



Fourth-order numerical method for the Riesz space fractional diffusion equation with a nonlinear source term

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Abstract This paper aims to propose a high-order and accurate numerical scheme for the solution of the nonlinear diffusion equation with Riesz space fractional derivative. To this end, we first discretize the Riesz fractional derivative with a fourth-order finite difference method, then we apply a boundary value method (BVM) of fourth-order for the time integration of the resulting system of ordinary differential equations. The proposed method has a fourth-order of accuracy in both space and time components and is unconditionally stable due to the favorable stability property of BVM. The numerical results are compared with analytical solutions and with those provided by other methods in the literature. Numerical experiments obtained from solving several problems including fractional Fisher and fractional parabolic-type sine-Gordon equations show that the proposed method is an efficient algorithm for solving such problems and can arrive at the high-precision.

Keywords. Compact finite difference method, Boundary value methods, Riesz space fractional derivatives, Unconditional stability, Diffusion equation.

2010 Mathematics Subject Classification. 65M06, 65M15, 35R11.

1. INTRODUCTION

In recent years, fractional differential equations have been successfully applied to modelling the various problems in fields of science and engineering. For instance, the fractional advection-dispersion equation is used in groundwater hydrology to model the transport of passive tracers carried by fluid flow in a porous medium and for solute transport in a subsurface material [17]. In spite of the fact that a considerable amount of research has been carried out on the theoretical analysis of these equations, analytic solution of many fractional differential equations cannot be obtained explicitly. So the numerical solution of fractional differential equations has become a valuable research topic [1, 3, 11, 12, 18, 22]. This inspires us to search for new and efficient numerical methods for solving these equations.

Received: 25 November 2019 ; Accepted: 05 April 2020.

In this paper we investigate the diffusion equation with Riesz space fractional derivative and nonlinear source term

$$\frac{\partial u(x, t)}{\partial t} = \kappa \frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha} + f(u, x, t), \quad a < x < b, \quad 0 \leq t \leq T, \tag{1.1}$$

with initial condition

$$u(x, 0) = \phi(x), \quad a < x < b, \tag{1.2}$$

and boundary conditions

$$u(a, t) = u(b, t) = 0, \quad 0 \leq t \leq T, \tag{1.3}$$

where $1 < \alpha < 2$, $\kappa > 0$ is diffusion coefficient and $\frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha}$ is the Riesz fractional operator which is the linear combination of the left and right Riemann-Liouville fractional derivatives, i.e

$$\frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha} = -\frac{1}{2 \cos(\alpha\pi/2)} [{}_a D_x^\alpha + {}_x D_b^\alpha] u(x, t), \tag{1.4}$$

where ${}_a D_x^\alpha$ is the left Riemann-Liouville fractional derivative defined as

$${}_a D_x^\alpha = \frac{1}{\Gamma(2 - \alpha)} \frac{\partial^2}{\partial x^2} \int_a^x (x - \xi)^{1-\alpha} u(\xi, t) d\xi,$$

and ${}_x D_b^\alpha$ is the right Riemann-Liouville fractional derivative defined as

$${}_x D_b^\alpha = \frac{1}{\Gamma(2 - \alpha)} \frac{\partial^2}{\partial x^2} \int_x^b (\xi - x)^{1-\alpha} u(\xi, t) d\xi.$$

Also $f(u, x, t)$ is the nonlinear source term that is a smooth function satisfies the Lipschitz condition

$$|f(u, x, t) - f(v, x, t)| \leq \mathcal{L}|u - v|,$$

in which $\mathcal{L} > 0$ is the Lipschitz constant.

The space fractional diffusion equations has many applications in fluid flow in porous materials, anomalous diffusion transport, chemistry [19, 20]. This equation is commonly used to model the growth and spreading of biological species.

