VIM-Padé technique for solving nonlinear and delay initial value problems

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Abstract	In this work, we employ a combination of variational iteration method (VIM) and Padé approximation method, called the VIM-Padé technique, to solve some nonlinear initial value problems and a delay differential equation (DDE). Some examples are provided to illustrate the capability and reliability of the technique. The obtained results by using the VIM are compared to the results of this technique. This comparison shows that VIM-Padé technique is more effective than VIM and yields faster convergence compared to the VIM.

1. Introduction

Many phenomena and problems in real world can be described by nonlinear initial value problems (NIVPs). Also, Delay differential equations appear mostly in the mathematical modeling of various science and engineering, particularly in the field of physics, chemistry, population dynamics, bio science and many other fields [9, 11, 16, 30, 37]. Although the exact solutions of NIVPs and DDEs are essential, there are many NIVPs and DDE which cannot be solved analytically. Due to this fact, finding the favorite approximate solution of nonlinear and delay problems is absolutely necessary. Recently, some suitable and effective methods have been emerged

for obtaining the approximate solution of NIVPs and DDEs. Homotopy analysis method [3–6, 17–19, 21, 22, 26–28, 31, 32, 35, 36], variational iteration method [12–15, 23, 25, 41], Adomian decomposition method [7, 8, 20, 29, 38, 39], and homotopy perturbation method [2] are examples of the methods. J. H. He in [24] proposed the variational iteration method (VIM) which has been employed by many mathematicians and authors [12–15, 23, 25, 41] in all over the world as a powerful and effective mathematical tool for obtaining approximate solutions of some problems. Some linear and nonlinear systems of ordinary differential equations (ODEs) are solved by using the VIM. The obtained approximate solutions in each iteration of the VIM form a functional sequence which converges to the exact solution of the problem. One of the important features of this method is that, in order to solve problems, it does not need to discretize the variables. Therefore, in some cases, computational round off errors have no effect on it and, as a result, large computer memory and time are not needed. The provided solution of the problem by the proposed scheme is in a closed form, while the techniques based on mesh points, finite difference method [33] for example, provide the approximation only at mesh points. In spite of these advantages, VIM has some drawbacks. The invalidity of this method in solving some problems and the slow convergence, especially in problems which are presented by a differential equation with complicated non-homogeneous function or functional-differential equation with proportional delay, are examples of the drawbacks. In the present paper we will apply a new modification of the mentioned method which is based on the use of Padé approximation [10, 40]. This modification will overcome the mentioned disadvantages and will accelerate the convergence rate. In this method, called VIM- Padé technique, to obtain more accurate analytical approximation, we have combined the VIM and Padé approximation method for solving some nonlinear and delay problems as follows: first, using the VIM, series solution will be obtained; then, we will apply the fit Padé approximation to this solution. Three examples will be presented to illustrate the efficiency of the VIM- Padé technique.

2. Basics ideas of the variational iteration method and VIM-Padé technique

2.1. Variational iteration method. Consider the following functional equation;

$$L(y) + N(y) = f(x),$$

where L and N are linear and nonlinear operators, respectively and $f(x)$ is a known analytical function. According to the variational iteration method, we consider the following correction functional

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi)[Ly_n(\xi) + N\tilde{y}_n(\xi) - f(\xi)]d\xi,$$

where $\lambda(\xi)$ is general Lagrange multiplier which can be obtained optimally by variational theory. Also $\tilde{y}_n$ is considered as a restricted variation, i.e. $\sigma\tilde{y}_n = 0$. In this method first, using integration by parts, the Lagrange multiplier λ will be determined. Using the determined Lagrange multiplier and the initial approximation

$y_0(x)$, the successive approximations y_{n+1} , $n \geq 0$ of the solution will be readily obtained. Therefore, the solution will be given by $y(x) = \lim_{n \to \infty} y_n(x)$.

This method gives rapidly convergent successive approximations of the exact solution if such a solution exists. Sometimes, the solutions obtained is possible have a restricted region of convergence for taking n to be as large as possible. Therefore, we used the Padé technique on the series obtained to increase the convergence region. The propose method illustrates that a low number of VIM iterations and Padé approximation can yield better solutions than VIM.

