method [27]. Figures 1, shows the plots of the matrix elements for $r = 2$ and $J = 5$ with $\epsilon = 10^{-4}, 10^{-3}$.
Table 4 shows the sparsity and L_2 error for $r = 2$, $J = 5, 6$ and different values of thresholding parameter, using the method presented in the previous section.

6. Conclusion

In this manuscript, the Alpert wavelet system is constructed and its operational matrix of the fractional integration is derived. We use this system to solve a class of nonlinear Fredholm integro-differential equation of fractional order numerically by simplifying the equation to an algebraic one.
The main advantage of the wavelet method for solving the equation is that after discreting the coefficients matrix of algebraic equations is sparse. The solution is

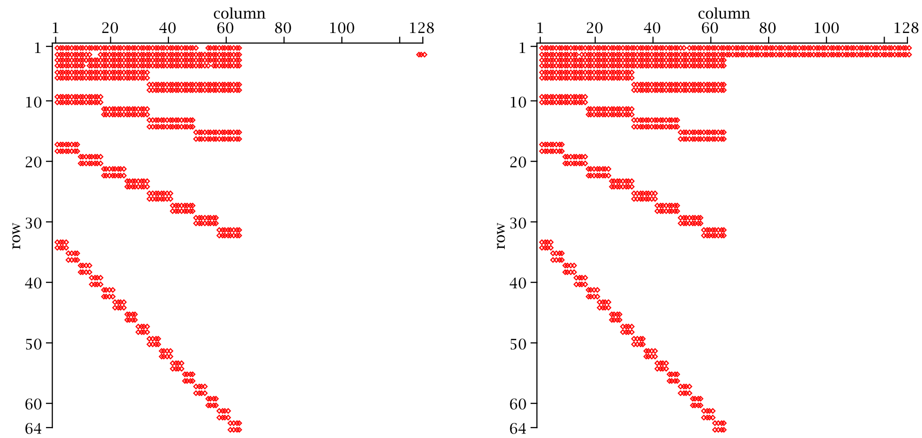

FIGURE 1. Compression matrices after thresholding with $\varepsilon = 10^{-4}$ (right) and $\varepsilon = 10^{-3}$ (left) using Alpert wavelets.

TABLE 3. Computational results for Example 2.

Methods	L_2 *Error*
CAS wavelet [23]	
$k = 2, M = 1$	$2.0862e - 03$
$k = 3, M = 1$	$6.3440e - 04$
Chebyshev wavelet [27]	
$k = 3, M = 2$	$6.0313e - 05$
Alpert multiwavelets	
$r = 3, \mathrm{J} = 5$	$1.4297e - 03$
$r = 3, \mathrm{J} = 6$	$5.0795e - 04$
$r = 4, \mathrm{J} = 6$	$5.8041e - 05$

convergent, even though the size of the increment may be large.
CAS wavelet [23] is constructed from the trigonometric polynomials and has periodicity. It is more suitable for solving the periodic problem. However, the problems we

TABLE 4. The percentage of compression and L_2 error with the use of Alpert wavelet system. Example 2.

	Parameter States (ε)	Compression S_ε	error L_2
	0	% 0	0.00216373
$J = 5$	10^{-4}	% 78.17	0.00216179
	10^{-3}	% 81.37	0.00216369
	10^{-2}	% 83.88	0.00235226
	0	% 0	0.00107926
$J = 6$	10^{-4}	% 88.28	0.00107954
	10^{-3}	% 89.30	0.00107989
	10^{-2}	% 91.61	0.00147190

usually deal with are non-periodic, and the examples considered here are also non-periodic. Compared with the CAS wavelet, the Alpert multiwavelets are constructed from the Lagrange interpolating polynomials. When solving the non-periodic problems, the Alpert wavelet has the superiorities (the calculation is easy implementation, and the approximation effect is better or our method is comparable to CAS wavelets).

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