There are several numerical approximations for the space fractional diffusion equation. Using the concept of fractional centred derivative, a CrankNicolson scheme of order $\mathcal{O}(\tau^2 + h^2)$ is suggested in [7] for linear form of Eq. (1.1). The resulting method for the Riesz space fractional diffusion equation is shown to be unconditionally stable and convergent. Another type of second order method is proposed in [16] using the backward difference formula. A Crank-Nicolson second order finite difference scheme for the Riesz space fractional-order parabolic-type sine-Gordon equation is given in [26]. Zhang and Liu [24] proposed an implicit finite difference method of order $\mathcal{O}(\tau + h)$ and proved the unconditional stability and convergence of method. Authors of [8], have given the fundamental solution of Riesz space fractional diffusion equation and proosed a finite difference scheme of order $\mathcal{O}(\tau + h^{2-\alpha})$. Based on the parameter spline function and improved matrix transform method, a numerical method is given in [25] with stability discussion of the difference schemes by the matrix method. A



second-order finite difference for Riesz space fractional diffusion equations with delay and a nonlinear source term is given in [27]. In [13], the space-fractional derivatives are approximated by a fourth order weighted and shifted Grunwald-Letnikov formula and authors proposed a method of order $\mathcal{O}(\tau^2 + h^4)$ for Riesz space fractional diffusion equation. Based on the idea presented in [13], a linearized quasi-compact finite difference scheme is proposed in [14] for semilinear space-fractional diffusion equations with a time delay. Also a tau approach for solution of the space fractional diffusion equation is given in [21].

The aim of this paper is to propose an unconditionally stable numerical method of order $\mathcal{O}(\tau^4 + h^4)$ for the solution of Eq. (1.1). We apply a fourth order finite difference scheme for space variable and a fourth order boundary value method for time component. Based on the linear multi-step formulas, boundary value methods (BVMs) are high-accuracy and unconditionally stable schemes for solving ordinary differential equations [4, 5, 6]. Unlike Runge-Kutta or other initial value methods, BVMs achieve the advantage of both high accuracy and good stability [2].

The rest of this paper is organized as follows: In Section 2 we introduce the BVM for the solution of ordinary and system of ordinary differential equations. In Section 3, we propose the new numerical method for the solution of Eq. (1.1). The results of numerical experiments are given in Section 4 and we compare them with analytical solutions and other existing methods for confirming the good accuracy of the proposed scheme. We conclude this article with a brief conclusive discussion in Section 5.

2. THE BOUNDARY VALUE METHODS

For positive integer numbers M and N , let $h = \frac{b-a}{M}$ denotes the step size of spatial variable, x , and $\tau = \frac{T}{N}$ denotes the step size of time variable, t . So we define

$$\begin{aligned} x_i &= a + ih & , & \quad i = 0, 1, 2, \dots, M, \\ t_k &= k\tau & , & \quad k = 0, 1, 2, \dots, N. \end{aligned}$$

BVMs constitute a class of methods for the solution of ordinary differential equations which is the generalization of the linear multi-step methods [4, 6]. These methods have high order of accuracy and are unconditionally stable methods. We briefly introduce these methods for the following initial value problem

$$\dot{y}(t) = f(t, y(t)), \quad y(0) = y_0, \quad t \geq 0. \quad (2.1)$$

In BVMs, at first we assume that the ODE is a boundary value problem and then impose extra initial and final conditions on the unknowns values of the boundaries. A k -step BVMs formula for the Eq. (3.5) can be written as

$$\sum_{i=0}^k \alpha_i y_{i+j} = \tau \sum_{i=0}^k \beta_i f_{i+j}, \quad , \quad j = 0, 1, \dots, N - k, \quad (2.2)$$

where τ is the step size of time and y_i is the approximate value for $y(i\tau)$, $t_i = i\tau$ and $f_i = f(y_i, t_i)$. The linear multi-step formula (3.6) must be with γ initial and $k - \gamma$ final