2.2. **VIM-Padé technique.** Sometimes, the solutions obtained of VIM as series have a finite domain of convergence even when n to be as large as possible. There are various methods to improve the convergence region of solution series. Of these, the Padé approximation method has attracted much attention that is applied to improve the accuracy of the VIM solution by a rational function [34]. Therefore, the Padé technique is used on the series obtained to rise the convergence domain.

The corresponding $[m, n]$ Padé approximation for solution series obtained from variational iteration method is given by

$$P[m, n]y_v(x) = \frac{\sum_{k=0}^{m} A_k x^k}{\sum_{k=0}^{n} B_k x^k}.$$

By using the normalization condition $B_0 = 1$, we can determine the A_k and B_k coefficients using formal power series [1].

3. EXAMPLES

In this section, we present two numerical examples to test the accuracy and efficiency of our proposed method.

In this Section we present a detailed computer simulations and discussion of our proposed method on the results for three problems.For all examples, the exact solutions are available. We performed all the symbolic and numerical computations by using Maple 18. The following error norms

$$RMSE = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (y_{ex}(x_i) - y_{ap}(x_i))^2},$$

$$L_2 = \sqrt{\int_{a}^{b} (y_{ex}(x) - y_{ap}(x))^2 dx},$$

are computed to show the accuracy of the VIM- Padé method and VIM method and are used to compare them.

Example 3.1. *Consider the following homogeneous nonlinear Emden-Fowler equation [34]:*

$$y'' + \frac{1}{x}y' - y^3 + 3y^5 = 0, \tag{3.1}$$

with initial conditions

$$y(0) = 1, y'(0) = 0.$$

According to variational iteration method (VIM); we construct a correct functional as follows

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi)[y_n''(\xi) + \frac{1}{\xi}y_n'(\xi) - \tilde{y}_n^3(\xi) + 3\tilde{y}_n^5(\xi)]d\xi \qquad (3.2)$$

where $\lambda(\xi)$ is a general Lagranges multiplier which can be identified optimally via variational theory; and $\tilde{y}_n$ is a restricted variation which $\delta\tilde{y}_n = 0$. To determine the optimal value of $\lambda(\xi)$ we take the variation for both sides with respect to $y_n(x)$ obtain

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_0^x \lambda(\xi)[y_n''(\xi) + \frac{1}{\xi}y_n'(\xi) - \tilde{y}_n^3(\xi) + 3\tilde{y}_n^5(\xi)]d\xi.$$

Or equivalently

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_0^x \lambda(\xi)[y_n''(\xi) + \frac{1}{\xi}y_n'(\xi)]d\xi.$$

By the integrating the integral at the right side by parts yields

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta[1 - \lambda'(x) + \frac{1}{x}\lambda(x)] + \delta\lambda(x)(y_n)_\xi(x)$$
$$+ \delta \int_0^x y_n(\xi)\left(\lambda''(\xi) - \frac{\xi\lambda'(\xi) - \lambda(\xi)}{\xi^2}\right)d\xi.$$

This yields the following stationary conditions

$$\begin{aligned} &\lambda(\xi = x) = 0; \\ &\lambda'(\xi = x) = 1; \\ &\lambda'' - \frac{\xi\lambda' - \lambda}{x^2} = 0. \end{aligned} \qquad (3.3)$$

Solving (3.3) gives $\lambda(\xi) = \xi\ln(\frac{\xi}{x})$. Substituting this value into Eq. (3.2) results the following iterate formula

$$y_{n+1}(x) = y_n(x) + \int_0^x \xi\ln(\frac{\xi}{x})\left[y_n''(\xi) + \frac{1}{\xi}y_n'(\xi) - y_n^3(\xi) + 3y_n^5(\xi)\right]d\xi, \quad n \geq 0 \ (3.4)$$