where \otimes is the Kronecker production and

$$\begin{aligned}
 y &= [y_1(t_1), y_2(t_1), \dots, y_m(t_1), y_1(t_2), y_2(t_2), \dots, y_m(t_2), \dots, y_1(t_N), y_2(t_N), \dots, y_m(t_N)]^T, \\
 g &= [g_1(t_1), g_2(t_1), \dots, g_m(t_1), g_1(t_2), g_2(t_2), \dots, g_m(t_2), \dots, g_1(t_N), g_2(t_N), \dots, g_m(t_N)]^T, \\
 f(y) &= [f_1(y(t_1)), f_2(y(t_1)), \dots, f_m(y(t_1)), f_1(y(t_2)), f_2(y(t_2)), \dots, f_m(y(t_2)), \\
 &\quad \dots, f_1(y(t_N)), f_2(y(t_N)), \dots, f_m(y(t_N))]^T, \\
 f_0 &= [f_1(y(t_0)), f_2(y(t_0)), \dots, f_m(y(t_0))]^T, \\
 g_0 &= [g_1(t_0), g_2(t_0), \dots, g_m(t_0)]^T, \\
 y_0 &= [y_1(t_0), y_2(t_0), \dots, y_m(t_0)]^T.
 \end{aligned}$$

3. PROPOSED NUMERICAL SCHEME

Recently a fourth order quasi-compact difference scheme to Riemann- Liouville fractional derivatives is proposed in [13]. Due to the relationship between the Riemann-Liouville fractional derivatives and the Riesz fractional derivative (1.4), we can obtain a fourth order approximation of the Riesz fractional derivatives. Let δ_x^2 be the second-order central difference operator, i.e

$$\delta_x^2 u(x) = \frac{u(x - h) - 2u(x) + u(x + h)}{h^2},$$

and the left and right shifted Grunwald difference quotient operators be

$$\begin{aligned}
 {}_L A_{h,p}^\alpha u(x) &= \frac{1}{h^\alpha} \sum_{j=0}^\infty g_j^{(\alpha)} u(x - (j - p)h), \\
 {}_R A_{h,p}^\alpha u(x) &= \frac{1}{h^\alpha} \sum_{j=0}^\infty g_j^{(\alpha)} u(x + (j - p)h),
 \end{aligned} \tag{3.1}$$

respectively, in which p is an integer and

$$g_0^{(\alpha)} = 1, \quad g_k^{(\alpha)} = \frac{(-1)^k}{k!} \alpha(\alpha - 1)\dots(\alpha - k + 1), \quad k = 1, 2, \dots$$



Let

$$\begin{aligned} {}_L\delta_x^\alpha u(x) &= \lambda_1 {}_L A_{h,1}^\alpha u(x) + \lambda_0 {}_L A_{h,0}^\alpha u(x) + \lambda_{-1} {}_L A_{h,-1}^\alpha u(x), \\ {}_R\delta_x^\alpha u(x) &= \lambda_1 {}_R A_{h,1}^\alpha u(x) + \lambda_0 {}_R A_{h,0}^\alpha u(x) + \lambda_{-1} {}_R A_{h,-1}^\alpha u(x), \end{aligned} \quad (3.2)$$

where

$$h > 0, \quad \lambda_1 = \frac{\alpha^2 + 3\alpha + 2}{12}, \quad \lambda_0 = \frac{4 - \alpha^2}{6}, \quad \lambda_{-1} = \frac{\alpha^2 - 3\alpha + 2}{12}.$$

Using the following quasi-compact operator

$$\mathcal{H}_x^\alpha u(x) = (I + d_\alpha \delta_x^2) u(x), \quad d_\alpha = \frac{-\alpha^2 + \alpha + 4}{24}, \quad (3.3)$$

where I is the identity operator, authors of [13] derived the following fourth-order approximations to Riemann-Liouville fractional derivatives.

Lemma 3.1. [13] *Let $u(x) \in L_1(\mathbb{R})$ and $u(x) \in C^{4+\alpha}(\mathbb{R})$, where*

$$C^{4+\alpha}(\mathbb{R}) = \left\{ u \mid \int_{-\infty}^{+\infty} (1 + |\tau|)^{4+\alpha} |\hat{u}(\tau)| d\tau < \infty \right\},$$

in which $\hat{u}(\tau) = \int_{-\infty}^{+\infty} e^{i\tau x} u(x) dx$ is the Fourier transformation of $u(x)$. Then we have

$${}_L\delta_x^\alpha u(x) = \mathcal{H}_x^\alpha ({}_{-\infty}D_x^\alpha u(x)) + \mathcal{O}(h^4),$$

$${}_R\delta_x^\alpha u(x) = \mathcal{H}_x^\alpha ({}_xD_{+\infty}^\alpha u(x)) + \mathcal{O}(h^4).$$

Using Lemma 3.1, we can obtain a fourth-order approximation to Riesz fractional derivatives. For $u(x) \in C[a, b]$ with $u(a) = u(b) = 0$, making zero-extension of $u(x)$ such that $u(x)$ is defined on \mathbb{R} and regarding to (3.1) and (3.2), we can write

$${}_L\delta_x^\alpha u(x) = \frac{1}{h^\alpha} \sum_{j=0}^{\lfloor \frac{x-a}{h} \rfloor} w_j^{(\alpha)} u(x - (j-1)h),$$

$${}_R\delta_x^\alpha u(x) = \frac{1}{h^\alpha} \sum_{j=0}^{\lfloor \frac{b-x}{h} \rfloor} w_j^{(\alpha)} u(x + (j-1)h),$$

where

$$w_0^{(\alpha)} = \lambda_1 g_0^{(\alpha)}, \quad w_1^{(\alpha)} = \lambda_1 g_1^{(\alpha)} + \lambda_0 g_0^{(\alpha)},$$

$$w_j^{(\alpha)} = \lambda_1 g_j^{(\alpha)} + \lambda_0 g_{j-1}^{(\alpha)} + \lambda_{-1} g_{j-2}^{(\alpha)}, \quad j \geq 2.$$



Define

$$\delta_x^\alpha u(x) = -\frac{1}{\cos\left(\frac{\alpha\pi}{2}\right)} [R\delta_x^\alpha u(x) + {}_R\delta_x^\alpha u(x)],$$

then from Lemma 3.1 we have

$$\delta_x^\alpha u(x) = \mathcal{H}_x^\alpha \left(\frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha} \right) + \mathcal{O}(h^4). \tag{3.4}$$

Consider the following fractional PDE,

$$\frac{\partial u(x, t)}{\partial t} = \kappa \frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha} + f(u(x, t)) + g(x, t), \quad a < x < b, \quad 0 \leq t \leq T. \tag{3.5}$$

Applying operator \mathcal{H}_x^α on both sides of Eq. (3.5) at point (x_j, t) , gives

$$\mathcal{H}_x^\alpha \frac{\partial u(x_j, t)}{\partial t} = \kappa \mathcal{H}_x^\alpha \frac{\partial^\alpha u(x_j, t)}{\partial |x|^\alpha} + \mathcal{H}_x^\alpha f(u(x_j, t)) + \mathcal{H}_x^\alpha g(x_j, t), \quad j = 1, 2, \dots, M-1. \tag{3.6}$$

Regarding to (3.4), there exists a series of bounded functions $\mathcal{R}_j(t)$ such that

$$\begin{aligned} H_x^\alpha \frac{d}{dt} u_j(t) = & -\frac{\kappa}{h^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} \left[\sum_{k=0}^{j+1} w_k^{(\alpha)} u_{j-k+1}(t) + \sum_{k=0}^{M-j+1} w_k^{(\alpha)} u_{j+k+1}(t) \right] \\ & + H_x^\alpha f(u_j(t)) + H_x^\alpha g_j(t) + h^4 R_j(t), \quad j = 1, 2, \dots, M-1. \end{aligned} \tag{3.7}$$

where $u_j(t) = u(x_j, t)$. Using boundary conditions (1.3), we can write (3.7) in the matrix-vector form

$$A_x \frac{d}{dt} U(t) = \kappa B_x U(t) + A_x f(U(t)) + A_x G(t) + h^4 \mathcal{R}(t), \tag{3.8}$$