Assuming $y_0(x) = 1$; as an initial approximation that satisfies the initial condition; from Eq. (3.4) we obtain the following successive approximations and corresponding $[m, n]$ Padé approximations for arbitrary m, n

$$y_1(x) = 1 - \frac{1}{2}x^2,$$
$$y_2(x) = 1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{3}{16}x^6 + \frac{29}{512}x^8 - \frac{3}{320}x^{10} + \frac{1}{1536}x^{12},$$
$$y_3(x) = 1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{5}{16}x^6 + \frac{1}{4}x^8 - \frac{477}{2560}x^{10} + \frac{15821}{122880}x^{12}$$
$$- \frac{5909}{71680}x^{14} + \frac{128307}{2621440}x^{16} + ..$$

The exact solution of (3.1) is given by

$$y(x) = \frac{1}{\sqrt{1 + x^2}}.$$

Therefore, the complete approximate solution can be readily obtained by the same iterative process.

Obviously, the above VIM solutions are expressed in series form. Comparisons of the second, third and forth-order VIM solutions (y_2, y_3 and y_4) with the exact solution are shown in Figure 1. We see that the VIM solutions error may enlarge with increase of x. So, we present a modification in VIM (using Padé approximation) to improve the error.
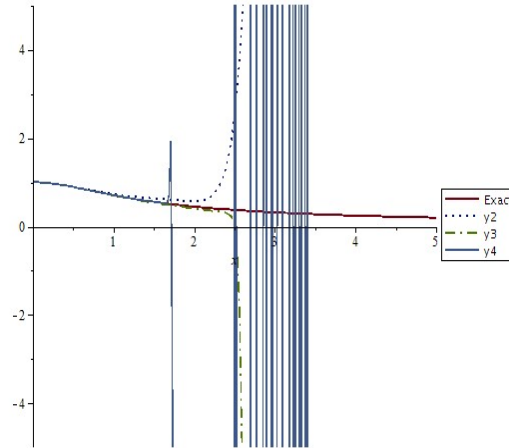

FIGURE 1. Comparison of the second, third and forth-order approximate solutions using VIM and the exact solution for example 3.1.

Now, we find the Padé approximations of the VIM solutions. For example, The $[2, 2]$ and $[6, 6]$ Padé approximations of $y_2(x)$, respectively are

$$P[2,2](y_2(x)) = \frac{1 + \frac{1}{4}x^2}{1 + \frac{3}{4}x^2}$$

$$P[6,6](y_2(x)) = \frac{1 + \frac{2275}{2064}x^2 + \frac{33}{80}x^4 - \frac{281}{82560}x^6}{1 + \frac{3307}{2064}x^2 + \frac{17309}{20640}x^4 + \frac{2633}{20640}x^6}.$$

As the same way, we compute the $[j, j]$ Padé approximations of $y_3(x)$ and $y_4(x)$. The second, third and forth-order VIM solutions and effective approximate solutions using present technique compared with the exact solution in Figure 2. From these figures, we can see that the proposed method solutions are in excellent agreement with the exact solutions. Moreover, the L_2 and RMS errors of $y_n(x)$, $(n = 2, 3, 4)$ for this example are listed in Table 1 to compare the accuracy of VIM and various VIM- Padé strategies.

Example 3.2. *Consider the following nonlinear Riccati equation [36] with the indicated initial condition:*

$$y'(x) = 1 + x^2 - y^2(x), \quad y(0) = 1. \tag{3.5}$$

TABLE 1. The L_2 and RMSE errors for $y_n(x), n = 2, 3, 4$ for Example 3.1 with VIM and VIM- Padé ($p[j, j]$) technique.