where

$$\begin{aligned} U(t) &= [u_1(t), u_2(t), \dots, u_{M-1}(t)]^T, \\ f(U(t)) &= [f(u_1(t)), f(u_2(t)), \dots, f(u_{M-1}(t))]^T, \\ G(t) &= [g_1(t), g_2(t), \dots, g_{M-1}(t)]^T, \\ \mathcal{R}(t) &= [\mathcal{R}_1(t), \mathcal{R}_2(t), \dots, \mathcal{R}_{M-1}(t)]^T, \\ A_x &= \text{tridiag}[d_\alpha, 1 - 2d_\alpha, d_\alpha], \\ B_x &= \text{toeplitz} \left[2w_1^{(\alpha)}, w_0^{(\alpha)} + w_2^{(\alpha)}, w_3^{(\alpha)}, w_4^{(\alpha)}, \dots, w_{M-1}^{(\alpha)} \right]. \end{aligned}$$



Let $y(t)$ be the approximation of $U(t)$. Neglecting the small term $h^4\mathcal{R}(t)$ in (3.8), we obtain the following semi-discretization scheme

$$A_x \frac{d}{dt} y(t) = \kappa B_x y(t) + A_x f(y(t)) + A_x G(t). \quad (3.9)$$

Using BVM approximation (2.10), we can obtain the solution of (3.9).

Remark 3.2. In a similar manners presented in [15, 23], one can show that the proposed method is locally stable and convergent.

4. NUMERICAL RESULTS

In this section we present the numerical results of the new method on several test problems. We tested the accuracy and stability of the method described in this paper by performing the mentioned scheme for different values of time and space step sizes. We calculated the computational order of the method presented in this article (denoted by C-order) with the following formula :

$$C - order = \frac{\log\left(\frac{e_1}{e_2}\right)}{\log\left(\frac{h_1}{h_2}\right)},$$

in which e_1 and e_2 are errors correspond to grids with mesh size h_1 and h_2 respectively. In all computations we put $h = \tau$ and report the maximum error in tables. Our calculations are run in Matlab software and on Intel Core 2 Duo CPU with 2.8 GHz speed and 2 GB RAM.

4.1. Test problem 1. We consider the following space fractional equation [16]

$$\frac{\partial u(x, t)}{\partial t} = \frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha} + 2^{10} t^3 x^4 (1-x)^4 + 2^7 t^4 \Psi(x, \alpha), \quad (x, t) \in (0, 1) \times (0, 1],$$

where

$$\Psi(x, \alpha) = \frac{1}{2 \cos(\alpha\pi/2)} [\psi_8(x, \alpha) - 4\psi_4(x, \alpha) + 6\psi_6(x, \alpha) - 4\psi_5(x, \alpha) + \psi_4(x, \alpha)],$$

with $\psi_k(x, \alpha) = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} [x^{k-\alpha} + (1-x)^{k-\alpha}]$. The exact solution is $2^8 t^4 x^4 (1-x)^4$. In Table 1 we compare the results of proposed method in this paper and the results of [16] for different values of fractional orders. The results of BDF2 method [16] are presented for 1000 spatial grid points ($h = 0.001$) while for our proposed method we put $h = \tau$.

Table 1: Comparison of errors for Test problem 1

τ	Present method		Method of [16] with $h = 0.001$	
	$\alpha = 1.2$	$\alpha = 1.8$	$\alpha = 1.2$	$\alpha = 1.8$
1/10	7.7477×10^{-4}	1.9121×10^{-3}	8.2078×10^{-3}	4.4906×10^{-3}
1/20	5.8424×10^{-5}	1.3206×10^{-4}	2.1717×10^{-3}	1.1768×10^{-3}
1/40	4.0838×10^{-6}	8.5625×10^{-6}	5.5796×10^{-4}	3.0113×10^{-4}
1/80	2.7261×10^{-7}	5.3669×10^{-7}	1.4147×10^{-4}	7.6346×10^{-5}
1/160	1.7775×10^{-8}	3.3057×10^{-8}	3.5729×10^{-5}	1.9441×10^{-5}



Table 2: Comparison of errors for Test problem 1

h	Present method		Method of [16] with $\tau = 0.001$	
	$\alpha = 1.2$	$\alpha = 1.8$	$\alpha = 1.2$	$\alpha = 1.8$
1/8	1.7453×10^{-3}	4.4174×10^{-3}	1.1075×10^{-2}	2.1003×10^{-2}
1/16	1.3555×10^{-4}	3.1500×10^{-4}	2.6458×10^{-3}	4.8526×10^{-3}
1/32	9.6798×10^{-6}	2.0757×10^{-5}	6.5515×10^{-4}	1.1922×10^{-4}
1/64	6.5381×10^{-7}	1.3124×10^{-6}	1.6355×10^{-4}	2.9688×10^{-5}
1/128	4.2881×10^{-8}	8.1187×10^{-8}	4.1034×10^{-5}	7.4233×10^{-5}

Tables 1,2 show that the proposed method has more accurate results in comparison with the method developed in [16]. In Table 3 we present the error, computational order and CPU time of method.

Table 3: Error, C-order and CPU time (s) of method for Test problem 1

τ	$\alpha = 1.1$		$\alpha = 1.7$		CPU time
	Error	C-order	Error	C-order	
1/10	5.9222×10^{-4}	–	1.7699×10^{-3}	–	0.004133
1/20	4.5650×10^{-5}	3.6974	1.2399×10^{-4}	3.8354	0.012522
1/40	3.2563×10^{-6}	3.8093	8.1009×10^{-6}	3.9360	0.112302
1/80	2.2103×10^{-7}	3.8809	5.0937×10^{-7}	3.9913	1.123820
1/160	1.4601×10^{-8}	3.9201	3.1424×10^{-8}	4.0188	13.62493

As we see from Table 3, the proposed method has approximately fourth order of accuracy in both space and time components which is compatible with theoretical ones.

4.2. Test problem 2. We consider the nonlinear space fractional Fisher equation

$$\frac{\partial u(x, t)}{\partial t} = \frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha} + u(1 - u) + f(x, t),$$

where

$$f(x, t) = 3t^2 \cos(t^3) - (\sin(t^3) + 1)(x - 1)^4 x^4 + [(\sin(t^3) + 1)(x - 1)^4 x^4]^2 + (1 + \sin(t^3))\Psi(x, \alpha),$$

The exact solution is $(1 + \sin(t^3))x^4(1 - x)^4$ and $x \in (0, 1)$. Tables 4,5 show the error, C-order and CPU time of present method for different values of T and α .

Table 4: Error, C-order and CPU time (s) of method for Test problem 2 at $T = 0.5$

$h = \tau$	$\alpha = 1.3$		$\alpha = 1.9$		CPU time
	Error	C-order	Error	C-order	
1/8	1.9471×10^{-5}	–	3.1473×10^{-5}	–	0.012414
1/16	1.6967×10^{-6}	3.5205	2.7475×10^{-6}	3.5179	0.049377
1/32	1.1649×10^{-7}	3.8645	1.7926×10^{-7}	3.9380	0.140291
1/64	7.5441×10^{-9}	3.9487	1.1350×10^{-8}	3.9813	1.228491
1/128	4.6715×10^{-10}	4.0134	7.0544×10^{-10}	4.0080	14.68158



Table 5: Error, C-order and CPU time (s) of method for Test problem 2 at $T = 1$

$h = \tau$	$\alpha = 1.3$		$\alpha = 1.9$		
	Error	C-order	Error	C-order	CPU time
1/8	3.1522×10^{-5}	—	5.6537×10^{-5}	—	0.048278
1/16	2.7466×10^{-6}	3.5206	4.4065×10^{-6}	3.6815	0.070451
1/32	1.8810×10^{-7}	3.8681	2.8613×10^{-7}	3.9449	0.300379
1/64	1.2172×10^{-8}	3.9499	1.8139×10^{-8}	3.9795	2.540614
1/128	7.5253×10^{-10}	4.0157	1.1273×10^{-9}	4.0082	30.00100

Tables 4,5 show the high accuracy and low CPU time of proposed method. Also the computational orders are in good agreement with theoretical ones.