|	L_2 *Error*			*RMSE Error*		
n	*VIM*	*VIM-Padé*		*VIM*	*VIM-Padé*	
2	*1522.3037*	$p[2, 2]$	*0.1508*	*766.9860*	$p[2, 2]$	*0.0754*
		$p[3, 3]$	*0.1508*		$p[3, 3]$	*0.0754*
		$p[4, 4]$	*0.5343*		$p[4, 4]$	*0.2672*
		$p[5, 5]$	*0.5343*		$p[5, 5]$	*0.2672*
		$p[6, 6]$	*0.2387*		$p[6, 6]$	*0.1195*
3	2.423×10^{15}	$p[2, 2]$	*0.1508*	9.3645×10^{14}	$p[2, 2]$	*0.0754*
		$p[3, 3]$	*0.1508*		$p[3, 3]$	*0.0754*
		$p[4, 4]$	*0.8061*		$p[4, 4]$	*1.5866*
		$p[6, 6]$	*0.1117*		$p[6, 6]$	*0.0559*
		$p[24, 24]$	*0.1014*		$p[24, 24]$	*0.0507*
		$p[28, 28]$	*0.0955*		$p[28, 28]$	*0.0477*
4	6.036×10^{76}	$p[2, 2]$	*0.1508*	2.4319×10^{75}	$p[2, 2]$	*0.0754*
		$p[4, 4]$	*0.0390*		$p[4, 4]$	*0.0195*
		$p[6, 6]$	*0.0423*		$p[6, 6]$	*0.0211*
		$p[20, 20]$	*0.0070*		$p[20, 20]$	*0.0035*

(a)   (b)
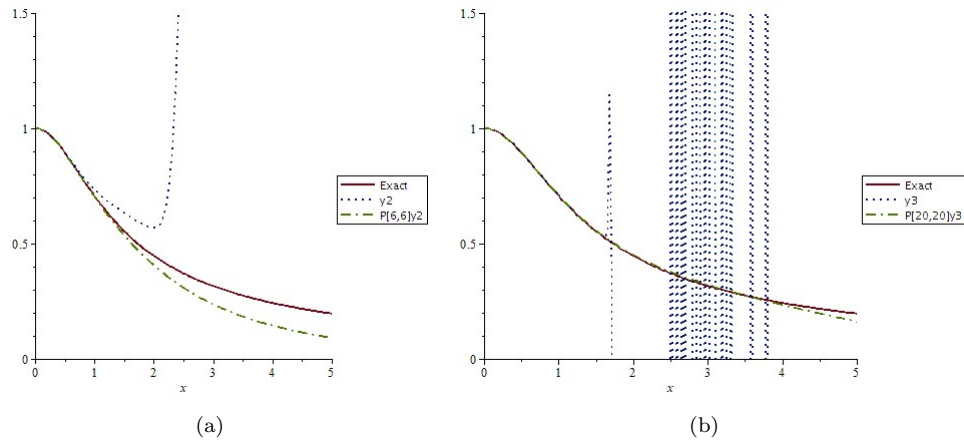
FIGURE 2. Comparison of the second (A) and third (B) approximate solutions the approximate solutions using VIM and VIM-Padé with the exact solution for example 3.1.

The exact solution of (3.5) *is given by*

$$y(x) = x + \frac{e^{-x^2}}{1 + \int_0^x e^{-t^2} dt}.$$
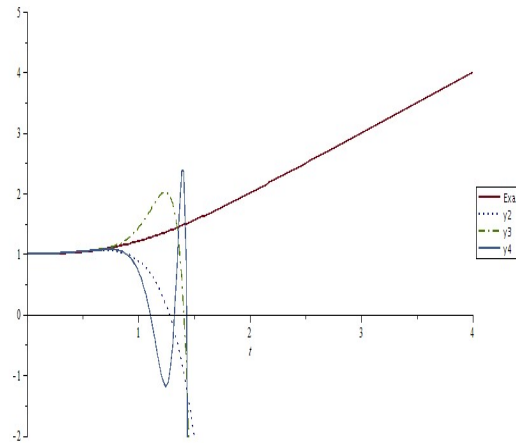
FIGURE 3. Comparison of the second, third and forth-order approximate solutions using VIM and the exact solution for example 3.2.

According to variational iteration method (VIM); we derive a correct functional as follows

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi)[y'_n(\xi) + \tilde{y}_n^2(\xi) - 1 - \xi^2]d\xi, \tag{3.6}$$

where the Lagranges multiplier $\lambda(\xi)$ is easily identified to be $\lambda = -1$. Therefore, we have the following variational iteration formula

$$y_{n+1}(x) = y_n(x) - \int_0^x [y'_n(\xi) + y_n^2(\xi) - 1 - \xi^2]d\xi. \tag{3.7}$$

Choosing $y_0(x) = 1$, the above iteration formula yields the following approximate solutions:

$$\begin{aligned} y_1(x) &= 1 + \frac{1}{3}x^3, \\ y_2(x) &= 1 + \frac{2}{3}x^3 - \frac{2}{3}x^4 - \frac{1}{9}x^7, \\ y_3(x) &= 1 + x^3 - 2x^4 + \frac{4}{3}x^5 + ... - \frac{1}{81}x^{15}, \\ &\quad . \\ &\quad . \end{aligned} \tag{3.8}$$

Now, the accuracy of VIM solution are improved by Padé technique. A few iterations of VIM and Padé can be applied for this problem with a high degree of accuracy. The graphs of second, third and forth-order approximate solutions by VIM and the exact solution are given in Fig.3. Comparing the VIM solutions with effective approximate solutions obtained by present technique are shown in Fig.4. Table 2 shows the results of computing the L_2 and $RMSE$ errors of approximate $y_n(x), n = 2, 3, 4$ for example 3.2 with VIM and various VIM- Padé strategies. In this table, similar to the
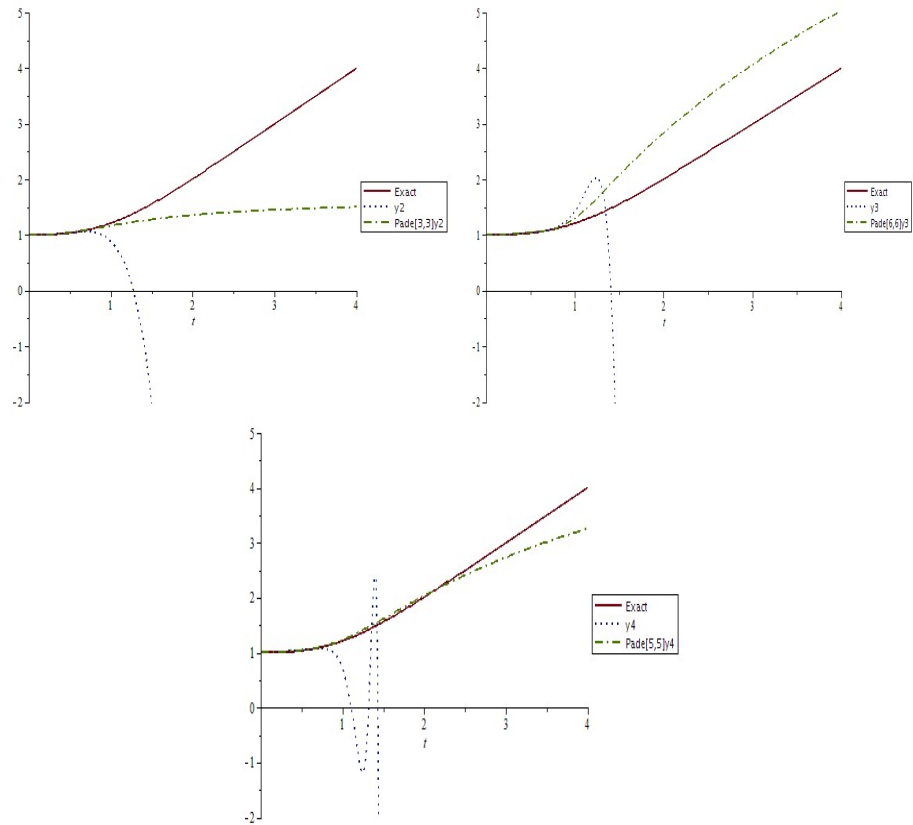
FIGURE 4. Comparison of the second (A), third (B) and forth-order (C) approximate solutions using VIM and VIM-Padé technique with the exact solution for example 3.2.

results of the table 1, we can observe that VIM- Padé strategy with regards to VIM one is more efficient.