4.3. Test problem 3. We consider the nonlinear space fractional parabolic-type sine-Gordon equation [26]

$$\frac{\partial u(x, t)}{\partial t} = \frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha} + \sin(u), \quad -5 \leq x \leq 5,$$

$$u(x, 0) = \frac{4 \exp(10x)}{(\exp(10x) + 1)^2}.$$

This problem does not have exact solution. Figures 1,2 show the approximate solution of this test problem for different values of α and T with $h = \tau = \frac{1}{40}$.

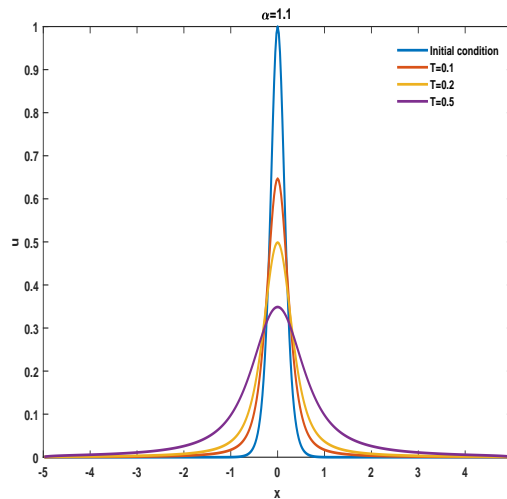


FIGURE 1. Plot of solitary wave solutions of Test problem 3 with $h = \tau = \frac{1}{40}$, $\alpha = 1.1$ and different values of T .



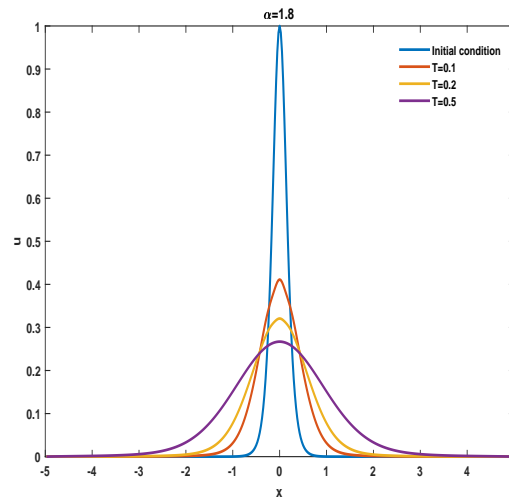


FIGURE 2. Plot of solitary wave solutions of Test problem 3 with $h = \tau = \frac{1}{40}$, $\alpha = 1.8$ and different values of T .

5. CONCLUSION

In this paper we proposed a high order numerical method for the solution of nonlinear diffusion equations with Riesz space fractional derivative. To this end, we applied a fourth-order quasi-compact finite difference approximation for space derivatives and a boundary value method of fourth-order in temporal direction. The numerical results are compared with analytical solutions and with those provided by other methods in the literature to show the high accuracy and efficiency of proposed method. Following the idea presented in [9, 10], the proposed method can be easily extended to two-dimensional Riesz space fractional diffusion equation.

REFERENCES

- [1] M. Abbaszadeh, *Error estimate of second-order finite difference scheme for solving the Riesz space distributed-order diffusion equation*, Appl. Math. Let., 88 (2019) 179-185.
- [2] P. Amodio, F. Mazzia, and D. Trigiante, *Stability of some boundary value methods for the solution of initial value problems*, BIT 33 (1993) 434-451.
- [3] M. R. Azizi and A. Khani, *Sinc operational matrix method for solving the Bagley-Torvik equation*, Comput. Methods Differ. Eq. 5 (2017) 56-66.
- [4] L. Brugnano and D. Trigiante, *Solving differential problems by multistep initial and boundary value methods*, Gordon and Beach Science Publishers, Amsterdam, 1998.
- [5] L. Brugnano and D. Trigiante, *Stability properties of some BVM methods*, Appl. Numer. Math. 13 (1993) 201-304.
- [6] L. Brugnano and D. Trigiante, *Boundary value methods: the third way between linear multistep and Runge-Kutta methods*, Computers Math. Applic. 36 (1998) 269-284.
- [7] C. Celik and M. Duman, *Crank-Nicolson method for the fractional diffusion equation with the Riesz fractional derivative*, J. Comput. Phys. 231 (2012) 1743-1750.