TABLE 2. The L_2 and RMSE errors for $y_n(x), n = 2, 3, 4$ for Example 3.2 with VIM and VIM- Padé ($p[j, j]$) technique.

| n	L_2 **Error**			**RMSE Error**		
	VIM	**VIM-Padé**		**VIM**	**VIM-Padé**	
2	1023.0619	$p[2, 2]$	3.0106	514.0983	$p[2, 2]$	1.5065
		$p[3, 3]$	2.3562		$p[3, 3]$	1.1793
		$p[4, 4]$	39.6056		$p[4, 4]$	106.5590
		$p[5, 5]$	6.5045		$p[5, 5]$	3.2552
		$p[6, 6]$	16.3156		$p[6, 6]$	8.1579
3	5.5411×10^6	$p[2, 2]$	3.0106	2.7907×10^6	$p[2, 2]$	1.5065
		$p[3, 3]$	2.6253		$p[3, 3]$	1.3139
		$p[4, 4]$	3.1206		$p[4, 4]$	1.5616
		$p[5, 5]$	3.8994		$p[5, 5]$	1.9503
		$p[6, 6]$	1.5475		$p[6, 6]$	0.7736
		$p[7, 7]$	11.1594		$p[7, 7]$	25.1438
4	2.4664×10^{14}	$p[2, 2]$	3.0106	1.2027×10^{14}	$p[2, 2]$	1.5065
		$p[3, 3]$	2.7307		$p[3, 3]$	1.3665
		$p[4, 4]$	2.7610		$p[4, 4]$	1.3817
		$p[5, 5]$	0.5310		$p[5, 5]$	0.2661
		$p[6, 6]$	1.8717		$p[6, 6]$	0.9370
		$p[7, 7]$	1.3951		$p[7, 7]$	0.6985
		$p[8, 8]$	3.2060		$p[8, 8]$	1.6042
		$p[9, 9]$	21.6977		$p[9, 9]$	22.6751
		$p[10, 10]$	52.7127		$p[10, 10]$	137.5276
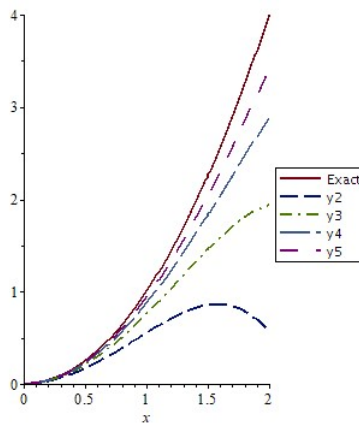
FIGURE 5. Comparison of the approximate solutions using VIM with the exact solution for Example 3.3.

Example 3.3. *Consider the second-order neutral functional-differential equation with proportional delay,*

$$y''(x) = \frac{3}{4}y(x) + y(\frac{x}{2}) + y'(\frac{x}{2}) + \frac{1}{2}y''(\frac{x}{2}) - x^2 - x + 1, \quad 0 < x < 1,$$
$$y(0) = y'(0) = 0.$$
(3.9)

Here, the Lagrange multiplier is found to be $\lambda = s - t$ ([18]). Therefore, by using the variational iteration method (VIM), the corresponding iteration formula assumes the form

$$y_{n+1}(x) = y_n(x) + \int_0^x (s - x)\left[y''(x) - \frac{3}{4}y(x) - y(\frac{x}{2}) - y'(\frac{x}{2})\right.$$
$$\left. - \frac{1}{2}y''(\frac{x}{2}) + x^2 + x - 1\right]ds.$$
(3.10)

TABLE 3. The L_2 and RMSE errors for $y_n(x), n = 2, 3, 4$ for Example 3.3 with VIM and VIM-Padé $(p[2, 2])$ technique.

	L_2 **Error**			**RMSE Error**		
n	2	3	4	2	3	4
VIM	30.9054	22.7472	13.8000	15.4985	11.4122	6.9251
VIM-Padé	10.4108	7.4326	4.7120	5.2136	3.7230	2.3608

Assuming $y_0(x) = 0$ as an initial approximation that satisfies the initial conditions, from the above iteration formula (3.10), we obtain the following successive approximations

$$y_1(x) = -\frac{1}{12}x^4 - \frac{1}{6}x^3 + \frac{1}{2}x^2,$$
$$y_2(x) = -\frac{13}{5760}x^6 - \frac{3}{320}x^5 - \frac{1}{16}x^4 - \frac{1}{8}x^3 + \frac{3}{4}x^2,$$
$$y_3(x) = -\frac{91}{2949120}x^8 - \frac{17}{92160}x^7 - \frac{343}{184320}x^6 - \frac{39}{5120}x^5 - \frac{7}{192}x^4$$
$$- \frac{7}{96}x^3 + \frac{7}{8}x^2,$$
(3.11)

Now, we apply Padé approximation to improve the accuracy of the VIM solution. A few approximations of VIM iterations and Padé can be used for this problem with a high degree of accuracy. The comparison some of these approximate solutions with the exact solution $y(x) = x^2$ is shown in Fig.5. In Fig.6 the comparison of these approximations and VIM- Padé technique solutions for them with the exact solution are shown. Fig.6 and Table 3 show that a low number of VIM iterations and Padé approximation $(p[2, 2])$ can yield better solutions than VIM.
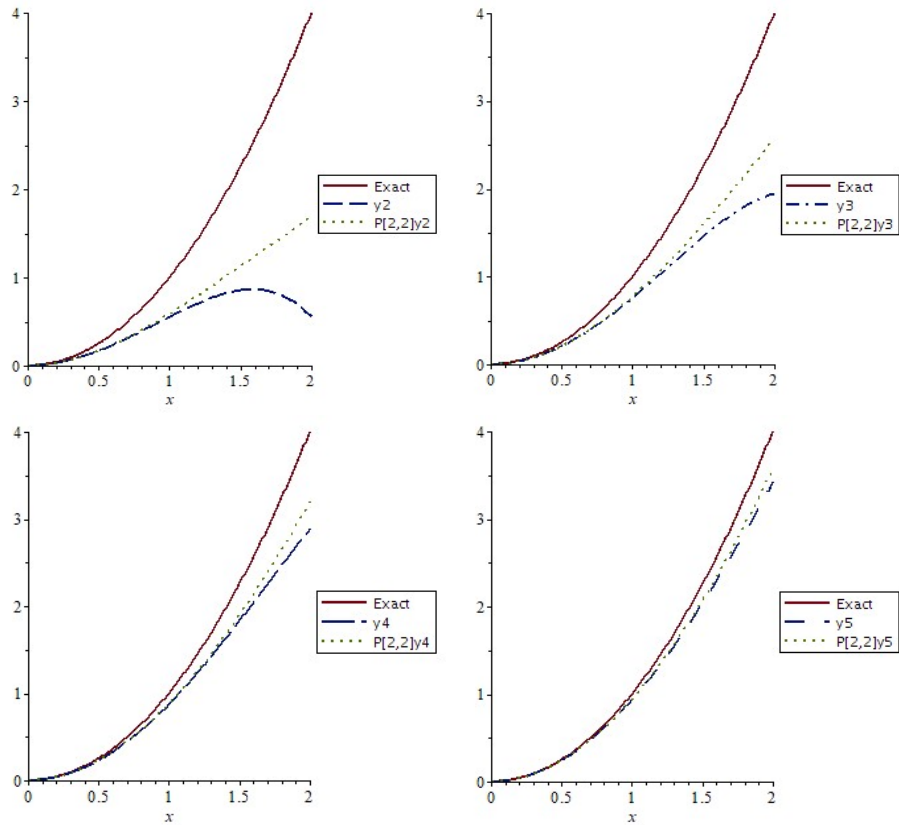
FIGURE 6. Comparison of the approximate solutions using VIM and VIM-Padé $[2, 2]$ with the exact solution for Example 3.3.

4. CONCLUSION

In this study, a combination of the VIM and the Padé approximation method, called the VIM-Padé technique, was effectively applied to find the approximate solution of two nonlinear differential equations and a functional-differential equation with proportional delay. In VIM the domain of convergence of the obtained truncated series solution is limited and we successfully used Padé technique with the VIM to increase the convergence region. We can conclude that the results obtained reveal that applying the VIM-Padé approximation method gives better results in comparison to the VIM.

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