- [8] J. Chen, F. Liu, I. Turner, and V. Anh, *The fundamental and numerical solutions of the Riesz space-fractional reaction-dispersion equation*, ANZIAM J., 50 (2008) 45-57.
- [9] M. Dehghan and A. Mohebbi, *High-order compact boundary value method for the solution of unsteady convection-diffusion problems*, Math. Comput. Simul. 79 (2008) 683-699.
- [10] M. Dehghan and A. Mohebbi, *The use of compact boundary value method for the solution of two-dimensional Schrodinger equation*, J. Comput. Appli. Math. 225 (2009) 124-134.
- [11] M. Dehghan and M. Abbaszadeh, *An efficient technique based on finite difference/finite element method for solution of two-dimensional space/multi-time fractional Bloch-Torrey equations*, Appl. Numer. Math., 131 (2018) 190-206.
- [12] M. Dehghan, M. Abbaszadeh, and W. Deng, *Fourth-order numerical method for the space-time tempered fractional diffusion-wave equation*, Appl. Math. Let., 73 (2017) 120-127.
- [13] Z. P. Hao, Z. Z. Sun, and W. R. Cao, *A fourth-order approximation of fractional derivatives with its applications*, J. Comput. Phys. 281 (2015), pp. 787-805.
- [14] Z. Hao, K. Fan, W. Cao, and Z. Sun, *A finite difference scheme for semilinear space-fractional diffusion equations with time delay*, Appl. Math. Comput. 275 (2016) 238-254.
- [15] F. Iavernaro and F. Mazzia, *Convergence and stability of multistep methods solving nonlinear initial value problems*, SIAM J. Sci. Comput. 18 (1997) 270-285.
- [16] H. L. Liao, P. Lyu, and S. Vong, *Second-order BDF time approximation for Riesz space-fractional diffusion equations*, Int. J. Comput. Math., 95 (2018) 144-158.
- [17] F. Liu, V. Anh, and I. Turner, *Numerical solution of the space fractional Fokker-Planck equation*, J. Comput. Appl. Math. 166 (2004) 209-219.
- [18] A. Mohebbi, *On the split-step method for the solution of nonlinear Schrodinger equation with the Riesz spacefractional derivative*, Comput. Methods Differ. Eq. 4 (2016) 54-69.
- [19] K. B. Oldham and J. Spanier, *The Fractional Calculus: Theory and Application of Differentiation and Integration to Arbitrary Order*, Academic Press, 1974.
- [20] I. Podulbny, *Fractional differential equations*, New York: Academic Press; 1999.
- [21] A. Saadatmandi and M. Dehghan, *A tau approach for solution of the space fractional diffusion equation*, Comput. Math. Appl. 62 (2011) 1135-1142.
- [22] A. Saadatmandi and M. A. Darani, *The operational matrix of fractional derivative of the fractional-order Chebyshev functions and its applications*, Comput. Methods Differ. Eq. 5 (2017) 67-87.
- [23] H. Wang, C. Zhang, and Y. Zhou, *A class of compact boundary value methods applied to semi-linear reactiondiffusion equations*, Appl. Math. Comput. 325 (2018) 69-81.
- [24] H. Zhang and F. Liu, *Numerical simulation of the Riesz fractional diffusion equation with a nonlinear source term*, J. Appli. Math. Informatics, 26 (2008) 1-14.
- [25] Y. X. Zhang and H. F. Ding, *Improved matrix transform method for the Riesz space fractional reaction dispersion equation*, J. Comput. Appl. Math. 260 (2014) 266-280.
- [26] Y. Zhou and Z. Luo, *A Crank-Nicolson finite difference scheme for the Riesz space fractional-order parabolic-type sine-Gordon equation*, Adv. Di. Equ. (2018) 2018:216.
- [27] S. Yang, *Finite difference method for Riesz space fractional diffusion equations with delay and a nonlinear source term*, J. Nonlinear Sci. Appl., 11 (2018) 17-25.

